Chapter 3 Problem Solutions

or student made this assumption from the beginning, then this answer follows almost by inspection.

**Problem 3.24**

The student should realize that both the Laplacian and the averaging process are linear operations, so it makes no difference which one is applied first.

**Problem 3.25**

The Laplacian operator is defined as

\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \]

for the unrotated coordinates and as

\[ \nabla^2 f = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \]

for rotated coordinates. It is given that

\[ x = x' \cos \theta - y' \sin \theta \quad \text{and} \quad y = x' \sin \theta + y' \cos \theta \]

where \( \theta \) is the angle of rotation. We want to show that the right sides of the first two equations are equal. We start with

\[
\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x'} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta.
\]

Taking the partial derivative of this expression again with respect to \( x' \) yields

\[
\frac{\partial^2 f}{\partial x'^2} = \frac{\partial^2 f}{\partial x^2} \cos^2 \theta + \frac{\partial f}{\partial x} \left( \frac{\partial f}{\partial y} \right) \sin \theta \cos \theta + \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial x} \right) \cos \theta \sin \theta + \frac{\partial^2 f}{\partial y^2} \sin^2 \theta.
\]

Next, we compute

\[
\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial y'} = \frac{\partial f}{\partial x} \sin \theta + \frac{\partial f}{\partial y} \cos \theta.
\]

Taking the derivative of this expression again with respect to \( y' \) gives

\[
\frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} \sin^2 \theta - \frac{\partial f}{\partial x} \left( \frac{\partial f}{\partial y} \right) \cos \theta \sin \theta - \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial x} \right) \sin \theta \cos \theta + \frac{\partial^2 f}{\partial y^2} \cos^2 \theta.
\]

Adding the two expressions for the second derivatives yields

\[
\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y^2}.
\]