### Table of Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Time-Domain: ( x(t) )</th>
<th>Frequency-Domain: ( X(j\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-at} u(t) ) (( a &gt; 0 ))</td>
<td>( \frac{1}{a + j\omega} )</td>
</tr>
<tr>
<td>( e^{bt} u(-t) ) (( b &gt; 0 ))</td>
<td>( \frac{1}{b - j\omega} )</td>
</tr>
<tr>
<td>( u(t + \frac{T}{2}) - u(t - \frac{T}{2}) )</td>
<td>( \frac{\sin(\omega T/2)}{\omega^2 / 2} )</td>
</tr>
<tr>
<td>( \frac{\sin(\omega t)}{\pi t} )</td>
<td>( [a(\omega - \omega_0) - a(\omega + \omega_0)] )</td>
</tr>
<tr>
<td>( \delta(t) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( \delta(t - t_0) )</td>
<td>( e^{-j\omega t_0} )</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>( \pi \delta(\omega) + \frac{1}{j\omega} )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 2\pi \delta(\omega) )</td>
</tr>
<tr>
<td>( e^{j\omega t} )</td>
<td>( 2\pi \delta(\omega - \omega_0) )</td>
</tr>
<tr>
<td>( A \cos(\omega_0 t + \phi) )</td>
<td>( \pi A e^{j\phi} \delta(\omega - \omega_0) + \pi A e^{-j\phi} \delta(\omega + \omega_0) )</td>
</tr>
<tr>
<td>( \cos(\omega_0 t) )</td>
<td>( \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) )</td>
</tr>
<tr>
<td>( \sin(\omega_0 t) )</td>
<td>( -j\pi \delta(\omega - \omega_0) + j\pi \delta(\omega + \omega_0) )</td>
</tr>
<tr>
<td>( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} )</td>
<td>( \sum_{k=-\infty}^{\infty} a_k \pi \delta(\omega - k\omega_0) )</td>
</tr>
<tr>
<td>( \sum_{k=-\infty}^{\infty} \delta(t - nT) )</td>
<td>( \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T}) )</td>
</tr>
</tbody>
</table>

### Table of Fourier Transform Properties

<table>
<thead>
<tr>
<th>Property Name</th>
<th>Time-Domain: ( x(t) )</th>
<th>Frequency-Domain: ( X(j\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( ax_1(t) + bx_2(t) )</td>
<td>( aX_1(j\omega) + bX_2(j\omega) )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*(t) )</td>
<td>( X^*(-j\omega) )</td>
</tr>
<tr>
<td>Time-Reversal</td>
<td>( x(-t) )</td>
<td>( X(-j\omega) )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( x(at) )</td>
<td>( \frac{1}{a}X(j\omega/a) )</td>
</tr>
<tr>
<td>Delay</td>
<td>( x(t - t_0) )</td>
<td>( e^{-j\omega t_0} X(j\omega) )</td>
</tr>
<tr>
<td>Modulation</td>
<td>( x(t)e^{j\omega_0 t} )</td>
<td>( X(j\omega - \omega_0) )</td>
</tr>
<tr>
<td>Modulation</td>
<td>( x(t) \cos(\omega_0 t) )</td>
<td>( \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0)) )</td>
</tr>
<tr>
<td>Differentiation</td>
<td>( \frac{d^2 x(t)}{dt^2} )</td>
<td>( (j\omega)^2 X(j\omega) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x(t) * h(t) )</td>
<td>( X(j\omega) H(j\omega) )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x(t) p(t) )</td>
<td>( \frac{1}{2} X(j\omega) + P(j\omega) )</td>
</tr>
</tbody>
</table>
1. [25 pts] Continuous–Time Signals and Systems:
Evaluate and simplify the following:

(a) [8 pts] \( \delta(t - 7) \ast [3e^{-4(t-2)} + 5\delta(t) + \cos(25\pi t)] \ast \delta(t + 3) \)

(b) [8 pts] \( \int_{-\infty}^{t} \tau^{-3} \delta(\tau - 5) d\tau \)

(c) [9 pts] \( \left[ \frac{d}{dt} \text{Re}\{2e^{j8\pi t}u(t - 4)\} \right] \ast \delta(t + 6) \)

ANSWER: Utilize properties and known results. In the first case, convolving with a shifted impulse simply introduces a shift. Also, convolution commutes.

\[
\delta(t - 7) \ast [3e^{-4(t-2)} + 5\delta(t) + \cos(25\pi t)] \ast \delta(t + 3) = \delta(t - 7) \ast \delta(t + 3) \ast [3e^{-4(t-2)} + 5\delta(t) + \cos(25\pi t)]
\]

\[
= \delta(t - 4) \ast [3e^{-4(t-2)} + 5\delta(t) + \cos(25\pi t)]
\]

\[
= 3e^{-4(t-6)} + 5\delta(t - 4) + \cos(25\pi (t - 4))
\]

In the integral, use the sifting property of an impulse:

\[
\int_{-\infty}^{t} \tau^{-3} \delta(\tau - 5) d\tau = 5^{5-3} \int_{-\infty}^{t} \delta(\tau - 5) d\tau
\]

\[
= 25u(t - 5)
\]

The real operator can be used to obtain a cosine, after which differentiation and convolution is performed. An alternative is to pass the differentiation operator inside the real operator. Both approaches work.

\[
\left[ \frac{d}{dt} \text{Re}\{2e^{j8\pi t}u(t - 4)\} \right] \ast \delta(t + 6) = \left[ \frac{d}{dt} (2\cos(8\pi t)u(t - 4)) \right] \ast \delta(t + 6)
\]

\[
= [-16\pi \sin(8\pi t)u(t - 4) + 2\cos(8\pi t)\delta(t - 4)] \ast \delta(t + 6)
\]

\[
= [2\cos(8\pi(4))\delta(t - 4) - 16\pi \sin(8\pi t)u(t - 4)] \ast \delta(t + 6)
\]

\[
= -16\pi \sin(8\pi t)u(t - 4) \ast \delta(t + 6)
\]

\[
= -16\pi \sin(8\pi(t + 6))u(t + 2)
\]
2. [25 pts] Impulse Responses, Convolution, and Frequency Response:

A LTI averaging system is characterized by the following input–output relation

\[ y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau) \, d\tau. \]

(a) [7 pts] Determine and plot the impulse response of the system, \( h(t) \).

(b) [7 pts] Determine the output through convolution when \( x(t) = 5 \cos(2\pi t/T) \), i.e., evaluate \( y(t) = x(t) \ast h(t) \).

(c) [7 pts] Determine and plot the frequency response of the system, \( H(j\omega) \).

(d) [4 pts] Justify your results for (b) using a frequency domain argument.

**ANSWER:**

To obtain the impulse response, replace \( x(t) \) with \( \delta(t) \),

\[ h(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} \delta(\tau) \, d\tau = \frac{1}{T}(u(t+T/2) - u(t-T/2)) \]

Apply the convolution definition and adjust the limits of integration to correspond to the nonzero values of the pulse shaped impulse response,

\[ y(t) = \frac{5}{T} \int_{-\infty}^{\infty} \cos(2\pi(t-\tau)/T)(u(\tau+T/2) - u(\tau-T/2)) \, d\tau \]

\[ = \frac{5}{T} \int_{\tau-T/2}^{\tau+T/2} \cos(2\pi(t-\tau)/T) \, d\tau \]

\[ = 0 \]

where the last line follows directly from the fact that the integration is over one period of the sinusoid, i.e., the area under one cycle of the sinusoid is zero.

Using known Fourier transform pairs (see table),

\[ h(t) = \frac{1}{T}(u(t+T/2) - u(t-T/2)) \iff H(j\omega) = \sin(\omega T/2) / (\omega T/2) \]

Note from the plot and equation that \( H(j\omega) \) has zero crossings at \( \omega = 2\pi k/T \), for \( k = \pm 1, \pm 2, \ldots \). Since \( X(j\omega) = 5\pi[\delta(\omega - 2\pi/T) + \delta(\omega + 2\pi/T)] \), the Fourier transform of the output is \( Y(j\omega) = X(j\omega)H(j\omega) = 0 \), which is in agreement with (b). In other words, the system has a null at the frequency of the input.
3. [25 pts] Fourier Transforms:
   Determine the following Fourier or inverse Fourier transforms. Give as simple an expression as possible.

   (a) [8 pts] Let \( y(t) = 3e^{-4t}u(t) \ast \delta(t - 3) \). Determine \( Y(j\omega) \).

   (b) [8 pts] Let \( X(j\omega) = \frac{\sin(j4\omega)}{\omega/2} + \frac{j\omega e^{j2\omega}}{3+j\omega} \). Determine \( x(t) \).

   (c) [9 pts] Prove
   \[
   A\cos(\omega_0 t + \phi) \Leftrightarrow \pi Ae^{j\phi} \delta(\omega - \omega_0) + \pi Ae^{-j\phi} \delta(\omega + \omega_0).
   \]
   The proof must be based on transform definition and mathematical arguments, not simple applications of Fourier transform properties. Hint: utilize the inverse Fourier transform.

ANSWER: For the first two parts, use known transforms and properties. In the first case, time domain convolution becomes frequency domain multiplication. Performing the convolution (timeshift) prior taking the Fourier transform also works.

\[
\begin{align*}
g(t) &= x(t) \ast h(t) = 3e^{-4t}u(t) \ast \delta(t - 3) \quad \Leftrightarrow \quad Y(j\omega) = X(j\omega)H(j\omega) \quad \Rightarrow \quad Y(j\omega) = \frac{3}{4 + j\omega} e^{j3\omega} = \frac{3e^{j3\omega}}{4 + j\omega}
\end{align*}
\]

In the second case, the first term is a sinc and therefore yields a rect. The second term has, by the properties, differentiation and timeshift in the time domain. These operations can be performed in either order. I choose differentiation first,

\[
\begin{align*}
X(j\omega) &= \frac{\sin(j4\omega)}{\omega/2} + \frac{j\omega e^{j2\omega}}{3+j\omega} \quad \Leftrightarrow \quad x(t) = [u(t+4) - u(t-4)] + \left[\frac{d}{dt}(e^{-3t}u(t))\right] \ast \delta(t - 2) \\
&= [u(t+4) - u(t-4)] + \left[e^{-3t}\delta(t) - 3e^{-3t}u(t)\right] \ast \delta(t - 2) \\
&= u(t+4) - u(t-4) + e^{-3(t-2)}\delta(t-2) - 3e^{-3(t-2)}u(t-2) \\
&= u(t+4) - u(t-4) + e^{-3(t-2)}(\delta(t-2) - 3u(t-2))
\end{align*}
\]

Lastly, as the hint suggested, it is easier to take the frequency domain representation and map it to the time domain using the inverse Fourier transform,

\[
\begin{align*}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \pi Ae^{j\phi} \delta(\omega - \omega_0) + \pi Ae^{-j\phi} \delta(\omega + \omega_0)e^{\omega t} d\omega \\
&= \frac{Ae^{j\phi}}{2} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{\omega t} d\omega + \frac{Ae^{-j\phi}}{2} \int_{-\infty}^{\infty} \delta(\omega + \omega_0)e^{\omega t} d\omega \\
&= \frac{Ae^{j\phi}e^{\omega_0 t}}{2} + \frac{Ae^{-j\phi}e^{-\omega_0 t}}{2} \\
&= A\cos(\omega_0 t + \phi)
\end{align*}
\]
4. [25 pts] Fourier Series, Spectrum, Modulation, and Filtering:

The above modulator has input \(x(t)\) that is periodic with period \(T = 10^{-3}\) seconds. The Fourier series coefficients describing the input are:

\[
a_k = \begin{cases} 
10 & k = 0 \\
\frac{2j}{k} & k \neq 0 
\end{cases}
\]

(a) [7 pts] Plot and label the spectrum \(X(j\omega)\).

(b) [7 pts] Plot and label the spectrum \(W(j\omega)\).

(c) [7 pts] Plot and label the spectrum \(Y(j\omega)\) for the case of the bandpass filter below.

\[
\begin{array}{c}
\text{H}(j\omega) \\
\text{1} \\
\text{0} \\
\text{3000} \\
\text{3000} \\
\text{5000} \\
\text{5000} \\
\text{0} \\
\end{array}
\]

(d) [4 pts] Express \(y(t)\) in terms of cosines.

ANSWER: The fundamental frequency of \(x(t)\) is \(f_o = 1\) kHz, and the signal can be expressed as

\[
x(t) = \sum_{-\infty}^{\infty} a_k e^{j2\pi kf_0t} = 10 + \sum_{k \neq 0} \frac{2j}{k} e^{j2000\pi kt}
\]

Taking the Fourier transform (from table)

\[
X(j\omega) = 20\pi \delta(\omega) + \sum_{k \neq 0} \frac{4j\pi}{k} \delta(\omega - 2000\pi k)
\]

The modulation introduces a spectral shift by \(\pm f_0\) (and scaling by 1/2),

\[
W(j\omega) = \frac{1}{2} X(j\omega - 4500\pi) + \frac{1}{2} X(j\omega + 4500\pi)
\]

\[
= 10\pi \delta(\omega - 4500\pi) + \sum_{k \neq 0} \frac{2j\pi}{k} \delta((\omega - 4500\pi) - 2000\pi k) +
\]

\[
10\pi \delta(\omega + 4500\pi) + \sum_{k \neq 0} \frac{2j\pi}{k} \delta((\omega + 4500\pi) - 2000\pi k)
\]
After filtering,

\[ Y(j\omega) = W(j\omega)H(j\omega) \]

\[ = 10\pi\delta(\omega - 4500\pi) + \frac{2j\pi}{k}\delta((\omega + 4500\pi) - 2000\pi k) \bigg|_{k=8} + \]

\[ 10\pi\delta(\omega + 4500\pi) + \frac{2j\pi}{k}\delta((\omega - 4500\pi) - 2000\pi k) \bigg|_{k=-8} \]

\[ = 10\pi\delta(\omega - 4500\pi) + 10\pi\delta(\omega + 4500\pi) + \frac{j\pi}{4}\delta(\omega - 3500\pi) - \frac{j\pi}{4}\delta(\omega + 3500\pi) \]

\[ = 10\pi\delta(\omega - 4500\pi) + 10\pi\delta(\omega + 4500\pi) + \frac{\pi e^{j\pi/2}}{4}\delta(\omega - 3500\pi) + \frac{\pi e^{-j\pi/2}}{4}\delta(\omega + 3500\pi) \]

Taking the inverse Fourier transform (see table) yields:

\[ y(t) = 10\cos(4500\pi t) + \frac{1}{4}\cos(3500\pi t + \pi/2) \]