ELEG 467/667 - Imaging and Audio Signal Processing

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Chapter IV(c)

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Spring 2013

The Discrete Fourier Transform

Given the sampled signal:

$$g_s(x) = g(x)\Sigma_m \delta(x - mX)$$

We can represent:

$$\begin{aligned}
G_s(u) &= \int_{-\infty}^{\infty} g_s(x) e^{-2\pi ux} dx \\
&= \int_{-\infty}^{\infty} g(x)\Sigma_m \delta(x - mX) e^{-2\pi ux} dx \\
&= \Sigma_m \int_{-\infty}^{\infty} g(x) \delta(x - mX) e^{-2\pi ux} dx \\
&= \Sigma_m g(mX) e^{-2\pi umX} \\
&= \Sigma_m g_m e^{-2\pi umX}
\end{aligned}$$

The Discrete Fourier Transform

If we take M samples of $G_s(u)$ over the period $u = 0$ to $u = 1/X$, or $u = \frac{n}{MX}$, for $n = 0, 1, ..., M-1$; this results in

$$G_m(n) \quad = \quad \Sigma_{m=0}^{M-1} g_m e^{-2\pi mn/M} \quad n = 0, 1, ...M-1.$$

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$$G_m(n) \quad = \quad \Sigma_{m=0}^{M-1} g_m e^{-2\pi mn/M} \quad n = 0, 1, ...M-1.$$

A more intuitive notation for the DFT is

$$G(u) = \Sigma_{x=0}^{M-1} g(x) e^{-2\pi ux/M} \quad u = 0, 1, ...M-1.$$

$$g(x) = \frac{1}{M} \Sigma_{u=0}^{M-1} G(u) e^{2\pi ux/M} \quad x = 0, 1, ...M-1.$$

- Note that the resolution in frequency depends on the duration of the signal sampled (duration$=MX$).
- Resolution in frequency: $\frac{1}{MX}$

The entire frequency range is given by

$$M\frac{1}{MX}=\frac{1}{X} \tag{1}$$

In 2-dimensions the DFT is

$$G(u,v) = \Sigma_{x=0}^{M-1}\Sigma_{y=0}^{N-1} g(x,y) e^{-2\pi(ux/M+vy/N)} \quad u = 0,1,...M-1; v = 0,1,...N-1.$$

$$g(x,y) = \frac{1}{MN}\Sigma_{u=0}^{M-1}\Sigma_{u=0}^{N-1} G(u,v) e^{2\pi(ux/M+vy/N)} \quad x = 0,1,...M-1; y = 0,1,...N-1.$$

Some Properties

- Translation

$$g(x,y)e^{j2\pi u_0 x/M} \quad \leftrightarrow \quad G(u-u_0)$$

Example: for $u_0 = M/2$

$$g(x,y)(-1)^x \quad \leftrightarrow \quad G(u-M/2)$$

Example in 2D: for $u_0 = M/2, v_0 = M/2$

$$g(x,y)(-1)^{(x+y)} \quad \leftrightarrow \quad G(u-M/2, v-M/2)$$

- Translation

$$g(x - x_0, y - y_0)$$

- Translation

$$g(x - x_0, y - y_0) \quad \leftrightarrow \quad G(u, v)e^{-j2\pi(x_0 u/M + y_0 v/M)}$$

- Translation

$$g(x - x_0, y - y_0) \quad \leftrightarrow \quad G(u,v)e^{-j2\pi(x_0 u/M + y_0 v/M)}$$

- Magnitude and Phase: Since the DFT is complex

$$G(u,v) = |G(u,v)|e^{j\varphi(u,v)}$$

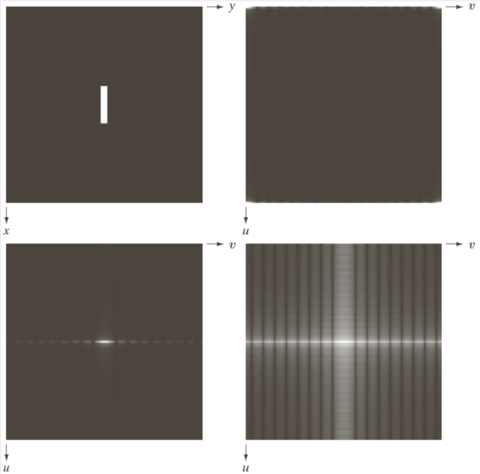
where:

$$|G(u,v)| = [(Re(u,v))^2 + (Im(u,v))^2]^{1/2} \triangleq \text{Fourier spectrum}$$

$$\varphi(u,v) = \arctan\left(\frac{Im(u,v)}{Re(u,v)}\right) \triangleq \text{Phase angle}$$

- Power spectrum

$$P(u,v) = |G(u,v)|^2 = [(Re(u,v))^2 + (Im(u,v))^2]$$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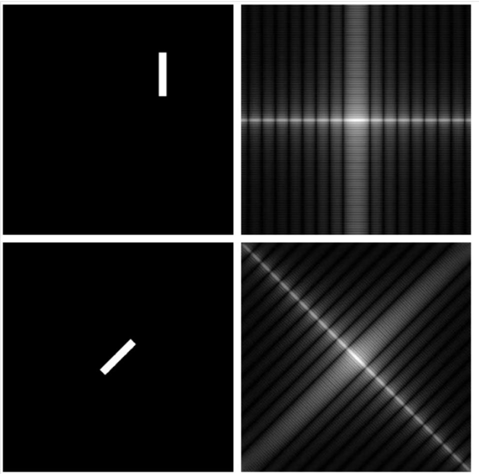
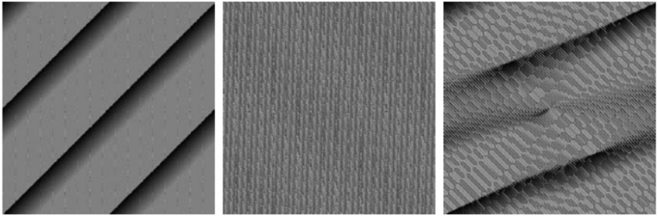
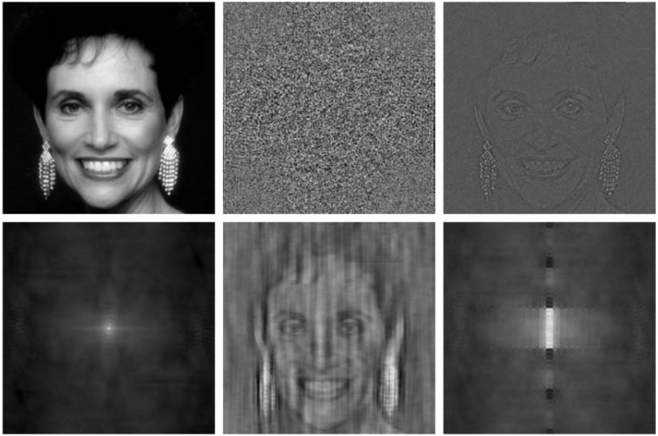




a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).



a	b	c
d	e	f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

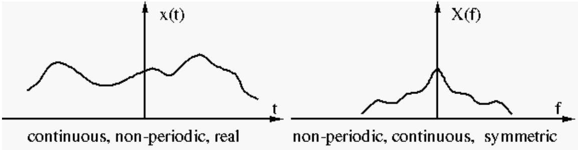
Four different forms of the Fourier transform

Non-periodic, continuous time function $x(t)$, continuous, non-periodic spectrum $X(f)$ This is the most general form of Fourier transform.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt, \quad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Alternatively, as $\omega = 2\pi f$, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$



Non-periodic, discrete time function $x[n]$, continuous, periodic spectrum $X_F(f)$
The discrete time function is a sample sequence. Time interval between consecutive samples $x[m]$ and $x[m+1]$ is $t_0 = 1/F$, where F is the sampling rate, which is also the period of the spectrum in the frequency domain.
The discrete time function can be written as

$$x(t) = \sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)$$

and its transform is:

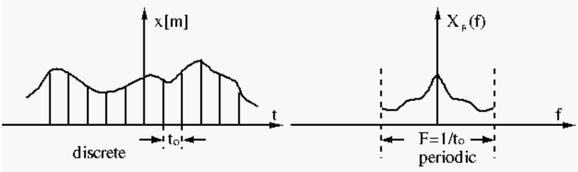
$$X_F(f) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f m t_0}, \qquad x[m] = \frac{1}{F} \int_{-F/2}^{+F/2} X_F(f) e^{j2\pi f m t_0} df$$

$$(m = 0, \pm 1, \pm 2, \cdots)$$

The spectrum is periodic:

$$X_F(f + kF) = X_F(f + k/t_0) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi(f+k/t_0)mt_0} = X_F(f)$$

(for $k = \pm 1, \pm 2, \cdots$) because $e^{\pm j2\pi mk} = 1$.



Periodic, continuous time function $x_T(t)$, discrete, non-periodic spectrum $X[n]$ This is the Fourier series expansion of periodic functions. The time period is T, and the interval between two consecutive frequency components is $f_0 = 1/T$, and its transform is:

$$X[n] = \frac{1}{T}\int_T x_T(t)e^{-j2\pi n f_0 t}dt, \qquad x_T(t) = \sum_{n=-\infty}^{\infty} X[n]e^{j2\pi n f_0 t}$$

$$n = 0, \pm 1, \pm 2, \cdots$$

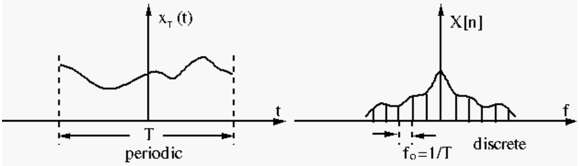
The discrete spectrum can also be represented as:

$$X(f) = \sum_{n=--\infty}^{\infty} X[n]\delta(f - nf_0)$$

The time function is periodic:

$$x_T(t + kT) = x_T(t + k/f_0) = \sum_{n=-\infty}^{\infty} X[n]e^{-j2\pi nf_0(t+k/f_0)} = x_T(t)$$

(for $k = \pm 1, \pm 2, \cdots$)



Periodic, discrete time function $x[m]$, discrete, periodic spectrum $X[n]$
This is the discrete Fourier transform (DFT).

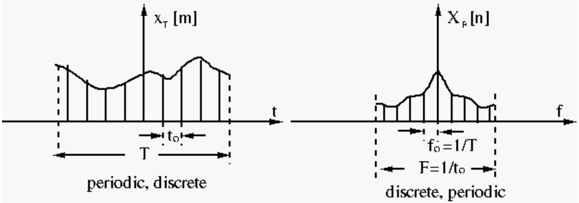
$$X[n] = \frac{1}{T}\sum_{m=0}^{N-1} x[m]e^{-j2\pi nmf_0t_0}, \qquad x[m] = \frac{1}{F}\sum_{n=0}^{N-1} X[n]e^{j2\pi nmf_0t_0}$$

$$m, n = 0, 1, \cdots, N-1$$

Here N is the number of samples in the period T, which is also the number of frequency components in the spectrum:

$$N = \frac{T}{t_0} = \frac{1/f_0}{1/F} = \frac{F}{f_0}$$

We therefore also have $TF = N$ and $t_0 f_0 = 1/N$.



The DFT can be redefined as

$$X[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{-mnj2\pi/N} = \sum_{m=0}^{N-1} w_N^{mn} x[m],$$

$$x[m] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{mnj2\pi/N} = \sum_{n=0}^{N-1} w_N^{-mn} X[n]$$

$$m, n = 0, 1, \cdots, N-1$$

where $w_N \triangleq e^{-j2\pi/N}/\sqrt{N}$. The time function and its spectrum are periodic: $x[m+kN] = x[m]$ and $X[n+kN] = X[n]$.

The forward and inverse DFT can be written as:

$$X[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{-mnj2\pi/N} = \sum_{m=0}^{N-1} w_N^{mn} x[m],$$

$$x[m] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{mnj2\pi/N} = \sum_{n=0}^{N-1} w_N^{-mn} X[n]$$

$$m, n = 0, 1, \cdots, N-1$$

Here we have defined

$$w^{mn} \triangleq \frac{1}{\sqrt{N}} (e^{-j2\pi/N})^{mn}, \qquad w^{*mn} = \frac{1}{\sqrt{N}} (e^{j2\pi/N})^{mn}$$

and w^{*mn} is its complex conjugate of w^{mn}. We further define an $N \times N$ matrix

$$\mathbf{W} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & w^{mn} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}_{N\times N}$$

where w^{mn} is the element in the mth row and nth column of $\mathbf{W}$.

W is symmetric ($w^{mn} = w^{nm}$)

$$\mathbf{W}^T = \mathbf{W}$$

and the rows (or columns) of **W** are orthogonal:

$$\langle \mathbf{w}_m, \mathbf{w}_n \rangle = \sum_{k=0}^{N-1} w^{*km} w^{kn} = \frac{1}{N} \sum_{k=0}^{N-1} (e^{j2\pi/N})^{mk} (e^{-j2\pi/N})^{nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} (e^{j2\pi/N})^{(m-n)k} \stackrel{*}{=} \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

as

- If $m = n$, $(e^{j2\pi/N})^{(n-m)k} = 1$ and $\langle \mathbf{w}_m, \mathbf{w}_n \rangle = 1$,
- If $m \neq n$, the summation becomes:

$$\sum_{k=0}^{N-1} (e^{j2\pi(n-m)/N})^k = \frac{1 - (e^{j2\pi(n-m)/N})^N}{1 - e^{j2\pi(n-m)/N}} = 0$$

We see that **W** is a unitary matrix (and symmetric):

$$\mathbf{W}^{*T} = \mathbf{W}^* = \mathbf{W}^{-1}$$

Matrix Form of the 1-D DFT

Define the two N-long vectors:

$$\mathbf{X} \triangleq \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}_{N\times 1}, \quad \mathbf{x} \triangleq \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}_{N\times 1}$$

The DFT can then be written more conveniently as a matrix-vector multiplication:

$$\mathbf{X} = \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & (e^{-j2\pi/N})^{mn} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \mathbf{Wx}$$

Matrix Form of the 1-D DFT

and

$$\mathbf{x} = \begin{bmatrix} x[0] \\ \cdot \\ \cdot \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & (e^{j2\pi/N})^{mn} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} X[0] \\ \cdot \\ \cdot \\ X[N-1] \end{bmatrix} = \mathbf{W}^{*}\mathbf{X} = \mathbf{W}^{-1}\mathbf{X}$$

The computational complexity of the 1-D DFT is $O(N^2)$, which, as we will see later, can be reduced to $O(N\, log_2 N)$ by the Fast Fourier Transform (FFT) algorithm.

Matrix Form of the 2D DFT

Reconsider the 2D DFT:

$$X[k,l] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \underbrace{[\frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x[m,n] e^{-j2\pi \frac{mk}{M}}]}_{X'[k,n]} e^{-j2\pi \frac{nl}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X'[k,n] e^{-j2\pi \frac{nl}{N}} \qquad \text{for } 0 \le m,k \le N-1, \;\; 0 \le n,l \le N-1$$

$$X'[k,n] \triangleq \frac{1}{\sqrt{M}} \sum_{m=0}^{N-1} x[m,n] e^{-j2\pi \frac{mk}{M}} \qquad (n = 0,1,\cdots,N-1)$$

The summation above is with respect to the row index m and the column index n is a fixed parameter, this expression is a one-dimensional Fourier transform of the nth column of $[x]$, which can be written in column vector (vertical) form as:

$$\mathbf{X'}_n = \mathbf{W}^* \mathbf{x}_n$$

for all columns $n = 0, \cdots, N-1$.

Matrix Form of the 2D DFT

Putting all these N columns together, we can write

$$\left[\mathbf{X'}_0, \cdots, \mathbf{X'}_{N-1}\right] = \mathbf{W}\left[\mathbf{x}_0, \cdots, \mathbf{x}_{N-1}\right]$$

or more concisely

$$\mathbf{X}' = \mathbf{W}\,\mathbf{x}$$

where **W** is a M by N Fourier transform matrix.

Matrix Form of the 2D DFT

$X[k,l] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X'[k,n] e^{-j2\pi \frac{nl}{N}}$ The sum is with respect to the column index n and the row index number k is fixed, this is a one-dimensional Fourier transform of the kth row of $\mathbf{X}'$, which can be written in row vector (horizontal) form as

$$\mathbf{X}_k^T = \mathbf{X'}_k^T \mathbf{W}^T, \quad (k = 0, \cdots, N-1)$$

Putting all these N rows together, we can write

$$\begin{bmatrix} \mathbf{X}_0^T \\ \cdot \\ \cdot \\ \mathbf{X}_{N-1}^T \end{bmatrix} = \begin{bmatrix} \mathbf{X'}_0^T \\ \cdot \\ \cdot \\ \mathbf{X'}_{N-1}^T \end{bmatrix} \mathbf{W}$$

($\mathbf{W}$ is symmetric: $\mathbf{W}^T = \mathbf{W}$), or more concisely

$$\mathbf{X} = \mathbf{X}'\mathbf{W}$$

Matrix Form of the 2D DFT

But since $\mathbf{X}' = \mathbf{W}\mathbf{x}$, we have

$$\mathbf{X} = \mathbf{W}\mathbf{x}\mathbf{W}$$

Hence the 2D DFT can be implemented by transforming all the rows of $\mathbf{x}$ and then transforming all the columns of the resulting matrix. The order of the row and column transforms is not important.
Similarly, the inverse 2D DFT can be written as

$$\mathbf{x} = \mathbf{W}^{*}\mathbf{X}\mathbf{W}^{*}$$

Again note that $\mathbf{W}$ is a symmetric Unitary matrix:

$$\mathbf{W}^{-1} = \mathbf{W}^{*T} = \mathbf{W}^{*}$$

The complexity of 2D DFT is $O(N^3)$ which can be reduced to $O(N^2 log_2 N)$ if FFT is used.

The Fast Fourier Transform - FFT (1D)

The DFT pair is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} \qquad k = 0, \ldots, N-1 \tag{2}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} \qquad n = 0, \ldots, N-1 \tag{3}$$

The computational complexity for each point of the DFT is:

- $(N-1)$ Complex multiplications
- $(N-1)$ Complex additions

Hence for N points in the sequence we have:

- $O[N(N-1)]$ Complex multiplications
- $O[N(N-1)]$ Complex additions

Consider the decimation in time FFT algorithm.

Divide the DFT in even and odd terms:

$$\begin{aligned} X(k) &= \sum_{r=0}^{(N/2)-1} x(2r) W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x(2r+1) W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x(2r) W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x(2r+1) W_N^{2rk} \end{aligned} \tag{4}$$

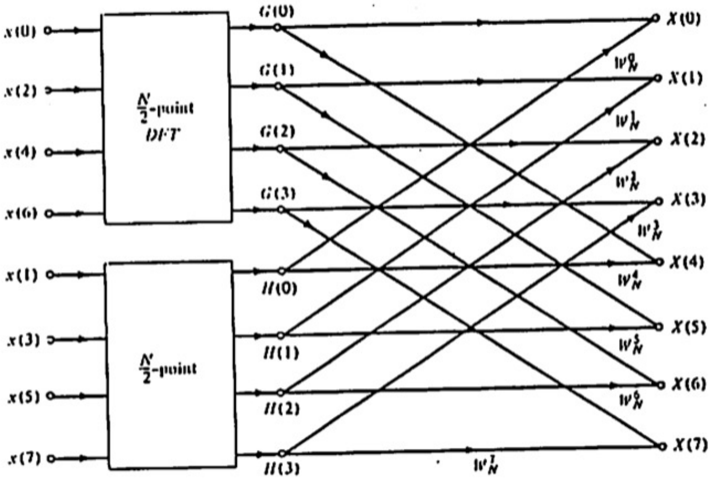
Notice $W_N^{2rk} = e^{-j\frac{2\pi}{N}2rk} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}$

Hence

$$X(k) = \underbrace{\sum_{r=0}^{(N/2)-1} x(2r) W_{N/2}^{rk}}_{\frac{N}{2}-point\ DFT} + W_N^k \underbrace{\sum_{r=0}^{(N/2)-1} x(2r+1) W_{N/2}^{rk}}_{\frac{N}{2}-point\ DFT} \qquad k = 0, 1, \cdots, N-1 \tag{5}$$

$$X(k) = G(k) + W_N^k H(k) \qquad k = 0, 1, \cdots, N-1 \tag{6}$$

But $G(k)$ and $H(k)$ are periodic in $\frac{N}{2}$.

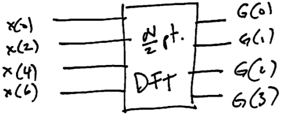


(a) Result of one decimation of the time samples

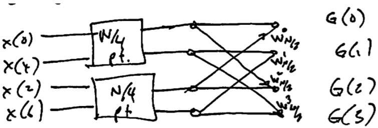
For instance

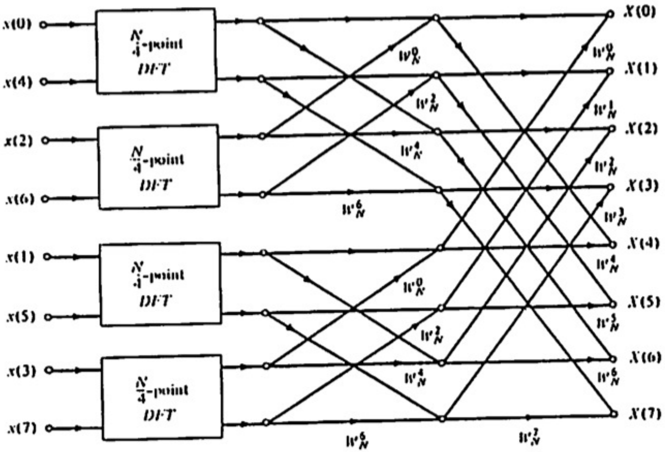
$$X(1) = G(1) + W_N^1 H(1) \qquad (N = 8)$$
$$X(5) = G(5) + W_N^5 H(5) \tag{7}$$
$$= G(1) + W_N^5 H(1)$$

Each of the $G(k)$ and $H(k)$ are $N/2$ DFT's; however, these can be computed using $N/4$ point DFT's and so on.
For instance the $N/2$ point DFT:



Can be found as:

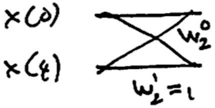




(b) Result of applying two decimations

FIGURE 10-4. Flow graphs showing the decimation-in-time decomposition of an N-point DFT computation ($N = 8$).

It self each 2 point DFT:



If $N = 2^b$ (a power of 2), then we have $log_2 N = b$ decompositions. At each stage we have N complex multiplications and additions. Hence the total number of complexity operations is:

- $O(Nlog_2 N)$ multiplications.
- $O(Nlog_2 N)$ additions

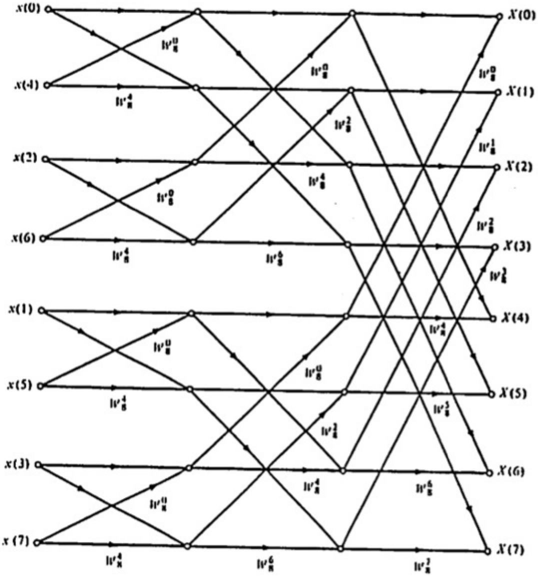


FIGURE 10-5. Complete flow graph for an FFT developed by applying decimation in time ($N = 8$).

CALCULATION OF THE 2-D DFT

1. Direct Calculation

The direct calculation of the 2-D DFT is the double sum:

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1,n_2) w_{N_1}^{n_1 k_1} w_{N_2}^{n_2 k_2}$$
$$0 \le k_1 \le N_1 - 1$$
$$0 \le k_2 \le N_2 - 1 \tag{8}$$

where $w_N = e^{\frac{-j2\pi}{N}}$ The evaluation of one sample of $X(k_1,k_2)$ requires $N_1 N_2$ complex multiplications and $N_1 N_2$ complex additions.

Thus, since there are $N_1 N_2$ points. The complexity is in the order of $[N_1^2 N_2^2]$.

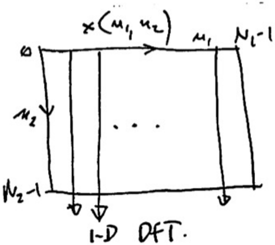
2. Row-Column Decomposition

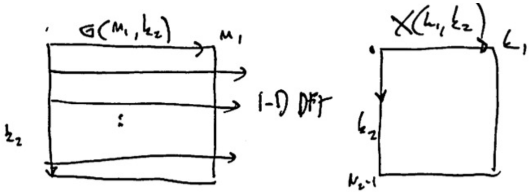
The 2-D DFT can be written as:

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1} \underbrace{\left[\sum_{n_2=0}^{N_2-1} x(n_1,n_2) w_{N_2}^{n_2 k_2} \right]}_{G(n_1,k_2)} w_{N_1}^{n_1 k_1} \tag{9}$$

$$X(k_1,k_2) = \sum_{n_1=0}^{N_1-1} G(n_1,k_2) w_{N_1}^{n_1 k_1} \tag{10}$$

Hence





The complexity here is as follow:

$$N_1(1D\,N_2\,pt.\,DFT_s) + N_2(1D\,N_1\,pt.\,DFT) = N_1 N_2^2 + N_2 N_1^2$$

or $\quad N_1 N_2(N_1 + N_2)$

3. Row column FFT

If N_1 and N_2 are powers of 2 then each $1D\,DFT$ can be computed with a $1D\,FFT$. Recall they each $N\,pt\,1D\,FFT$ has a complexity $N\log N$.

Hence, the complexity is reduced to:

$$N_1 N_2 \log N_2 + N_2 N_1 \log N_1 = N_1 N_2 \log(N_2 N_1) \tag{11}$$

To get a feeling for a numerical savings involved consider a 1024×1024 $2D$ DFT.

$$C_{direct} = 2^{40} \approx 10^{12} \text{ complex multiplications}$$
$$C_{r/c\,direct} = 2^{31} \approx 10^{9} \text{ complex multiplications}$$
$$C_{r/c\,FFT} = 10 \times 2^{20} \approx 10^{7} \text{ complex multiplications}$$

If it would take 1 day to process a $2D$ direct, then it would take 1 sec with the r/c FFT!!

Linear Convolution Via DFT

- Recall in one dimension

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \; k = 0, 1, \ldots, N-1.$$

- The $N \times N$ unitary DFT matrix $\mathbb{W}$ is given by

$$\mathbb{W} = \left\{ \frac{1}{\sqrt{N}} w_N^{un} \right\}$$

- Circular convolution Theorem: If

$$x_2(n) = \sum_{k=0}^{N-1} h(n-k)_c x_1(k), \; 0 \leq n \leq N-1$$

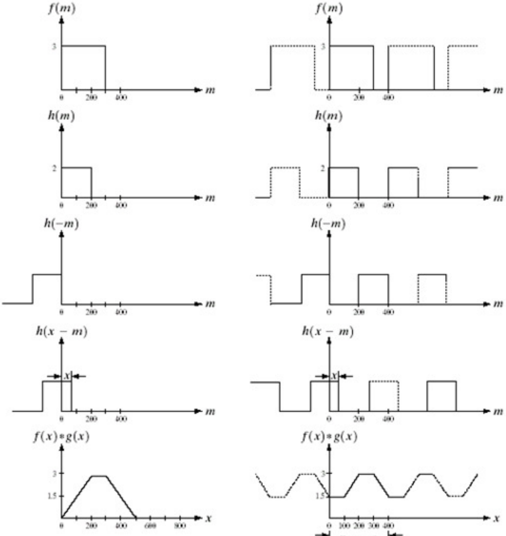
then

$$DFT\{x_2(n)\}_N = DFT\{h(n)\}_N DFT\{x_1(n)\}_N$$

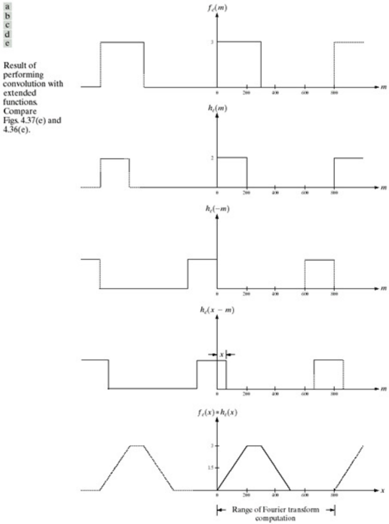
Linear Convolution via DFT (I)

a	f
b	g
c	h
d	i
e	j

Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.



Linear Convolution via DFT (II)



Linear Convolution via DFT Algorithm

The linear convolution of two sequences $\{h(n)\}_{n=0}^{P-1}$ and $\{x(n)\}_{n=0}^{N-1}$ can be obtained by the following algorithm:

1. Define $M \geq P + N$
2. Define $\tilde{h}(n)$ and $\tilde{x}(n)$ as the M zero extended sequences of $h(n)$ and $x(n)$ respectively
3. Compute $\hat{Y}(k) = \hat{H}(k)\hat{X}(k)$, where $\hat{H}(k) = DFT\{\tilde{h}(n)\}_M$ and $\hat{X}(k) = DFT\{\tilde{x}(n)\}_M$
4. Take the inverse DFT of $\hat{Y}(k)$ to obtain $y(n)$

Two Dimensional DFT

The two dimensional DFT of an $N \times N$ image is a separable transform defined as

$$X(u,v) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) w_N^{km} w_N^{ln}, \;\; 0 \leq k,l \leq N-1$$

and the inverse transform is defined as

$$x(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X(u,v) w_N^{-km} w_N^{-ln}, \;\; 0 \leq m,n \leq N-1.$$

Two Dimensional Linear Convolution

- The DFT of the two dimensional circular convolution of two arrays is the product of their DFTs, *i.e.*, if

$$y(m,n) = \sum_{m'=0}^{N-1} \sum_{n'=0}^{N-1} h(m-m', n-n')_c u(m', n'),\ 0 \le m,n \le N-1$$

  then

$$DFT\{y(m,n)\}_N = DFT\{h(m,n)\}_N DFT\{u(m,n)\}_N$$

Two Dimensional Linear Convolution

- The DFT of the two dimensional circular convolution of two arrays is the product of their DFTs, *i.e.*, if

$$y(m,n)=\sum_{m'=0}^{N-1}\sum_{n'=0}^{N-1}h(m-m',n-n')_c u(m',n'),\ 0\leq m,n\leq N-1$$

then

$$DFT\{y(m,n)\}_N = DFT\{h(m,n)\}_N DFT\{u(m,n)\}_N$$

- Extensions to linear filtering can be done using zero padding

Two dimensional Example of Zero Padding

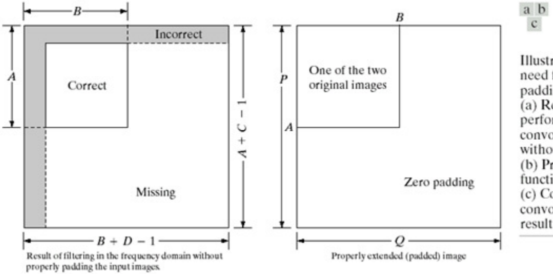
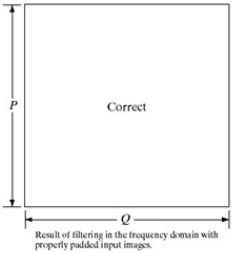


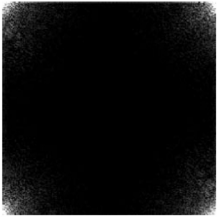
Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.



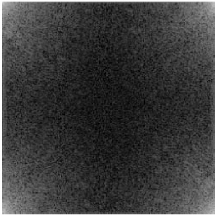
$$P = A + C - 1$$
$$Q = B + D - 1$$

Example of Image DFT (I)

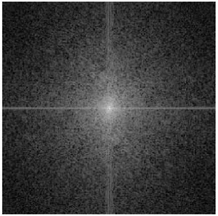
(*a*) Original



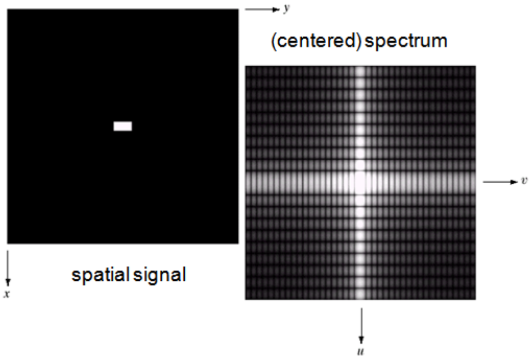
(*b*) Clipped magnitude, nonordered



(*c*) Log magnitude, nonordered



(*d*) Log magnitude, ordered

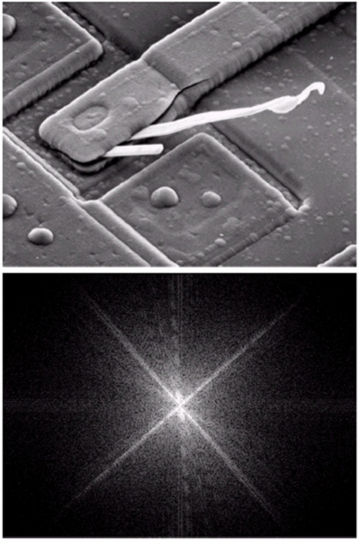


We can center the DFT by premultiplying image U by the array $(-1)^{m+n}$

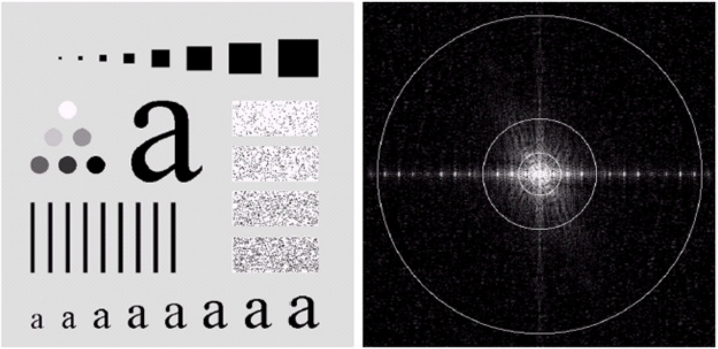
$$x(k+N/2, l+N/2) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m,n)(-1)^{m+n} w_N^{km} w_N^{ln}$$

Example of Image DFT (III)

- Scanning Electron Microscope (SEM) image of IC board
- Edges correspond to high frequencies
- Note directionality of edges



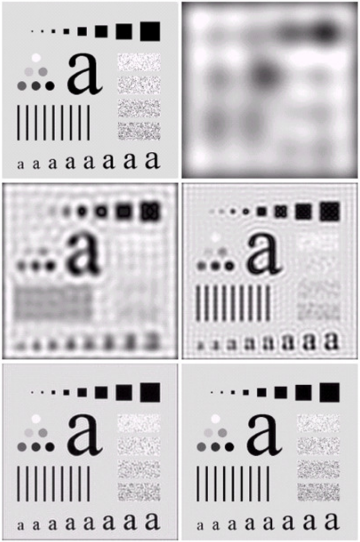
Energy Compaction (I)



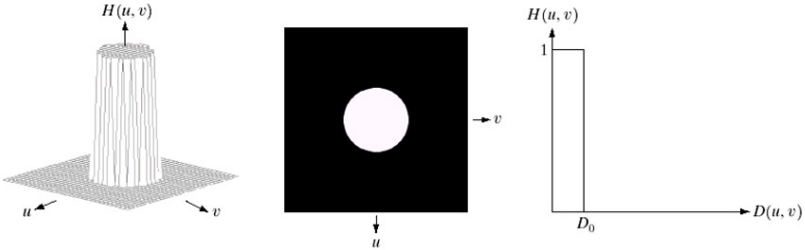
a b

FIGURE (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Energy Compaction (II)



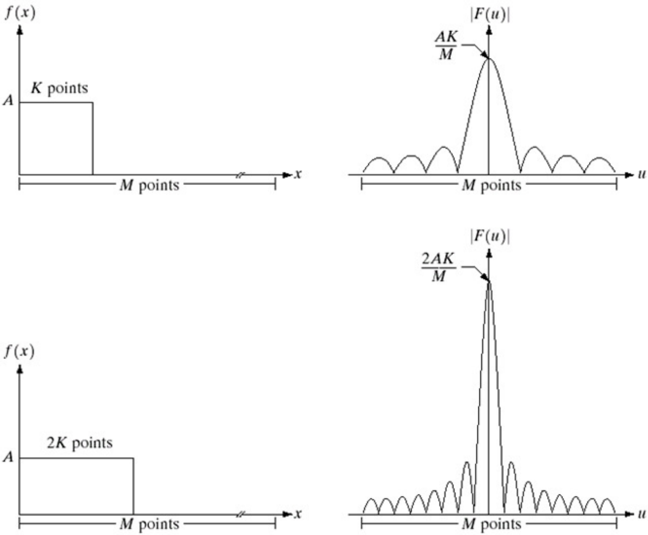
Ideal Low Pass Filters (I)



a b c

FIGURE (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Low Pass Filters (II)



a b
c d

FIGURE (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

ILPF Example

- Isolated spatial domain points represent fine details
- Convolution simply replicates sincs
  - Width of sinc controls blurring
  - Positive and negative values of sinc cause ringing
  - One dimensional signals are scan lines

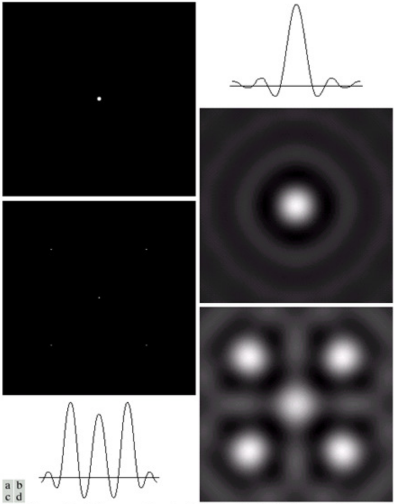
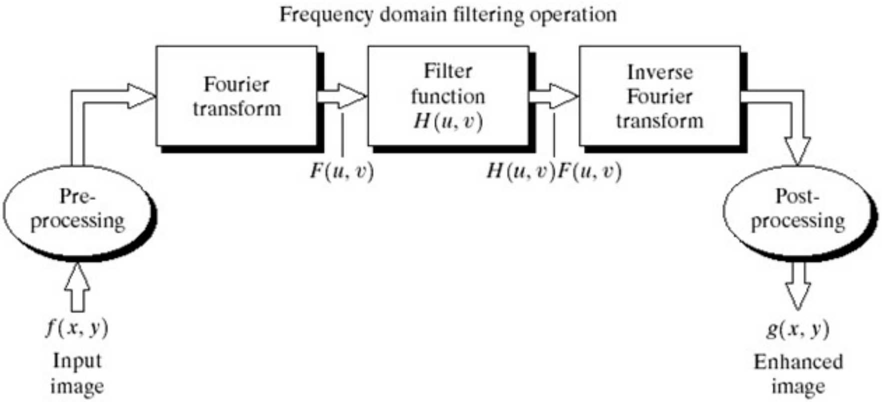


FIGURE (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Filtering in the Frequency Domain



Low Pass and High Filtering Example

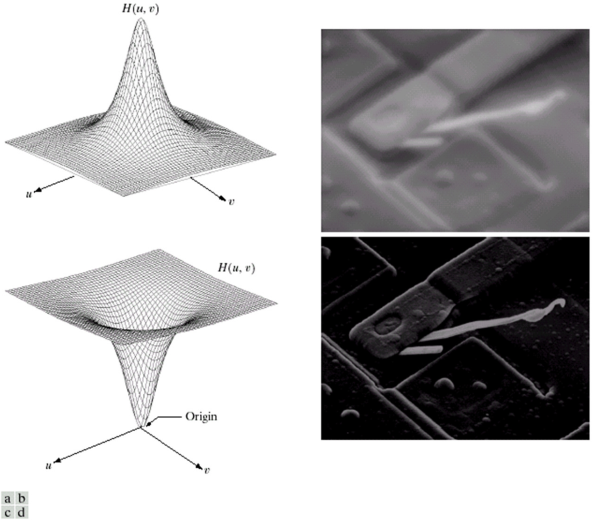
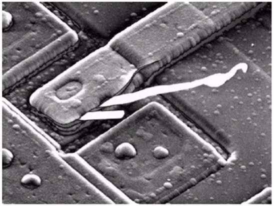


FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).



DC shifted highpass filter output

Low Pass Filters

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

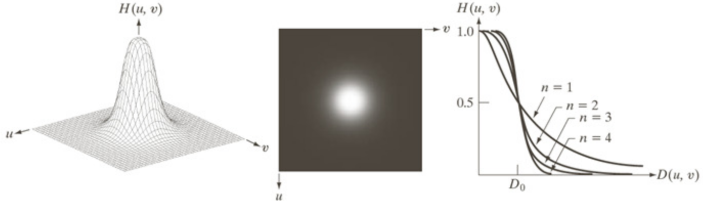
- $D(u,v)$ is the distance from point (u,v) to the origin
- Ideal filter can be implemented digitally but has undesired effects
- Butterworth filter is a smooth approximation to ideal filter
- Gaussian filter is a smooth function both in space and frequency domains

Butterworth Low Pass Filter

Frequency response:

$$H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}$$

- Order: n, Cutoff frequency: D_0
- Smooth transfer function
    - Minimizes ringing
    - Order controls transition bandwidth

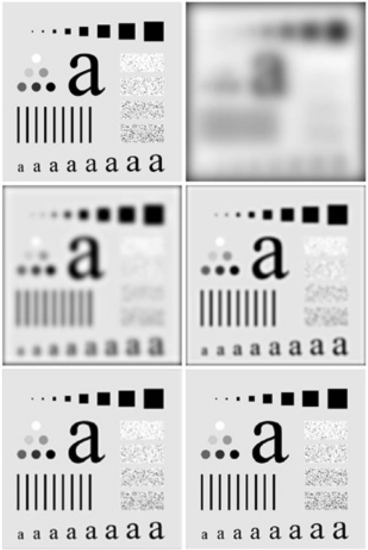


a b c

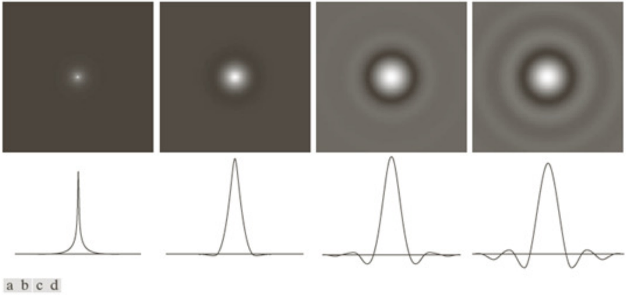
FIGURE (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Filter Example

- Size 500×500
- Filter order:2
- $D_0 = 5, 15, 30, 80$ and 230
- Significantly reduced ringing compared to ILPF



Spatial Domain Representation of Butterworth Filter



- Cutoff frequency:5
- Increasing filter order: $1, 2, 5$ and 20
  - Impulse response spreads, oscillations introduced
  - Smoothing and ringing introduced

Gaussian Low Pass Filter

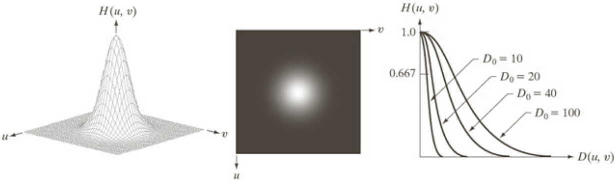


FIGURE (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

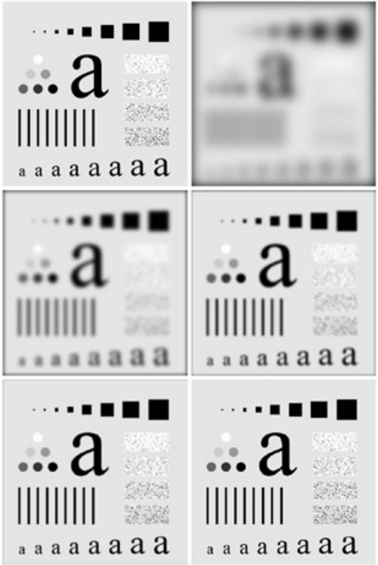
Frequency response:

$$H(u,v) = \exp - D^2(u,v)/2D_0^2$$

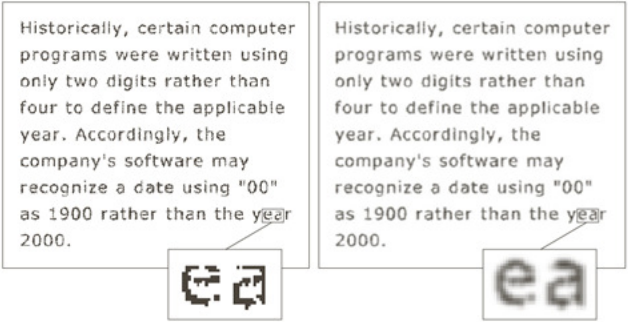
- Spatial domain also a gaussian function
- No ringing
- Less cutoof/transition control

Gaussian Low Pass Filter Example

- $D_0 = 5, 15, 30, 80$ and 230
- Not as much smoothing
- More gradual transition band
- No ringing



Application Example



- Poor resolution sampled text
  - Scanned material, faxes
  - Broken text
- Result of Gaussian low pass filtering: broken character segments are joined

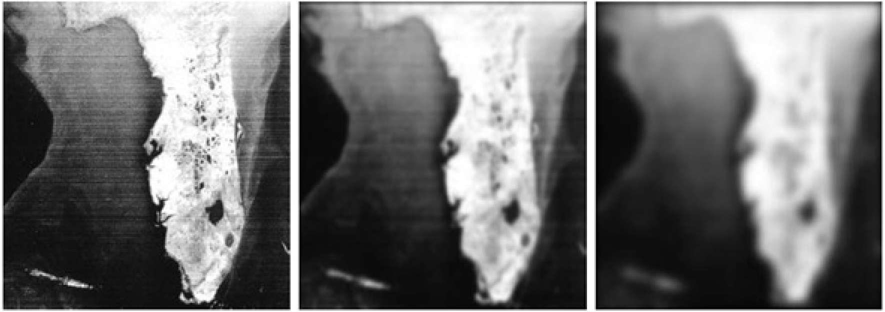
Cosmetic Smoothing of Images

a b c

(a) Original, (b) GLPF with $D_0 = 100$, and (c) GLPF with $D_0 = 80$

- Note reduction in skin lines

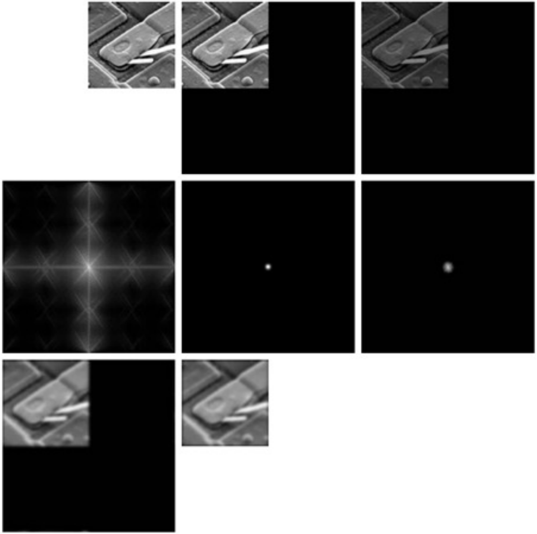
Enhancement of Poorly Acquired Images



a b c

FIGURE (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

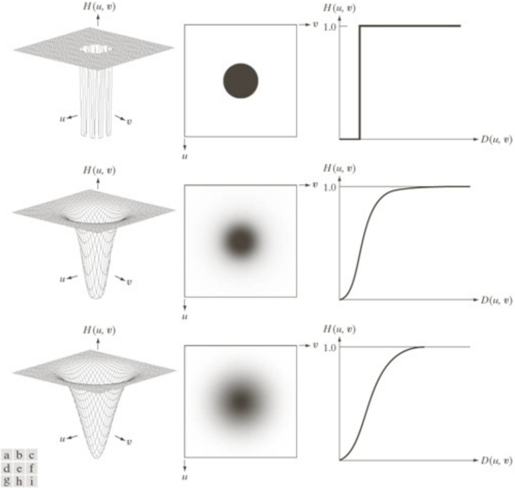
Gaussian Filter with Zero Padding Example



- Simple highpass representation

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

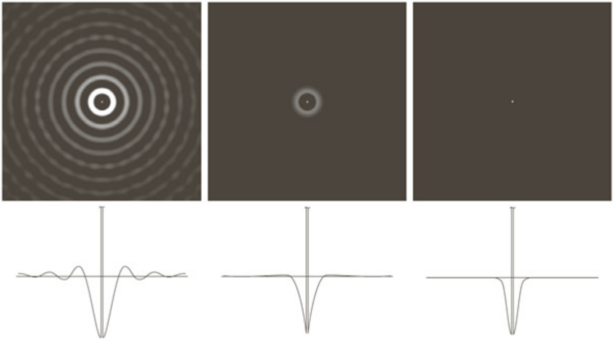
- Spectrally centered examples
  - Ideal
  - Butterworth
  - Gaussian



High Pass Filters

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$



a b c

FIGURE Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

High Pass Filtering Example

Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and 160.

Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160.

Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160.

Application Example

- Thumb print
- Result of highpass filtering
- Result of thresholding

Laplacian Operator

- Recall that

$$\mathfrak{F}\left\{\nabla^2 f(x,y)\right\} = -(u^2+v^2)F(u,v)$$

- Then, the Laplacian operator is a filter with frequency response

$$H(u,v) = -(u^2+v^2)$$

- If spectral centering is used then

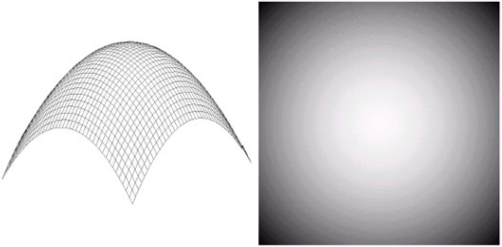
$$\nabla^2 f(x,y) \leftrightarrow -\left[(u-N/2)^2+(v-N/2)^2\right]F(u,v)$$

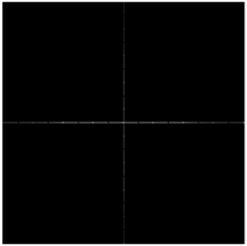
- A sharpened image is given by:

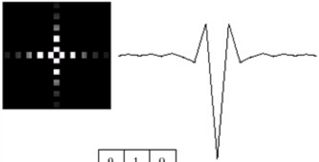
$$\begin{aligned} g(x,y) &= f(x,y) - \nabla^2 f(x,y) \\ &= \mathfrak{F}^{-1}\left\{\left[1-\left((u-N/2)^2+(v-N/2)^2\right)\right]F(u,v)\right\} \end{aligned}$$

Laplacian in the Frequency and Spatial Domain

- Highpass filter
- Spatial response
  - Restricted to axis
- Rotation invariant response
  - diagonals can be added



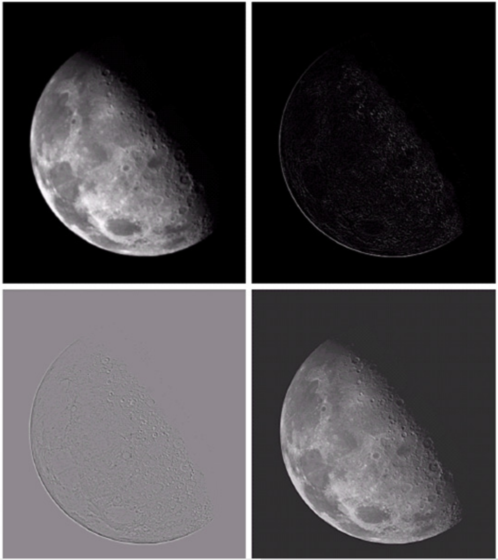




0	1	0
1	−4	1
0	1	0

Laplacian Example

- Shown
  - Original
  - Laplacian
  - Scaled Laplacian
  - Enhanced result



Unsharp Masking

- High-boost filtering

$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y)$$

- Rearranging

$$f_{hb}(x,y) = (A-1)f(x,y) + f_{hp}(x,y)$$

- Composite frequency response

$$H_{hb}(u,v) = (A-1) - H_{hp}(u,v)$$

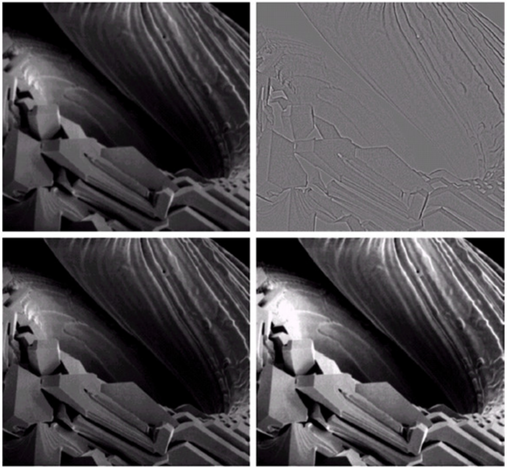
- High frequency emphasis

$$H_{hfe}(u,v) = a + bH_{hp}(u,v)$$

High-Boost Filtering Example

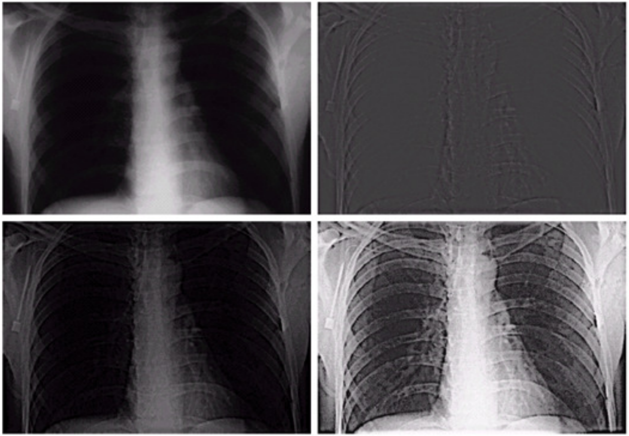
a b
c d

(a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)



High Frequency Emphasis Example

High frequency emphasis, $a = 0.5$, $b = 2.0$



a b
c d

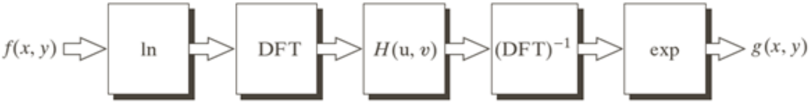
(a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Homomorphing Filtering

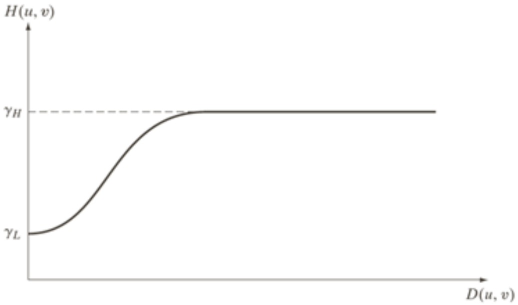
- Recall illumination and reflectance image model

$$f(x,y) = i(x,y)r(x,y)$$

- Not directly separable in the frequency domain
- Solution:



Homomorphing Filtering

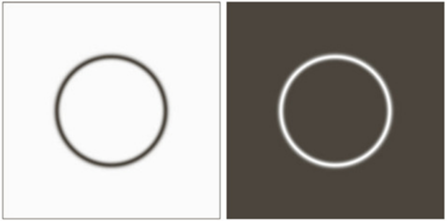


- Illumination component
  - Slow spatial variations (low frequencies)
- Reflectance component
  - Varies abruptly, especially at object borders (high frequencies)
- Homomorphic filter characteristics
  - Attenuate illumination component (low frequencies)

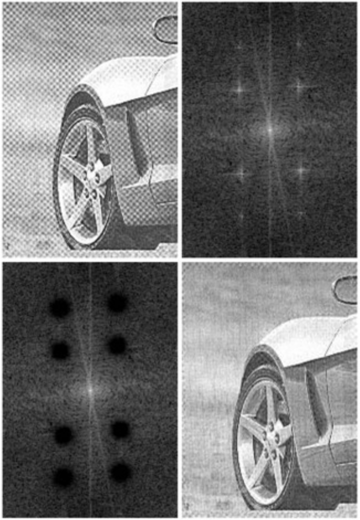
Bandreject/Bandpass Filters

Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

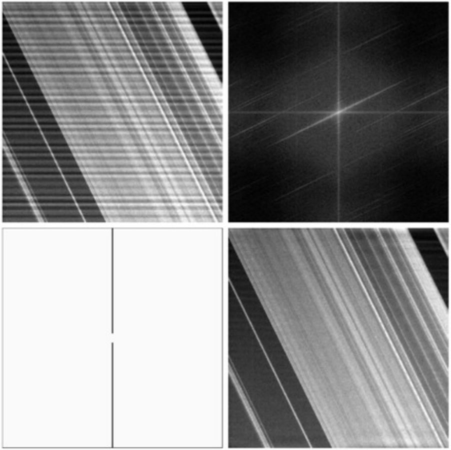


Bandreject Example (I)

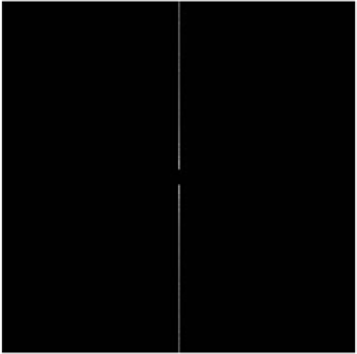


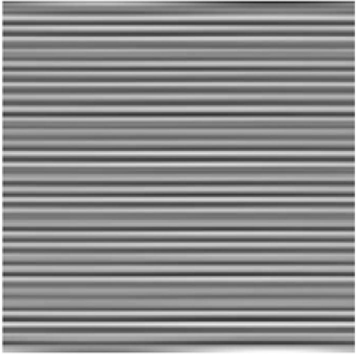
Bandreject Example (II)

- Image example with nearly periodic interference
- Spectrum: energy in the vertical axis represents the interference pattern
- Notch vertical filter
- Result of filtering



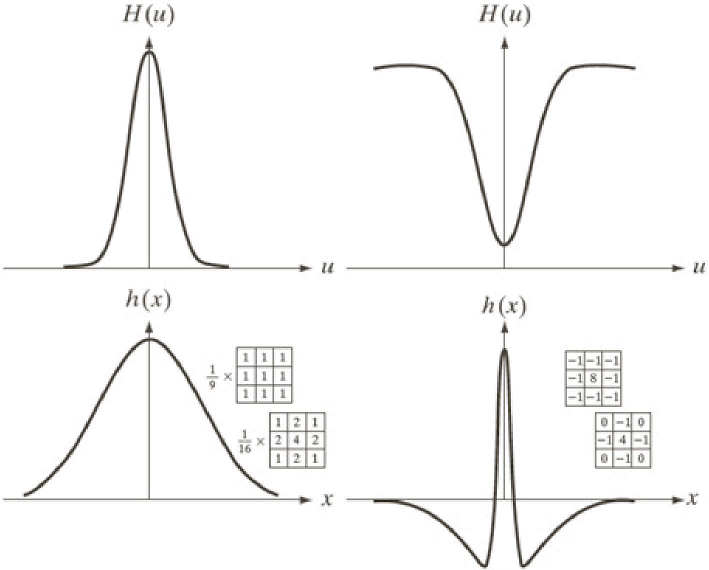
Bandpass Example





- Same image as before
- Result: interference pattern

Implementation Examples



Correlation Example

- Correlation measures statistical similarity
- Common application: template matching
- Zero pad image and template
- Multiply DFTs (conjugate image DFTs)
- Invert results
- Find peaks location