ELEG404/604: Imaging & Deep Learning

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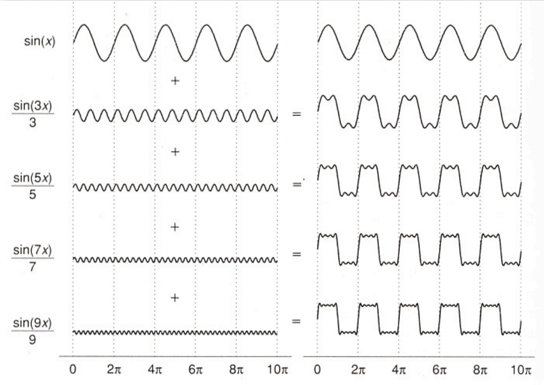
Chapter III

Frequency Representations



(Foley)

- a sum of sine waves, each of different wavelength (*frequency*) and height (*amplitude*), can approximate arbitrary functions
- to adjust horizontal position (*phase*), replace with cosine waves, or use a mixture of sine and cosine waves

Frequency Representations

- Fourier series: any continuous, integrable, periodic function can be represented as an infinite series of sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

- a sum of sine waves, each of different wavelength (*frequency*) and height (*amplitude*), can approximate arbitrary functions
- to adjust horizontal position (*phase*), replace with cosine waves, or use a mixture of sine and cosine waves

Fourier Analysis

The use of the Fourier Transform is of fundamental importance in the analysis and design of signal processing systems. Before we look into 2-D we first review 1-D Fourier analysis. In particular, recall that the 1-D Fourier Transform of a signal is:

$$G(u) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ut}\,\mathrm{d}t$$

$$g(t) = \int_{-\infty}^{\infty} G(u)e^{j2\pi ut}\,\mathrm{d}u$$

$g(t) \leftrightarrow G(u)$ is the Fourier Transform pair

The Fourier Transform pair satisfies several useful properties:

$$(a) \quad Scaling \qquad g(t/T) \leftrightarrow TG(uT)$$

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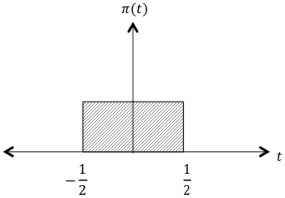
$$
\begin{array}{lll}
(a) & Scaling & g(t/T) \leftrightarrow TG(uT) \\
(b) & Shift & g(t-T) \leftrightarrow G(u)e^{-j2\pi T} \\
(c) & Duality & G(t) \leftrightarrow g(-u) \\
(d) & Linearity & ag(t)+bh(t) \leftrightarrow aG(u)+bH(u) \\
(e) & Convolution & g(t)*h(t) \leftrightarrow G(u)H(u)
\end{array}
$$

Common 1-D Transforms:

Common 1-Dimensional transforms:

$$\pi(t) = rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$



$$rect(t) \leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u}$$

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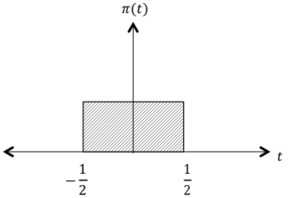

$$rect(t) \leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u}$$
$$\delta(t) \leftrightarrow 1$$

Common 1-D Transforms:

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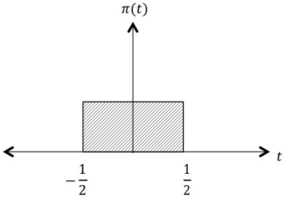

$$\begin{aligned} rect(t) &\leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u} \\ \delta(t) &\leftrightarrow 1 \\ e^{-\alpha|t|}\alpha > 0 &\leftrightarrow \frac{2\alpha}{\alpha^2+(2\pi u)^2} \end{aligned}$$

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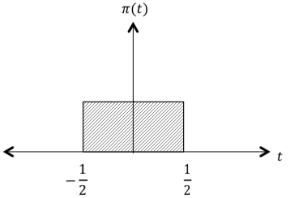

$$\begin{array}{ll} rect(t) & \leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u} \\ \delta(t) & \leftrightarrow 1 \\ e^{-\alpha|t|}\alpha > 0 & \leftrightarrow \frac{2\alpha}{\alpha^2+(2\pi u)^2} \\ \sum_{m=-\infty}^{\infty} \delta(t - mT) & \leftrightarrow \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(u - \frac{m}{T}) \end{array}$$

Common 1-D Transforms:

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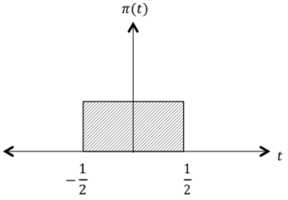


$$\begin{aligned}
&rect(t) && \leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u} \\
&\delta(t) && \leftrightarrow 1 \\
&e^{-\alpha|t|} \alpha > 0 && \leftrightarrow \frac{2\alpha}{\alpha^2+(2\pi u)^2} \\
&\sum_{m=-\infty}^{\infty} \delta(t-mT) && \leftrightarrow \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(u-\frac{m}{T}) \\
&cos(2\pi u_0 t) && \leftrightarrow \frac{1}{2}[\delta(u-u_0)+\delta(u+u_0)]
\end{aligned}$$

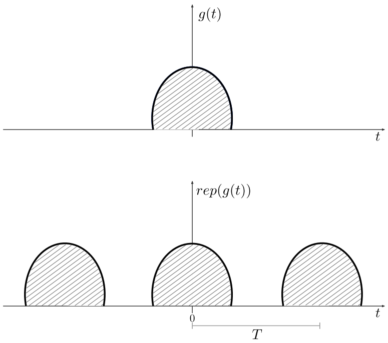
Example

We know:

$$\sum_{m=-\infty}^{\infty} \delta(t-m) \leftrightarrow \frac{1}{T}\sum_{m}\delta(u-\frac{m}{T})$$

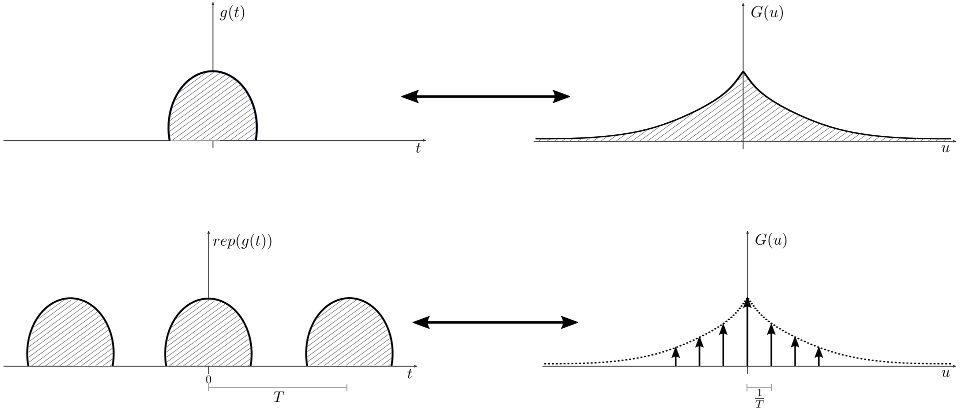
Ok, now we look at:

$$\begin{aligned} \text{rep}_T(g(t)) &= \sum_m g(t-mT) \\ &= g(t) * \textstyle\sum_m \delta(t-mT) \end{aligned}$$



The Fourier Transform of $\text{rep}_T(g(t))$ is then

$$
\begin{aligned}
F\{\text{rep}_T(g(t))\} &= F\{g(t) \circledast \textstyle\sum_m \delta(t - mT)\} \\
&= G(u)\frac{1}{T}\sum_m \delta(u - \frac{m}{T}) \\
&= \frac{1}{T}\sum_m G(\frac{m}{T})\delta(u - \frac{m}{T}) \\
&= \frac{1}{T}\text{comb}_{\frac{1}{T}}\left[G(u)\right]
\end{aligned}
$$



$g(t)$
t
$G(u)$
u
$rep(g(t))$
0
T
t
$G(u)$
u
$\frac{1}{T}$

hence;

$$\text{rep}_T[g(t)] = g(t) * \frac{1}{T}\sum_n e^{(j2\pi n u_0 t)}$$

Taking Fourier Transforms:

$$\begin{aligned} F\{\text{rep}_T[g(t)]\} &= G(u)\cdot\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(u-\frac{n}{T}) \\ &= \frac{1}{T}\sum_n G(u)\delta(u-\frac{n}{T}) \end{aligned}$$

hence;

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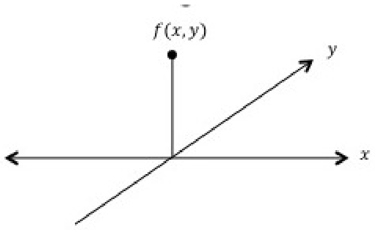
Taking Fourier Transforms:

$$\begin{aligned} F\{\text{rep}_T[g(t)]\} &= G(u)\cdot\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(u-\frac{n}{T}) \\ &= \frac{1}{T}\sum_n G(u)\delta(u-\frac{n}{T}) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} G(\frac{n}{T})\delta(u-\frac{n}{T}) \\ &= \tfrac{1}{T} comb_{\frac{1}{T}}[G(u)] \end{aligned}$$

2-Dimensional Systems

In 2-Dimensions the input and output of a system are functions of 2 independent variables. For instance, in image processing, the variables are the spatial coordinates (x,y) and the value of the function is the intensity of the image at that point.



The impulse response $\delta(x,y) \rightarrow \boxed{\text{System}} \rightarrow h(x,y)$ characterizes the output for any other input $\ell(x,y)$: [linear-time inv.]

$$\begin{aligned}\ell(x,y) \rightarrow \boxed{\text{System}} \rightarrow g(x,y) &= h(x,y) * \ell(x,y) \\ &= \int\int_{-\infty}^{\infty} h(x-\sigma, y-\beta)\ell(\sigma,\beta)\mathrm{d}\sigma\mathrm{d}\beta\end{aligned}$$

Continuous-Space Fourier-Transform

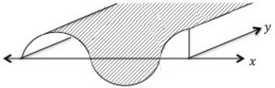
The 2-D Fourier transform pair is:

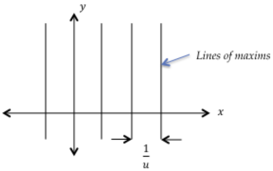
$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)e^{j2\pi(ux+vy)}\mathrm{d}u\mathrm{d}v \\
G(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(ux+vy)}\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

The idea is to superimpose infinite terms of the form $cos2\pi(ux+vy)$ to form any 2-D image, where lines of constant amplitude are given by:

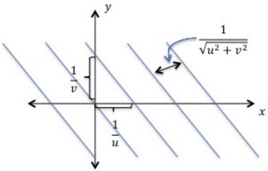
$$2\pi(ux+vy) = k$$

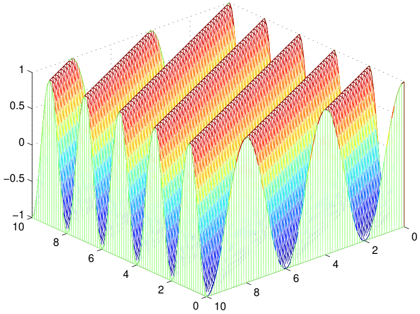
Example: $v=0$ (vertical frequency is zero) $\rightarrow cos2\pi(ux+0)$





In general, $cos 2\pi(ux + vy)$has patterns like this

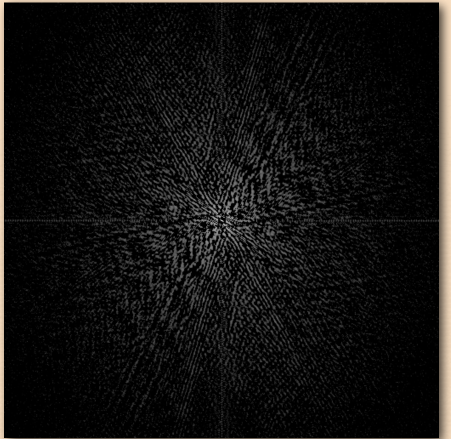




Fourier Transforms of Images

A Typical Photograph

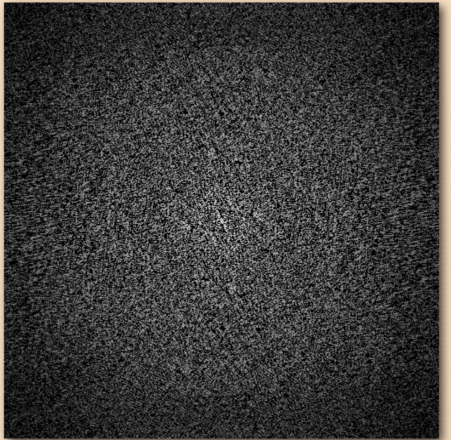
image



spectrum

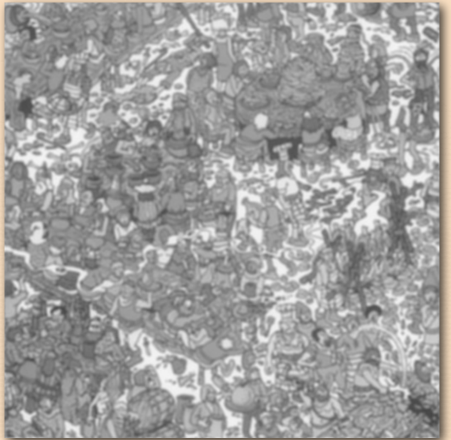
An Image with Higher Frequencies

image

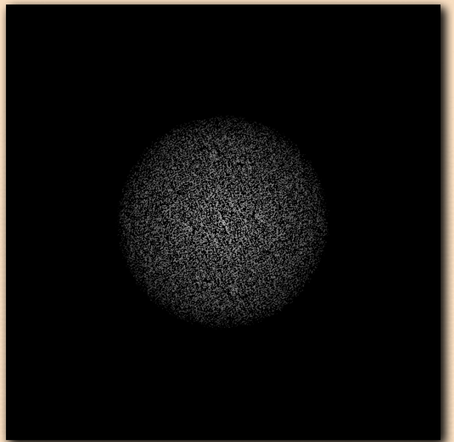


spectrum

Blurring in the Fourier Domain



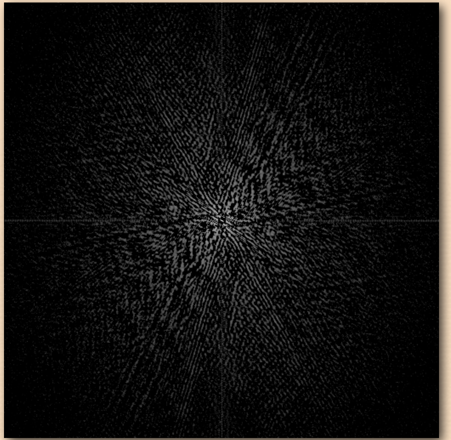
image



spectrum

Original Flower

image



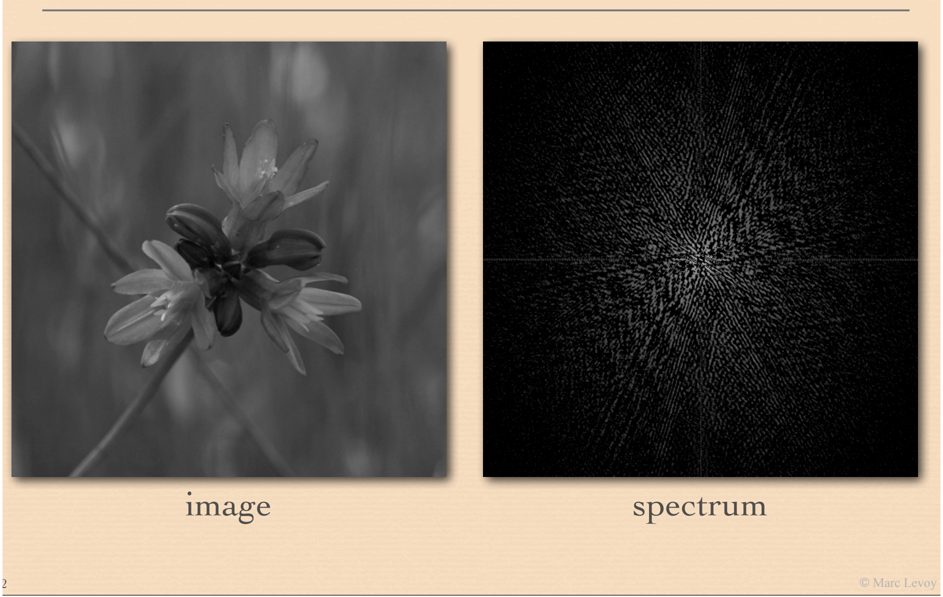
spectrum

Sharpening in the Fourier Domain

image spectrum

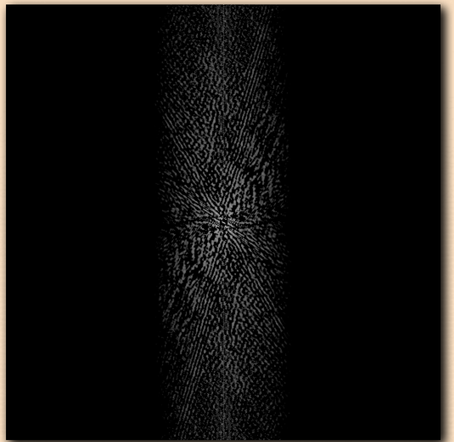
1

What Does this Filtering Operation Do?



What Does this Filtering Operation Do?

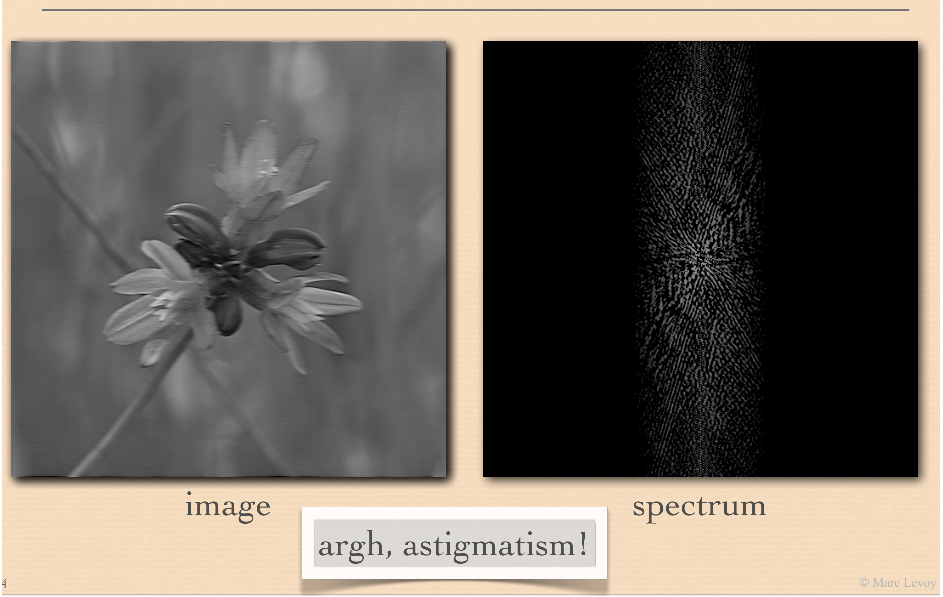
image



spectrum

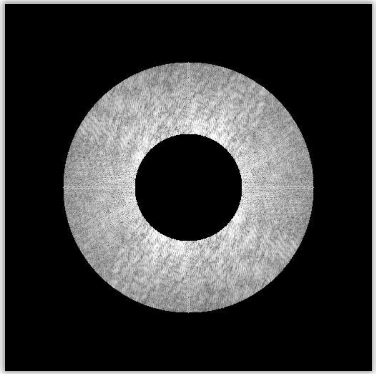
3

Blurring in x, sharpening in y



Filtering – Band-pass Filter

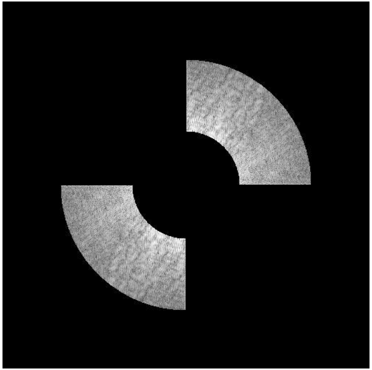
←



Filtering – Oriented Band-pass Filter

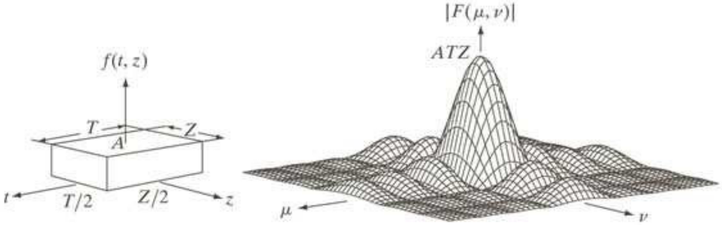
- edges with specific orientation (e.g., hat) are gone!

←



Example: Find the Fourier transform of $rect[x,y] = rect[x]rect[y]$

$$G(u,v) = \int rect(x)e^{-j2\pi ux}\mathrm{d}x \int rect(y)e^{-j2\pi vy}\mathrm{d}y$$
$$G(u,v) = sinc(u)sinc(v)$$



Common 2-Dimensional Functions

$$1)\quad \text{rect}[x,y] = \text{rect}[x]\text{rect}[y] = f(x,y) = \begin{cases} 1 & |x|,|y| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

Common 2-Dimensional Functions

$$1)\quad \text{rect}[x,y] = \text{rect}[x]\text{rect}[y] = f(x,y) = \begin{cases} 1 & |x|,|y| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$2)\quad \text{sinc}[x,y] = \text{sinc}[x]\text{sinc}[y] = \frac{\sin \pi x}{\pi x}\frac{\sin \pi y}{\pi y}$$

Common 2-Dimensional Functions

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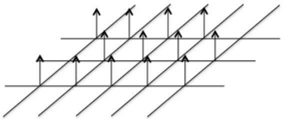
$$2)\quad \mathrm{sinc}[x,y] = \mathrm{sinc}[x]\mathrm{sinc}[y] = \frac{\sin \pi x}{\pi x}\frac{\sin \pi y}{\pi y}$$

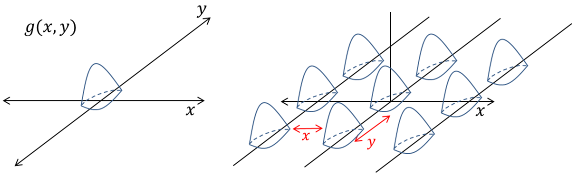
$$3)\quad \mathrm{comb}[x,y] = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$

where,

$$\delta(x-m, Y-n) = \delta(x-m)\delta(y-n)$$



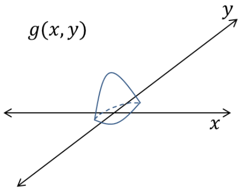
$$4) \quad \text{rep}_{XY}[g(x,y)] \;=\; \sum_m \sum_n g(x - mX, y - nY)$$

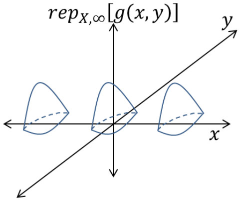


$$5)\quad \text{comb}_{XY}[g(x,y)] = \sum_m \sum_n g(mX,nY)\delta(x-mX,y-nY) = g(x,y)\sum_m\sum_n \delta(x-mX,y-nY)$$

Some notation:

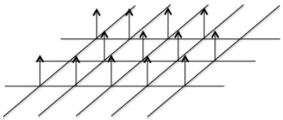
$$\text{rep}_{X,\infty}[g(x,y)] = \sum_k g(x-kX,y)$$
$$\text{rep}_{\infty,Y}[g(x,y)] = \sum_k g(x,y-kY)$$





Example: 1. Take F.T of

$$\begin{aligned} \text{comb}[x,y] &= \sum_m \sum_n \delta(x-m, y-n) \\ &= \sum_m \sum_n \delta(x-m)\delta(y-n) \end{aligned}$$



$$
\begin{aligned}
Fourier\{\text{comb}[x,y]\} &= F\left\{\sum_n \delta(x-m)\sum_m \delta(y-n)\right\} \\
&= \sum \delta(u-m)\sum \delta(v-n) \\
&= \text{comb}[u,v]
\end{aligned}
$$

$$
\text{comb}[x,y] \longleftrightarrow \text{comb}[u,v]
$$

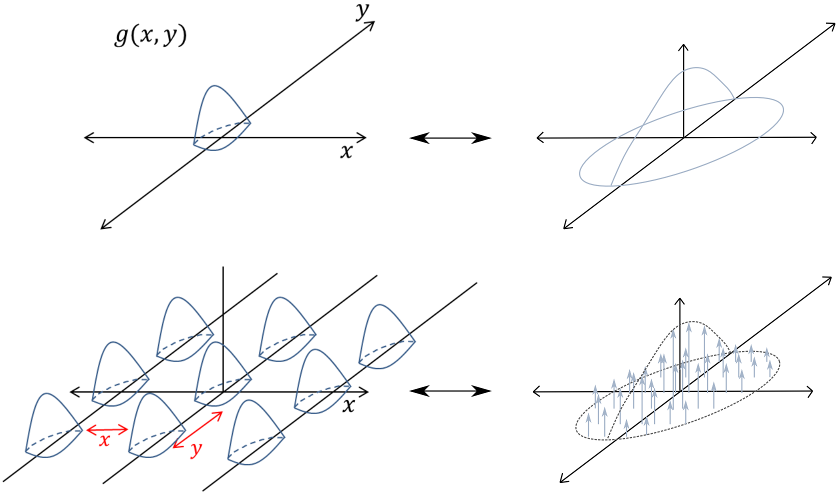
$$
\begin{aligned}
Fourier\{\text{comb}[x,y]\} &= F\left\{\sum_n \delta(x-m)\sum_m \delta(y-n)\right\} \\
&= \sum \delta(u-m)\sum \delta(v-n) \\
&= \text{comb}[u,v]
\end{aligned}
$$

$$
\text{comb}[x,y] \longleftrightarrow \text{comb}[u,v]
$$

In general

$$
\text{comb}_{XY}[x,y] \longleftrightarrow \frac{1}{X}\frac{1}{Y}\text{comb}_{\frac{1}{X}\frac{1}{Y}}[u,v]
$$

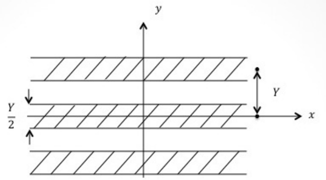
$$\begin{aligned} rep_{XY}[g(x,y)] &= g(x,y) * comb_{XY}[x,y] \\ \therefore F[rep()] &= G(u,v) \cdot \frac{1}{X}\frac{1}{Y}\sum_{mn} \delta\left(u - \frac{m}{X}, v - \frac{n}{Y}\right) \\ &= \frac{1}{XY}\sum\sum G\left(\frac{m}{X}, \frac{n}{Y}\right)\delta\left(u - \frac{m}{X}, v - \frac{n}{Y}\right) \\ &= \frac{1}{XY}\text{comb}_{\frac{1}{X}\frac{1}{Y}}[G(u,v)] \end{aligned}$$



g(x,y)
y
x
x
x
y

Example: Take F.T of

$$rep_{\infty,Y}[rect(2y/Y)]$$

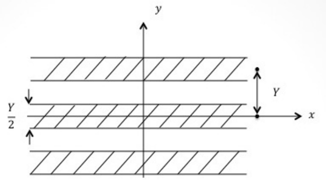


note:

$$\text{rect}[2y/Y] \quad = \quad \underset{\text{function of } x}{\underset{\uparrow}{1}} \quad \underset{\text{function of } y}{\underset{\uparrow}{\text{rect}[2y/Y]}}$$

Example: Take F.T of

$$rep_{\infty,Y}[rect(2y/Y)]$$



note:

$$\text{rect}[2y/Y] = \underset{\text{function of } x}{\underset{\uparrow}{1}} \quad \underset{\text{function of } y}{\underset{\uparrow}{\text{rect}[2y/Y]}}$$

$$1 \longleftrightarrow \delta(u)$$
$$\text{rect}[2y/Y] \longleftrightarrow \frac{Y}{2}\text{sinc}[\frac{Y}{2}v]$$

hence,

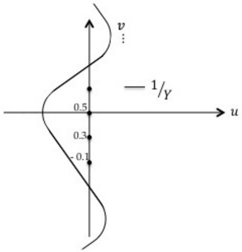
$$1 \cdot \text{rect}[2y/Y] \longleftrightarrow \delta(u)\tfrac{Y}{2}\text{sinc}[\tfrac{Y}{2}v]$$

So,

$$G(u,v) = \tfrac{1}{Y}\text{comb}_{\frac{1}{Y}}[\delta(u)\tfrac{Y}{2}\text{sinc}[\tfrac{Y}{2}v]]$$

or

$$G(u,v) = \delta(u)(\frac{1}{2}\sum_k sinc[\frac{k}{2}]\delta(v - \frac{k}{Y})$$



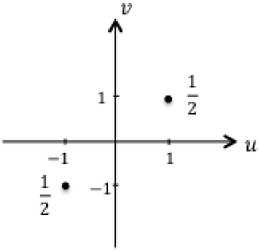
<u>EX3</u>

$$\begin{array}{rcll} g(x,y) & = & \cos[2\pi(x+y)] & \\ \delta(x,y) & \longleftrightarrow & 1 & \\ \delta(x+1,y+1) & \longleftrightarrow & e^{j2\pi(u+v)} & \\ & & & \\ e^{j2\pi[x+y]} & \longleftrightarrow & \delta(-u+1,-v+1) & \text{duality} \\ & = & \delta(u-1,v-1) & \text{even function} \end{array}$$

$$Similarity \quad e^{-j2\pi[x+y]} \quad \longleftrightarrow \quad \delta(u+1, v+1)$$

$$\therefore$$

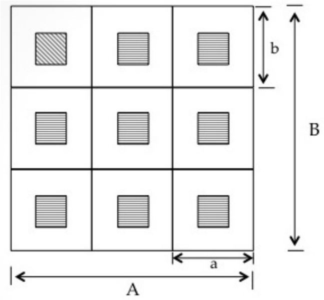
$$\cos[2\pi(x+y)] \longleftrightarrow \frac{1}{2}[\delta(u-1, v-1) + \delta(u+1, v+1)]$$



Example Spectrum of block.

Recall

- $rect[x,y] \leftrightarrow sinc(u,v)$
- $\delta(x,y) \leftrightarrow 1$
- $rep_{X,Y}[g(x,y)] \leftrightarrow \frac{1}{XY} comb_{\frac{1}{X},\frac{1}{Y}}[G(u,v)]$



Approach 1: Sum of F.T of each term but easier next way.

Approach 2:

Consider $\text{rect}[\frac{x}{c}, \frac{y}{d}]$

From 1. $\text{rect}[\frac{x}{c}, \frac{y}{d}] \longleftrightarrow cd\ \text{sinc}[cu, dv]$

$$rep_{ab}[\text{rect}[\frac{x}{c}, \frac{y}{d}]] \quad \longleftrightarrow \quad (\frac{cd}{ab})\text{comb}_{\frac{1}{a},\frac{1}{b}}[\text{sinc}[cu, dv]]$$

infinite pattern transform

Approach 2:

$$\text{Consider} \quad \mathrm{rect}[\tfrac{x}{c}, \tfrac{y}{d}]$$
$$\text{From 1.} \quad \mathrm{rect}[\tfrac{x}{c}, \tfrac{y}{d}] \longleftrightarrow cd\ \mathrm{sinc}[cu, dv]$$

$$rep_{ab}[\mathrm{rect}[\tfrac{x}{c}, \tfrac{y}{d}]] \quad \longleftrightarrow \quad (\tfrac{cd}{ab})\mathrm{comb}_{\frac{1}{a},\frac{1}{b}}[\mathrm{sinc}[cu, dv]]$$

infinite pattern transform

So:

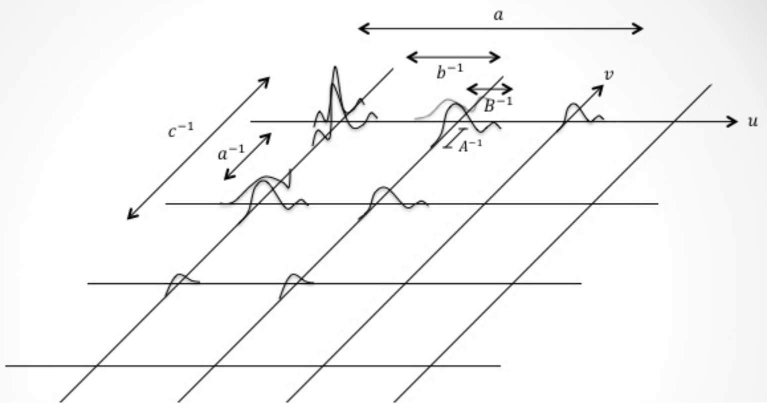
$$g(x,y) \quad = \quad \mathrm{rep}_{a,b}[\mathrm{rect}(\tfrac{x}{c}, \tfrac{y}{d})]\mathrm{rect}(\tfrac{x}{A}, \tfrac{y}{B})$$

So $(x \longleftrightarrow *)$

$$\boxed{G(u,v) = (\tfrac{cd}{ab})\mathrm{comb}_{\frac{1}{a},\frac{1}{b}}[\mathrm{sinc}(cu, dv)] * AB\mathrm{sinc}[Au, Bv]}$$

To see what this is, let's write it out.

$$
\begin{aligned}
G(u,v) &= \left(\frac{ABcd}{ab}\right)\sum_m\sum_n \text{sinc}\left(\frac{cm}{a},\frac{dn}{b}\right)\delta\left(u-\frac{m}{a},v-\frac{n}{b}\right)*\text{sinc}(Au,Bv) \\
&= \left(\frac{ABcd}{ab}\right)\sum_m\sum_n \text{sinc}\left(\frac{cm}{a},\frac{dn}{b}\right)\text{sinc}\left[A\left(u-\frac{m}{a}\right),B\left(v-\frac{n}{b}\right)\right]
\end{aligned}
$$



- Overall drop off controlled by pulse slope
- Interval between replications controlled by period
- Pulse width controlled by duration

Two Dimensional Convolution Example:

Let $f_1(x,y)$ and $f_2(x,y)$ be as shown below, where the shaded area represents a value of 1, else 0.

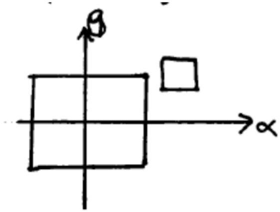
a. In the first case, $f_1(x,y) = \mathrm{rect}(x/A, y/A)$ and $f_2(x,y) = \mathrm{rect}(x/B, y/B)$ where $A < B$.
b. In the second case, $f_1(x,y) = f_2(x,y) = \mathrm{circ}(x,y)$.

Convolve the two sets of functions and plot the results.

- The $2-D$ convolution is given by

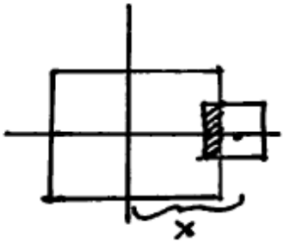
$$g(x,y) = \int\int f_1(x-\alpha, y-\beta) f_2(\alpha,\beta) d\alpha d\beta$$

- for $|y|,|x| > \frac{A+B}{2}$, $g(x,y) = 0$.



- for $|y|, |x| < \frac{B-A}{2}$, $\qquad g(x,y) = A^2$.
- for $\frac{B-A}{2} < |x| < \frac{A+B}{2}$;

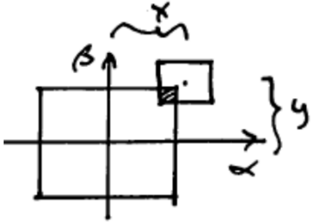
$$|y| < \frac{B-A}{2} \qquad g(x,y) = A[\tfrac{B}{2} - (x - \tfrac{A}{2})].$$



- for $\frac{B-A}{2} < |y| < \frac{A+B}{2}$;

$$|x| < \frac{B-A}{2} \qquad g(x,y) = A[\tfrac{B}{2} - (y - \tfrac{A}{2})].$$

- for $\frac{B-A}{2} < |x|, |y| < \frac{A+B}{2}$; $\quad g(x,y) = [\frac{B}{2} - (x - \frac{A}{2})][\frac{B}{2} - (y - \frac{A}{2})]$.

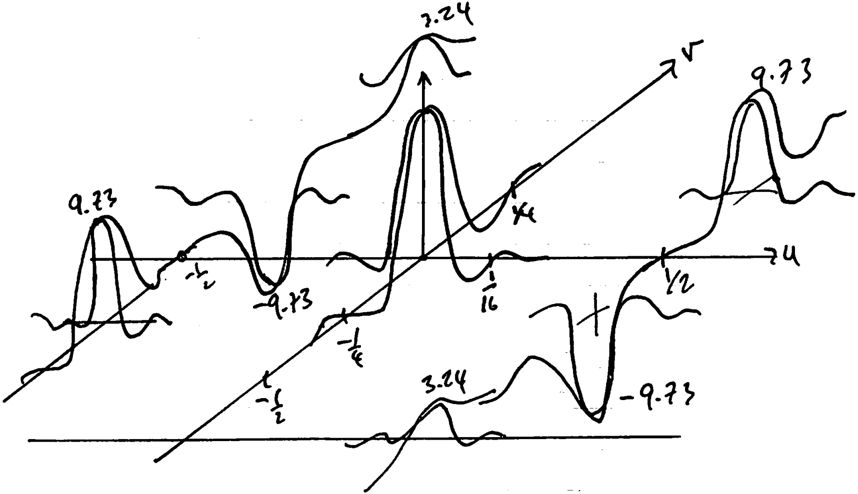


Example

$$g(x,y) = \text{rep}_{2,4}[\text{rect}(\frac{x+1/2}{1},\frac{y-1}{1}) + \text{rect}(\frac{x-1/2}{1},\frac{y+1}{2})] * \text{rect}(\frac{x}{16},\frac{y}{8})$$

$$G(u,v) = \frac{1}{8}\text{comb}_{\frac{1}{2},\frac{1}{4}}[2\text{sinc}(u,2v)(e^{-j2\pi(-\frac{u}{2}+v)} + e^{j2\pi(-\frac{u}{2}+v)})] * 2\text{sinc}(16u,8v)$$

$$G(u,v) = 24\sum_k\sum_\ell \text{sinc}(\frac{k}{2},\frac{\ell}{2})\cos[\pi(\frac{k-\ell}{2})]\text{sinc}[16(u-\frac{k}{2}),8(v-\frac{\ell}{4})]$$


3.24
v
9.73
9.73
1/4
u
-1/2
-9.73
1/16
1/2
-1/4
-1/2
3.24
-9.73