LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

\[ x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

- SPECTRUM Representation
  - Graphical Form shows DIFFERENT Freqs

FREQUENCY DIAGRAM

- Plot Complex Amplitude vs. Freq

\[ 4e^{-j\pi/2}, 7e^{j\pi/3}, 10, 7e^{-j\pi/3}, 4e^{j\pi/2} \]

Another FREQ. Diagram

- Frequency is the vertical axis
- Time is the horizontal axis

Figure 3.18  Sheet-music notation is a time–frequency diagram.
MOTIVATION

* Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

PHASOR BEAT EXAMPLE

* Beat notes/patterns occur sinusoids of SIMILAR frequency are added

Fur Elise WAVEFORM

* Beat Notes

Speech Signal: BAT

* Nearly Periodic in Vowel Region
  - Period is (Approximately) \( T \approx 0.0065 \text{ sec} \)
Euler’s Formula Reversed

Solve for cosine (or sine)

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]
\[ e^{-j\omega t} = \cos(-\omega t) + j \sin(-\omega t) \]
\[ e^{j\omega t} - e^{-j\omega t} = 2 \cos(\omega t) \]

\[ \cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t}) \]

INVERSE Euler’s Formula

Solve for cosine (or sine)

\[ \cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t}) \]
\[ \sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) \]

PHASOR EXAMPLE

SPECTRUM Interpretation

Cosine = sum of 2 complex exponentials:

\[ A \cos(7t) = \frac{A}{2} e^{j7t} + \frac{A}{2} e^{-j7t} \]

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big
**NEGATIVE FREQUENCY**

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let’s assume 400Hz $\rightarrow$ 60 mph
  - $+400\text{Hz}$ means towards the radar
  - $-400\text{Hz}$ means away (opposite direction)
  - Think of a train whistle

**SPECTRUM of SINE**

- Sine = sum of 2 complex exponentials:
  \[ A\sin(7t) = \frac{A}{2j} e^{j7t} - \frac{A}{2j} e^{-j7t} \]
  \[ = \frac{1}{2} Ae^{-j0.5\pi} e^{j7t} + \frac{1}{2} Ae^{j0.5\pi} e^{-j7t} \]

  - Positive freq. has phase = $-0.5\pi$
  - Negative freq. has phase = $+0.5\pi$

**GRAPHERICAL SPECTRUM**

**EXAMPLE of SINE**

\[ A\sin(7t) = \frac{1}{2} Ae^{-j0.5\pi} e^{j7t} + \frac{1}{2} Ae^{j0.5\pi} e^{-j7t} \]

**SPECTRUM ---> SINUSOID**

- Add the spectrum components:
Gather \((A, \omega, \phi)\) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 \(-\pi/2\)
  - 7 \(\pi/3\)
  - 10 0
  - 4 \(+\pi/3\)

Note the conjugate phase

DC is another name for zero-freq component
DC component always has \(\phi=0\) or \(\pi\) (for real \(x(t)\))

Add Spectrum Components-1

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 \(-\pi/2\)
  - 7 \(\pi/3\)
  - 10 0
  - 4 \(+\pi/3\)

\(x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}\)

Add Spectrum Components-2

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 \(-\pi/2\)
  - 7 \(\pi/3\)
  - 10 0
  - 4 \(+\pi/3\)

\(x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}\)

Use Euler’s Formula to get REAL sinusoids:

\(A \cos(\omega t + \varphi) = \frac{1}{2} Ae^{-j\varphi} e^{j\omega t} + \frac{1}{2} Ae^{j\varphi} e^{-j\omega t}\)
**FINAL ANSWER**

\[ x(t) = 10 + 14 \cos(2\pi(100)t - \pi/3) + 8 \cos(2\pi(250)t + \pi/2) \]

So, we get the general form:

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

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**Example: Synthetic Vowel**

- **Sum of 5 Frequency Components**

<table>
<thead>
<tr>
<th>( f_k ) (Hz)</th>
<th>( X_k )</th>
<th>Mag</th>
<th>Phase (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>(771 + j12202)</td>
<td>12.226</td>
<td>1.508</td>
</tr>
<tr>
<td>400</td>
<td>(-8865 + j28048)</td>
<td>29.416</td>
<td>1.876</td>
</tr>
<tr>
<td>500</td>
<td>(48001 - j8995)</td>
<td>48.836</td>
<td>-0.185</td>
</tr>
<tr>
<td>1600</td>
<td>(1657 - j13520)</td>
<td>13.621</td>
<td>-1.449</td>
</tr>
<tr>
<td>1700</td>
<td>4723 + j0</td>
<td>4723</td>
<td>0</td>
</tr>
</tbody>
</table>

*Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound “nh”.*

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**Summary: GENERAL FORM**

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

\[ x(t) = X_0 + \sum_{k=1}^{N} \text{Re}\{X_k e^{j2\pi f_k t}\} \]

\[ \text{Re}\{z\} = \frac{1}{2} z + \frac{1}{2} z^* \]

\[ X_k = A_k e^{j\varphi_k} \]

Frequency = \( f_k \)

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**SPECTRUM of VOWEL**

- **Note:** Spectrum has 0.5\( X_k \) (except \( X_{DC} \))
- **Conjugates in negative frequency**
SPECTRUM of VOWEL (Polar Format)

Vowel: Magnitude Spectrum

Vowel: Phase Angle Spectrum

Vowel Waveform (sum of all 5 components)

Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals 1/fs0.