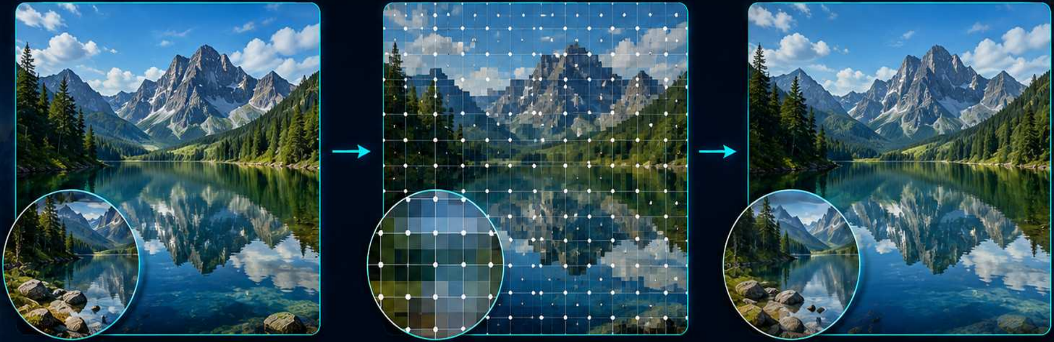
Chapter 6: Sampling and Interpolation

Digital Imaging & Photography



Continuous scene　　Sampled on a 2-D grid　　Interpolated image

Gonzalo R. Arce

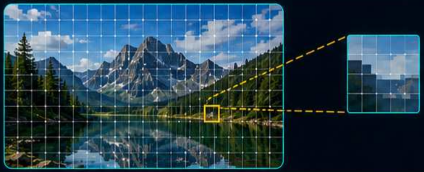
Department of Electrical and Computer Engineering, University of Delaware

Image Sampling

- Discrete-time image processing represents images by samples on a 2-D lattice.
- CCD / CMOS sensors measure the scene at regularly spaced detector locations.
- Typical sampled image sizes range from 256×256 to 2000×2000 and beyond.

Sampling the scene

→ Each grid point corresponds to one measured sample (pixel).



One sample (pixel) represents the average light captured over the detector area.

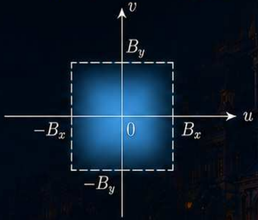
Sampling a Continuous Image

- Although images are not generally band-limited, we often approximate them as band-limited signals.

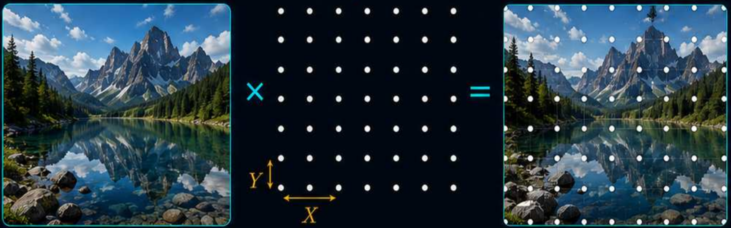
$$g_s(x, y) = \sum_m \sum_n g(mX, nY)\,\delta(x - mX, y - nY)$$

- Assume the continuous image $g(x, y)$ is band-limited, so $G(u, v) = 0$ outside $|u| \leq B_x$ and $|v| \leq B_y$.

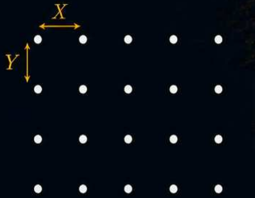
Frequency domain of $g(x, y)$



Continuous image $g(x, y)$ | 2-D impulse lattice | Sampled image $g_s(x, y)$



Sampling lattice (spacings X, Y)

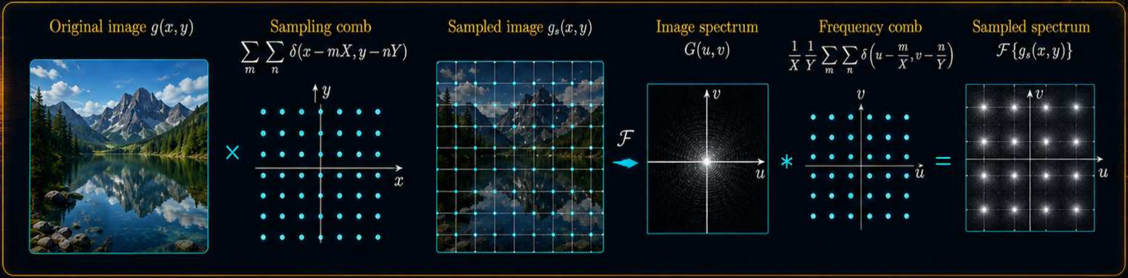


Fourier Transform of the Sampled Image

$$\mathcal{F}\{g_s(x,y)\} = \mathcal{F}\left\{g(x,y)\sum_m\sum_n \delta(x-mX, y-nY)\right\}$$

$$= G(u,v) * \frac{1}{X}\frac{1}{Y}\sum_m\sum_n \delta\left(u-\frac{m}{X}, v-\frac{n}{Y}\right)$$

Sampling in the spatial domain multiplies the image by an impulse lattice; in the frequency domain this becomes convolution with a frequency comb.

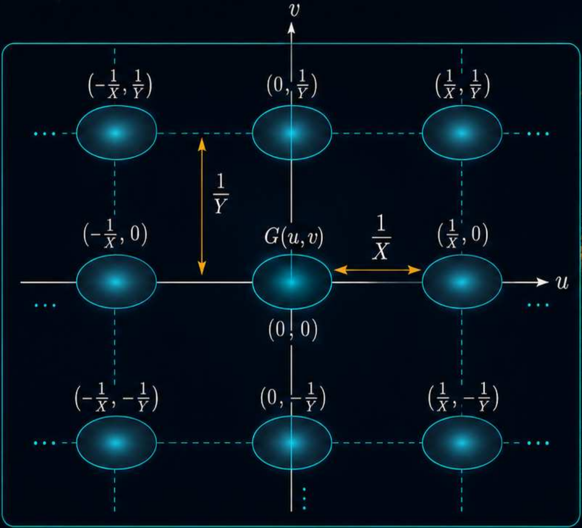


Replicas in the Frequency Domain

$$\mathcal{F}\{g_s(x,y)\} = \frac{1}{XY}\sum_m \sum_n G\left(u-\frac{m}{X}, v-\frac{n}{Y}\right)$$

$$= \frac{1}{XY}\,\mathrm{rep}_{1/X,\,1/Y}\left[G(u,v)\right]$$

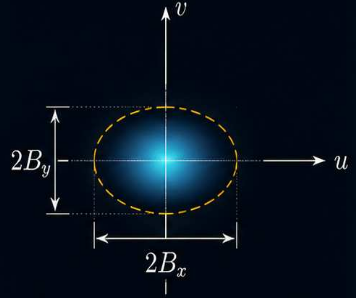
- The original spectrum $G(u,v)$ is replicated periodically at spacing $1/X$ in u and $1/Y$ in v.



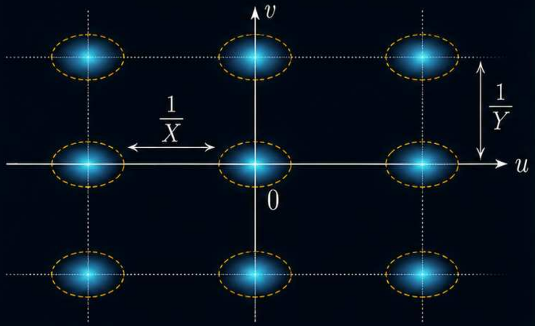
Spectrum of the Sampled Image

✧ A band-limited spectrum $G(u,v)$ becomes a periodic array of replicas after sampling. ✧

Original spectrum $G(u,v)$



Replicated spectrum of $g_s(x,y)$



Sampling in space $\leftrightarrow$ periodic replication in frequency.
Closer replicas imply greater risk of overlap and aliasing.

Avoiding Aliasing and Ideal Reconstruction

Sampling conditions (no aliasing)

$$B_x < \frac{1}{2X}, \qquad B_y < \frac{1}{2Y}$$

Ideal reconstruction filter

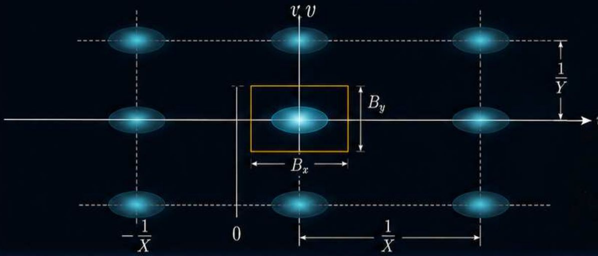
$$H(u,v) = \begin{cases} XY, & |u| < \frac{1}{2X},\ |v| < \frac{1}{2Y} \\ 0, & \text{else} \end{cases}$$

Reconstruction kernel (2-D sinc)

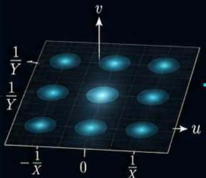
$$h(x,y) = sinc\left[\frac{x}{X}, \frac{y}{Y}\right]$$

- Sampling intervals must be large enough to avoid overlap of spectral replicas.
- Ideal reconstruction is obtained with a 2-D sinc kernel in the spatial domain.

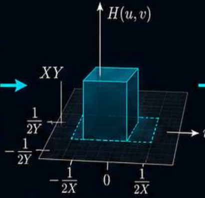
2-D replicated spectrum $G_s(u,v)$



Sampled spectrum $G_s(u,v)$



Ideal rectangular LPF $H(u,v)$



Reconstructed image $g(x,y)$

Ideal Sinc Reconstruction

✧ Ideal reconstruction from samples

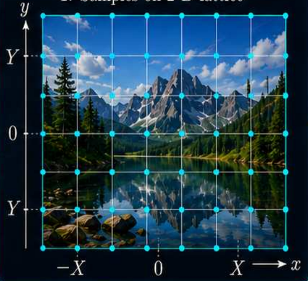
A bandlimited image can be perfectly reconstructed from its uniformly spaced 2-D samples.

$$\hat{g}(x,y) = g_s(x,y) * \operatorname{sinc}\left[\frac{x}{X}, \frac{y}{Y}\right]$$

$$= \left[\sum_m \sum_n g(mX, nY)\,\delta(x - mX, y - nY)\right] * \operatorname{sinc}\left[\frac{x}{X}, \frac{y}{Y}\right]$$

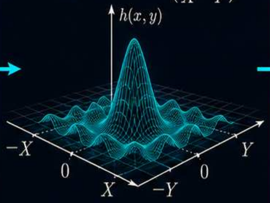
$$\hat{g}(x,y) = \sum_m \sum_n g(mX, nY)\operatorname{sinc}\left(\frac{x - mX}{X}, \frac{y - nY}{Y}\right)$$

1. Samples on 2-D lattice



2. 2-D sinc kernel (interpolator)

$$h(x,y) = \operatorname{sinc}\left(\frac{x}{X}, \frac{y}{Y}\right)$$



3. Reconstructed image $\hat{g}(x,y)$

✧ Each sample contributes a shifted sinc basis function.

✧ The reconstructed image is the superposition of all shifted sinc kernels.

Practical Reconstruction with Gaussian Kernels

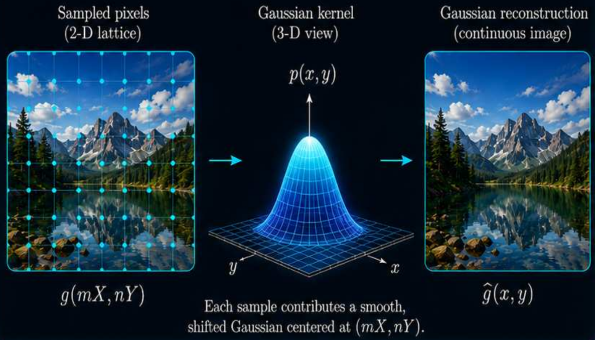
- The ideal LPF is difficult to obtain; practical systems often use Gaussian smoothing kernels.

- Gaussian kernel (normalized)

$$p(x,y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]$$

- Gaussian reconstruction (continuous estimate)

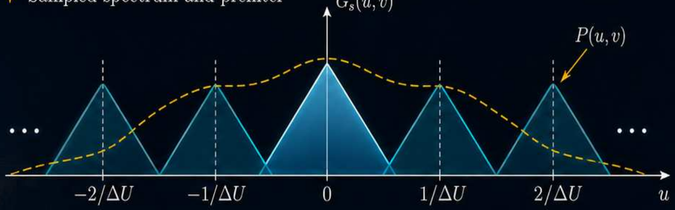
$$\hat{g}(x,y) = \sum_m \sum_n g(mX, nY) \left(\frac{1}{2\pi\sigma^2}\right) \times \exp\left[-\frac{(x-mX)^2+(y-nY)^2}{2\sigma^2}\right]$$



Narrower kernels preserve detail but risk aliasing; wider kernels reduce aliasing but increase blur.

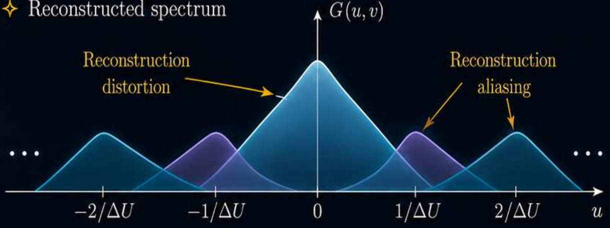
Aliasing and Reconstruction Distortion

Sampled spectrum and prefilter



- Nonideal filtering distorts the desired spectrum.
- Overlapping replicas leak into the reconstructed image.
- Aliasing appears as false low-frequency structure.

Reconstructed spectrum



Original (no aliasing) | Aliased reconstruction

Aliasing introduces false patterns and jagged detail.

Why Aliasing Matters in Resizing

Original

High-resolution reference

Resized without prefiltering

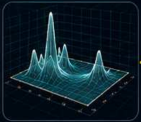
Moiré patterns and jagged edges from folded high frequencies

Resized with anti-aliasing

Clean edges and natural detail after prefiltering

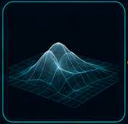
- Interpolation and decimation change image size while preserving sampling structure.
- Without prefiltering, fine detail folds into false patterns.
- Anti-aliasing is essential in commercial imaging applications.

The anti-aliasing pipeline



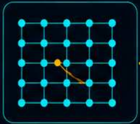
1. Prefilter

Low-pass filter removes high frequencies that cannot be preserved at the new sampling rate.



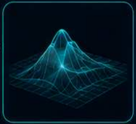
2. Sample / Resize

Take samples on the new lattice (interpolate or decimate) to change image size.

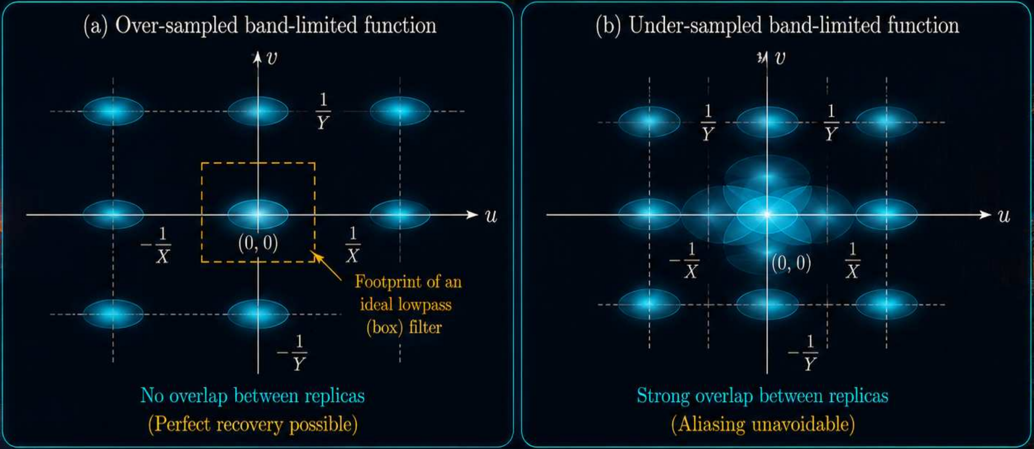


3. Reconstruct

Reconstruct a continuous image from the new samples (e.g., via sinc or other kernel).



Aliasing in 2-D Frequency Space



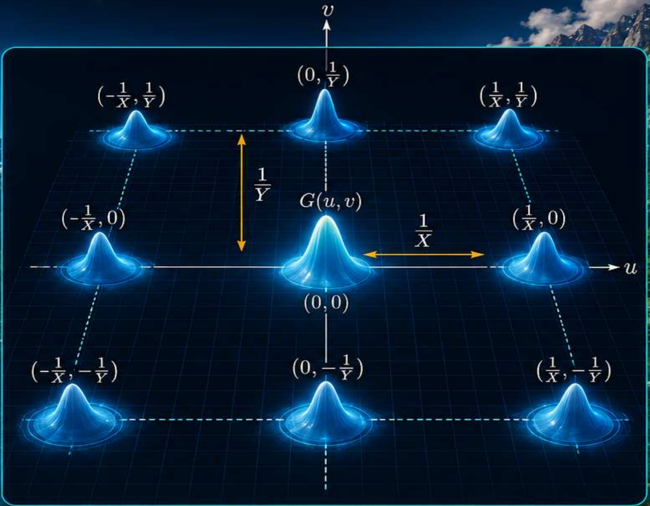
- **Over-sampling** keeps spectral replicas separate, enabling perfect recovery.
- **Under-sampling** causes overlap, making aliasing unavoidable.

Replicas in the Frequency Domain

$$\mathcal{F}\{g_s(x,y)\} = \frac{1}{XY}\sum_m \sum_n G\left(u-\frac{m}{X}, v-\frac{n}{Y}\right)$$

$$= \frac{1}{XY}\,\mathrm{rep}_{1/X,1/Y}\,[G(u,v)]$$

- The original spectrum $G(u,v)$ is replicated periodically at spacing $1/X$ in u and $1/Y$ in v.

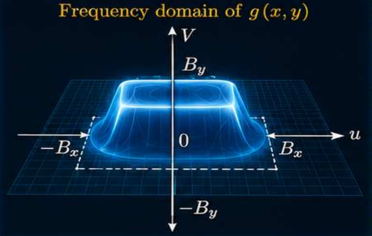


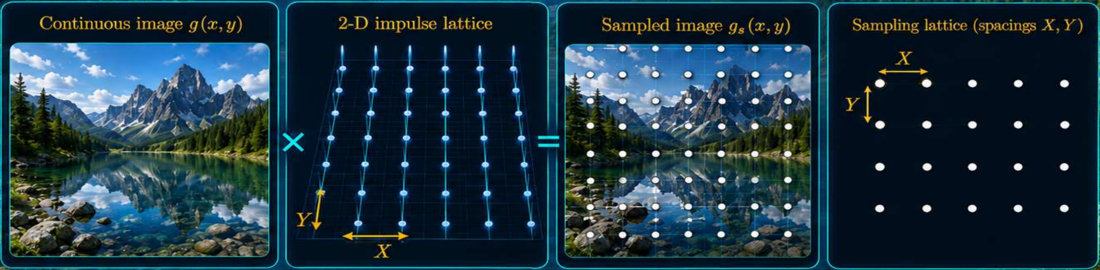
Sampling a Continuous Image

- Although images are not generally band-limited, we often approximate them as band-limited signals.

$$g_s(x,y) = \sum_m \sum_n g(mX, nY)\,\delta(x - mX, y - nY)$$

- Assume the continuous image $g(x,y)$ is band-limited, so $G(u,v) = 0$ outside $|u| \le B_x$ and $|v| \le B_y$.

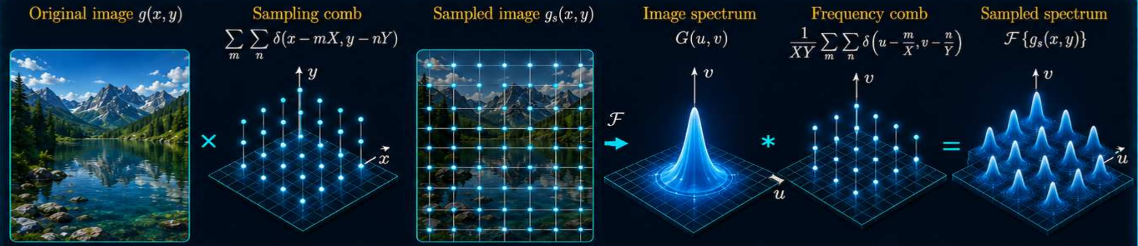




Fourier Transform of the Sampled Image

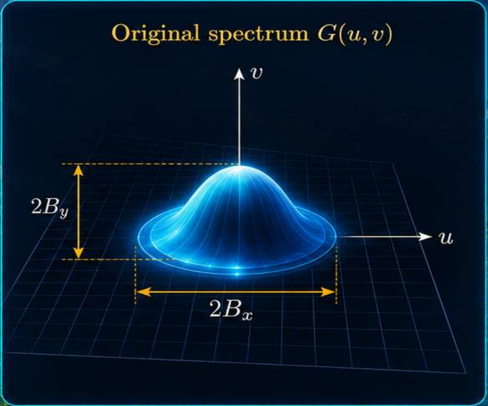
$$\mathcal{F}\{g_s(x,y)\} = \mathcal{F}\left\{g(x,y)\sum_m\sum_n \delta(x-mX, y-nY)\right\}$$
$$= G(u,v) * \frac{1}{XY}\sum_m\sum_n \delta\left(u-\frac{m}{X}, v-\frac{n}{Y}\right)$$

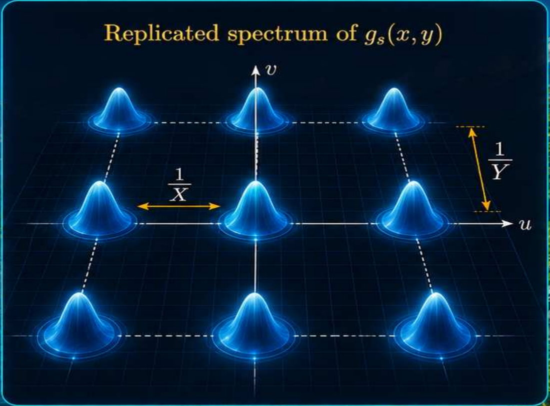
Sampling in the spatial domain multiplies the image by an impulse lattice;
in the frequency domain this becomes convolution with a frequency comb.



Spectrum of the Sampled Image

A band-limited spectrum $G(u, v)$ becomes a periodic array of replicas after sampling.





Sampling in space $\leftrightarrow$ periodic replication in frequency.
Closer replicas imply greater risk of overlap and aliasing.

Avoiding Aliasing and Ideal Reconstruction

Sampling conditions (no aliasing)

$$B_x < \frac{1}{2X}, \qquad B_y < \frac{1}{2Y}$$

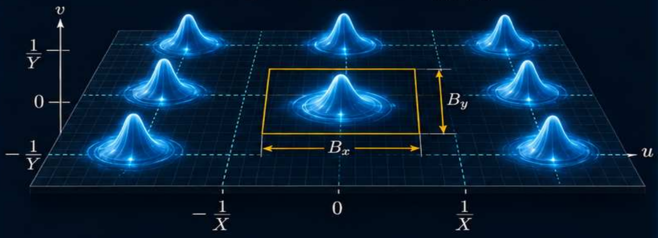
Ideal reconstruction filter

$$H(u,v) = \begin{cases} XY, & |u| < \frac{1}{2X},\ |v| < \frac{1}{2Y} \\ 0, & \text{else} \end{cases}$$

Reconstruction kernel (2-D sinc)

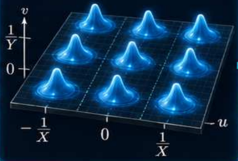
$$h(x,y) = sinc\left[\frac{x}{X}, \frac{y}{Y}\right]$$

2-D replicated spectrum $G_s(u,v)$

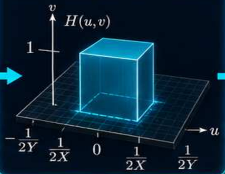


- Sampling intervals must be large enough to avoid overlap of spectral replicas.
- Ideal reconstruction is obtained with a 2-D sinc kernel in the spatial domain.

Sampled spectrum $G_s(u,v)$



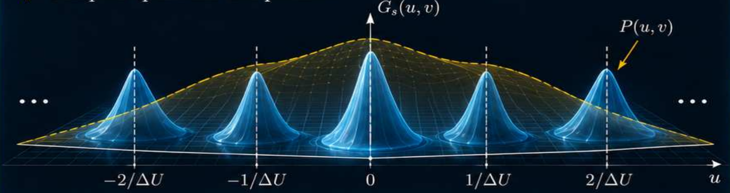
Ideal rectangular LPF $H(u,v)$



Reconstructed image $g(x,y)$

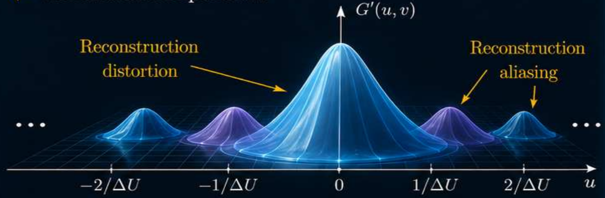
Aliasing and Reconstruction Distortion

✧ Sampled spectrum and prefilter



- ✧ Nonideal filtering distorts the desired spectrum.
- ✧ Overlapping replicas leak into the reconstructed image.
- ✧ Aliasing appears as false low-frequency structure.

✧ Reconstructed spectrum



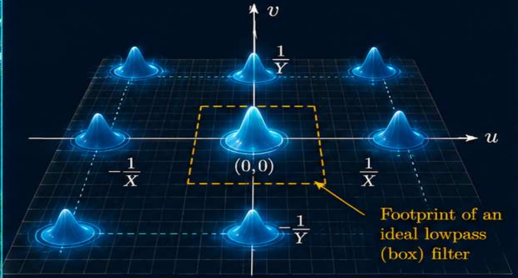
Original (no aliasing)

Aliased reconstruction

Aliasing introduces false patterns and jagged detail.

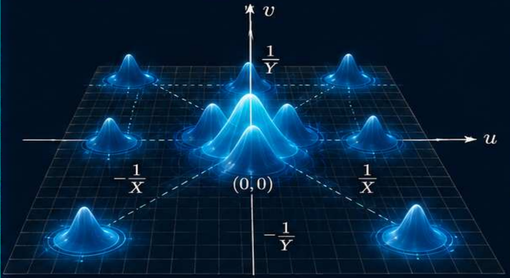
Aliasing in 2-D Frequency Space

(a) Over-sampled band-limited function



No overlap between replicas
(Perfect recovery possible)

(b) Under-sampled band-limited function



Strong overlap between replicas
(Aliasing unavoidable)

- **Over-sampling** keeps spectral replicas separate, enabling perfect recovery.
- **Under-sampling** causes overlap, making aliasing unavoidable.

Aliasing in Resampled Images

1. Original (No Aliasing)

- Full-resolution image.
- All spatial frequencies preserved.

2. Resized by Point Sampling (Aliasing / Moiré Visible)

- Downsampled using point sampling.
- High frequencies fold into lower frequencies → aliasing (moiré, jaggies).

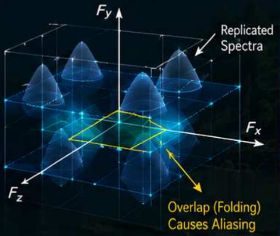
3. Prefiltered (Blurred) Before Resizing (No Aliasing)

- Low-pass filtered (prefiltered) to suppress high frequencies.
- Downsampled image is smooth and free of aliasing.

Why Aliasing Happens

- Sampling replicates the spectrum. When replicas overlap, high frequencies fold (alias) into the baseband.

3-D Frequency Domain Illustration



How to Avoid Aliasing

- Prefilter (low-pass filter) the image to suppress high frequencies before sampling (resizing). This prevents spectral overlap and eliminates aliasing.

Key Takeaway: Aliasing occurs when high frequencies exceed the new sampling rate and fold into the image. Prefiltering removes those high frequencies before sampling, producing clean, stable results.

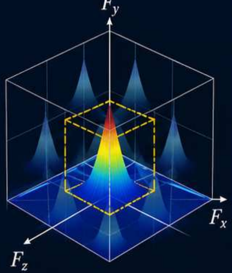
Aliasing in Natural Images

Original (fine detail)

Sampled (aliasing)

Fine detail above Nyquist
folds into false low-frequency structure.

3-D Spectrum
(ideal sampling)



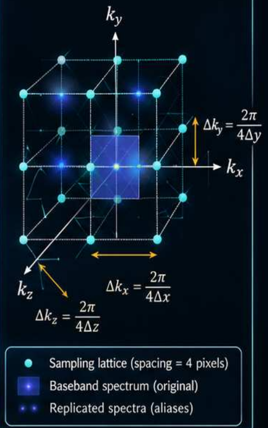
Simulation of Point Sampling

Sample every 4^{th} pixel in x and y, then upsize by pixel replication

1. High-Resolution Reference (Original)

2. Point-Sampled (every 4th pixel) + Pixel Replication

3. 3-D Sampling Lattice and Spectrum Replicas



Sampling every 4^{th} pixel in x and y keeps 1/16 of the samples.
Pixel replication (nearest-neighbor upsampling) produces blocky images.
In frequency, replicas appear every Δk_x, Δk_y, Δk_z causing aliasing if they overlap the baseband.

Example of Aliasing

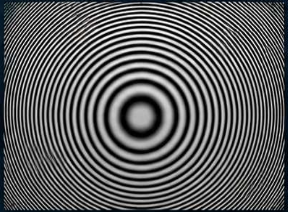
Visual aliasing
Occurs when a continuous image containing high spatial frequencies is sampled below the Nyquist rate.

Spurious spatial-frequency components
High-frequency components fold (alias) into the baseband of the sampled image.

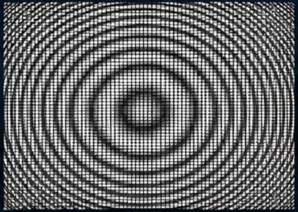
Low-frequency false patterns
Aliasing creates incorrect low-frequency structures that were not in the original.

Moiré patterns
Interaction of fine details with the sampling grid can produce strong moiré (beating) artifacts.

Original (continuous)
Fine radial pattern



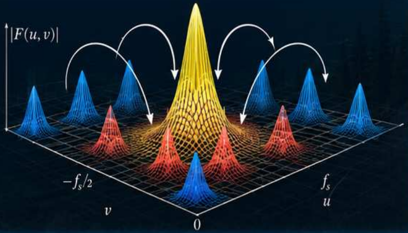
Sampled (aliased)
Lower sampling rate



High spatial frequencies become incorrect low-frequency patterns.

3-D Frequency View

- Original spectrum $|F(u,v)|$
- Sampling replicas
- Aliased (folded) components



High-frequency content (yellow) folds into the baseband (red), corrupting low frequencies.

Aliasing in Photography

Point sampling: The sensor measures radiance only at discrete grid points (pixels) and ignores everything between them.

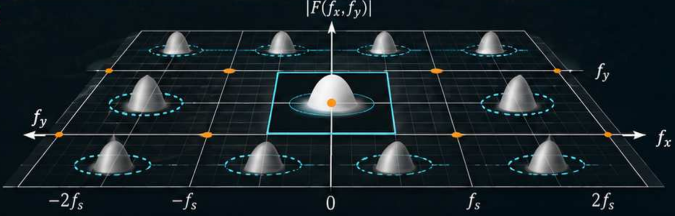
$$g[m,n] = f(x,y)\,\delta(x - m\Delta_x,\, y - n\Delta_y)$$

$$f_s = \frac{1}{\Delta_x}, \frac{1}{\Delta_y} \quad \text{(cycles per unit length)}$$

Spatial domain pipeline

Frequency domain view (2-D)

- Original spectrum $F(f_x, f_y)$
- Sampling replicas
- Baseband (captured)
- Nyquist region $|f_x| \le \frac{f_s}{2},\ |f_y| \le \frac{f_s}{2}$

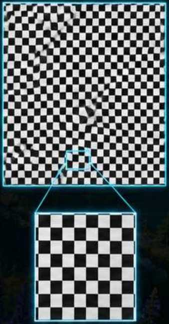


Aliasing:

If scene frequencies exceed $\pm\frac{f_s}{2}$, replicas overlap the baseband and fold into it, causing moiré and false detail.

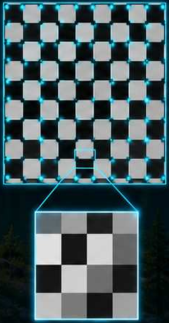
Aliasing from Undersampling

1. Fine Pattern (Original)



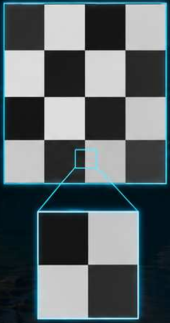
Contains high spatial frequencies

2. Sampled Pattern (Undersampled)



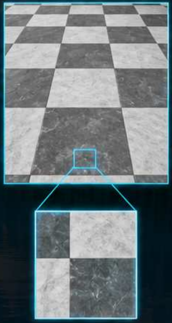
Too few samples to capture high-frequency detail

3. Aliased Result (Low-Frequency Pattern)



High frequencies fold into low frequencies (aliasing)

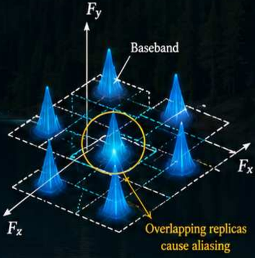
4. Apparently Normal Image (Incorrect)



Looks fine, but is the wrong image (lost detail)

Why Aliasing Occurs

Undersampling replicates the spectrum. High-frequency replicas overlap the baseband.



- - - Spectral replicas (due to sampling)

- - - Sampling region (Nyquist limit)

Sampling below the Nyquist rate causes high-frequency content to masquerade as low-frequency content. The result can look reasonable—but it is the wrong image.