Problem 9.12

The proof, which consists of proving that
\[ \{ x \in \mathbb{Z}^2 \mid x + b \in A, \text{ for every } b \in B \} = \{ x \in \mathbb{Z}^2 \mid (B)_x \subseteq A \}, \]
follows directly from the definition of translation because the set \((B)_x\) has elements of the form \(x + b\) for \(b \in B\). That is, \(x + b \in A\) for every \(b \in B\) implies that \((B)_x \subseteq A\). Conversely, \((B)_x \subseteq A\) implies that all elements of \((B)_x\) are contained in \(A\), or \(x + b \in A\) for every \(b \in B\).

Problem 9.13

(a) Let \(x \in A \ominus B\). Then, from the definition of erosion given in the problem statement, for every \(b \in B, x + b \in A\). But, \(x + b \in A\) implies that \(x \in (A)_-b\). Thus, for every \(b \in B, x \in (A)_-b\), which implies that \(x \in \bigcap_{b \in B} (A)_-b\). Suppose now that \(x \in \bigcap_{b \in B} (A)_-b\). Then, for every \(b \in B, x \in (A)_-b\). Thus, for every \(b \in B, x + b \in A\) which, from the definition of erosion, means that \(x \in A \ominus B\).

(b) Suppose that \(x \in A \ominus B = \bigcap_{b \in B} (A)_-b\). Then, for every \(b \in B, x \in (A)_-b\), or \(x + b \in A\). But, as shown in Problem 9.12, \(x + b \in A\) for every \(b \in B\) implies that \((B)_x \subseteq A\), so that \(x \in A \ominus B = \{ x \in \mathbb{Z}^2 \mid (B)_x \subseteq A \}\). Similarly, \((B)_x \subseteq A\) implies that all elements of \((B)_x\) are contained in \(A\), or \(x + b \in A\) for every \(b \in B\) or, as in (a), \(x + b \in A\) implies that \(x \in (A)_-b\). Thus, if for every \(b \in B, x \in (A)_-b\), then \(x \in \bigcap_{b \in B} (A)_-b\).