Problem 4.3

Ideal low-pass filter with variable bandwidth. The transfer function of the matched filter for a rectangular pulse of duration \( \tau \) and amplitude \( A \) is given by

\[
H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT)
\]  

(1)

The amplitude response \( |H_{opt}(f)| \) of the matched filter is plotted in Fig. 1(a). We wish to approximate this amplitude response with an ideal low-pass filter of bandwidth \( B \). The amplitude response of this approximating filter is shown in Fig. 1(b). The requirement is to determine the particular value of bandwidth \( B \) that will provide the best approximation to the matched filter.

We recall that the maximum value of the output signal, produced by an ideal low-pass filter in response to the rectangular pulse occurs at \( t = T/2 \) for \( BT \leq 1 \). This maximum value, expressed in terms of the sine integral, is equal to \((2A/\pi)\text{Si}(\pi BT)\). The average noise power at the output of the ideal low-pass filter is equal to \( BN_0 \). The maximum output signal-to-noise ratio of the ideal low-pass filter is therefore

\[
(SNR)_0' = \frac{(2A/\pi)^2\text{Si}^2(\pi BT)}{BN_0}
\]  

(2)

Thus, using Eqs. (1) and (2), and assuming that \( AT = 1 \), we get

\[
\frac{(SNR)'_0}{(SNR)_0} = \frac{2}{\pi^2BT} \text{Si}^2(\pi BT)
\]

This ratio is plotted in Fig. 2 as a function of the time-bandwidth product \( BT \). The peak value on this curve occurs for \( BT = 0.685 \), for which we find that the maximum signal-to-noise ratio of the ideal low-pass filter is 0.84 dB below that of the true matched filter. Therefore, the "best" value for the bandwidth of the ideal low-pass filter characteristic of Fig. 1(b) is \( B = 0.685/T \).
Figure 1

Figure 2