Problem 9.10

The approach is to prove that
\[
\left\{ x \in \mathbb{Z}^2 \left| (\hat{B})_x \cap A \neq \emptyset \right. \right\} \equiv \left\{ x \in \mathbb{Z}^2 \left| x = a + b \right. \text{ for } a \in A \text{ and } b \in B \right\}.
\]
The elements of \((\hat{B})_x\) are of the form \(x - b\) for \(b \in B\). The condition \((\hat{B})_x \cap A \neq \emptyset\) implies that for some \(b \in B\), \(x - b \in A\), or \(x - b = a\) for some \(a \in A\) (note in the preceding equation that \(x = a + b\)). Conversely, if \(x = a + b\) for some \(a \in A\) and \(b \in B\), then \(x - b = a\) or \(x - b \in A\), which implies that \((\hat{B})_x \cap A \neq \emptyset\).

Problem 9.11

(a) Suppose that \(x \in A \oplus B\). Then, for some \(a \in A\) and \(b \in B\), \(x = a + b\). Thus,
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The proof, which consists of proving that

\[ x \in (A)_b \] and, therefore, \[ x \in \bigcup_{b \in B} (A)_b. \]

On the other hand, suppose that \( x \in \bigcup_{b \in B} (A)_b. \)

Then, for some \( b \in B, x \in (A)_b. \) However, \( x \in (A)_b \) implies that there exists an \( a \in A \) such that \( x = a + b. \) But, from the definition of dilation given in the problem statement, \( a \in A, b \in B, \) and \( x = a + b \) imply that \( x \in A \odot B. \)

(b) Suppose that \( x \in \bigcup_{b \in B} (A)_b. \) Then, for some \( b \in B, x \in (A)_b. \) However, \( x \in (A)_b \) implies that there exists an \( a \in A \) such that \( x = a + b. \) But, if \( x = a + b \) for some \( a \in A \) and \( b \in B, \) then \( x - b = a \) or \( x - b \in A, \) which implies that \( x \in (\hat{B})_x \cap A \neq \emptyset. \) Now, suppose that \( x \in (\hat{B})_x \cap A \neq \emptyset. \) The condition \( (\hat{B})_x \cap A \neq \emptyset \) implies that for some \( b \in B, x - b \in A \) or \( x - b = a \) (i.e., \( x = a + b \)) for some \( a \in A. \) But, if \( x = a + b \) for some \( a \in A \) and \( b \in B, \) then \( x \in (A)_b \) and, therefore, \( x \in \bigcup_{b \in B} (A)_b. \)

Problem 9.12

The proof, which consists of proving that

\[ \{ x \in \mathbb{Z}^2 | x + b \in A, \text{ for every } b \in B \} \equiv \{ x \in \mathbb{Z}^2 | (B)_x \subseteq A \}, \]

follows directly from the definition of translation because the set \((B)_x\) has elements of the form \( x + b \) for \( b \in B. \) That is, \( x + b \in A \) for every \( b \in B \) implies that \((B)_x \subseteq A. \)

Conversely, \((B)_x \subseteq A\) implies that all elements of \((B)_x\) are contained in \( A, \) or \( x + b \in A \) for every \( b \in B. \)

Problem 9.13

(a) Let \( x \in A \odot B. \) Then, from the definition of erosion given in the problem statement, for every \( b \in B, x + b \in A. \) But, \( x + b \in A \) implies that \( x \in (A)_{-b}. \) Thus, for every \( b \in B, x \in (A)_{-b}, \) which implies that \( x \in \bigcap_{b \in B} (A)_{-b}. \) Suppose now that \( x \in \bigcap_{b \in B} (A)_{-b}. \) Then, for every \( b \in B, x \in (A)_{-b}. \) Thus, for every \( b \in B, x + b \in A \) which, from the definition of erosion, means that \( x \in A \odot B. \)

(b) Suppose that \( x \in A \odot B = \bigcap_{b \in B} (A)_{-b}. \) Then, for every \( b \in B, x \in (A)_{-b}, \) or \( x + b \in A. \) But, as shown in Problem 9.12, \( x + b \in A \) for every \( b \in B \) implies that \((B)_x \subseteq A, \) so that \( x \in A \odot B = \{ x \in \mathbb{Z}^2 | (B)_x \subseteq A \}. \) Similarly, \((B)_x \subseteq A \) implies that all elements of \((B)_x\) are contained in \( A, \) or \( x + b \in A \) for every \( b \in B \) or, as in (a), \( x + b \in A \) implies that \( x \in (A)_{-b}. \) Thus, if for every \( b \in B, x \in (A)_{-b}, \) then \( x \in \bigcap_{b \in B} (A)_{-b}. \)