Chapter 5: Fourier Spectral Analysis

Frequency representations, Fourier series, Fourier transforms, and key properties

Spatial Domain (Image)

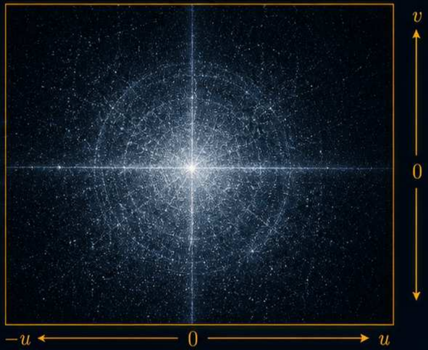
$f(x, y)$

2D Fourier Transform

$\mathcal{F}$

Frequency Domain (Magnitude Spectrum)

$|F(u, v)|$



$$F(u, v) = \iint_{-\infty}^{\infty} f(x, y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

Frequency Representations

A signal as a sum of sinusoidal components with different frequency, amplitude, and phase

$$x(t) = \sum_k A_k \cos(2\pi f_k t + \phi_k)$$

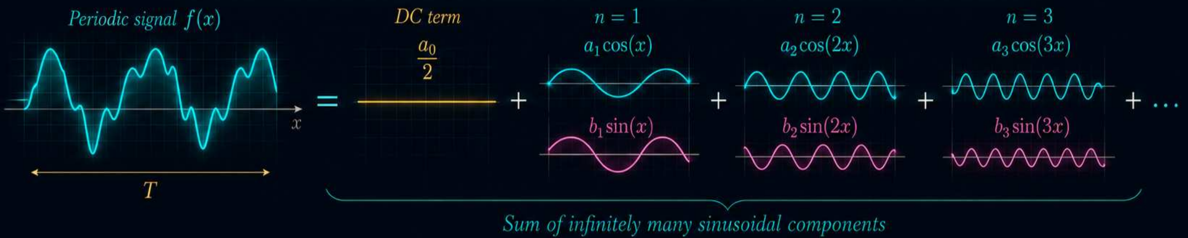
- *frequency* controls oscillation rate
- *amplitude* controls contribution strength
- *phase* controls horizontal shift

complex signals can be expressed as *weighted combinations* of *simple oscillatory components*.

Fourier Series

Any continuous, integrable, periodic function can be represented by an infinite series of sines and cosines.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$



Frequency determines wavelength.

Amplitude determines contribution.

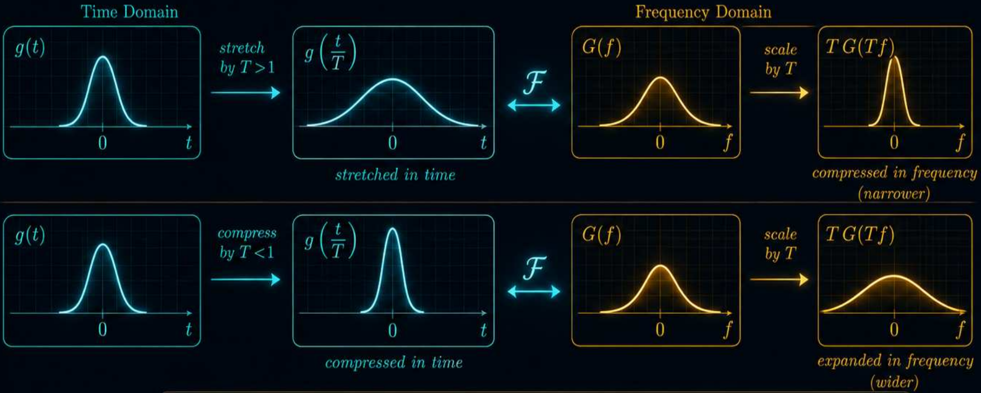
Phase can be modeled through sine/cosine combinations.

Infinite Series more terms give a more accurate approximation.

Fourier Transform Properties: Scaling

Scaling in time leads to inverse scaling in frequency.

$$g\left(\frac{t}{T}\right) \longleftrightarrow T\,G(Tf)$$



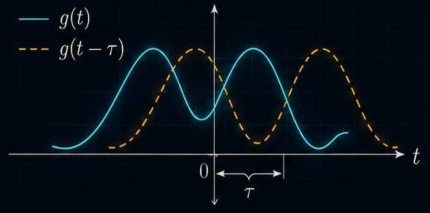
- time stretching narrows the spectrum
- time compression widens the spectrum
- the factor T preserves transform consistency

If the signal is scaled in time by T, its spectrum is scaled in frequency by $\frac{1}{T}$, and multiplied by T to *preserve* the total transform energy content.

Fourier Transform Properties: Shift

$$g(t-\tau) \longleftrightarrow G(f)e^{-j2\pi f\tau}$$

Time Domain

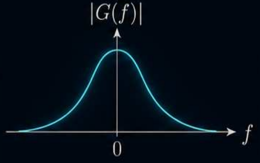


Shifting a signal to the right by τ seconds.

$\mathcal{F}$

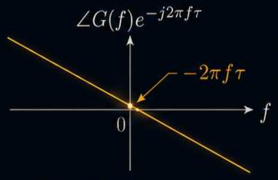
Frequency Domain

Magnitude *(unchanged)*



The magnitude spectrum remains the same.

Phase *(linear shift)*

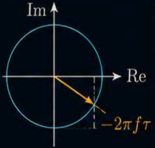


A linear phase term appears with slope $-2\pi\tau$.

A time shift introduces a linear phase factor in the frequency domain.

Complex exponential (unit magnitude)

$$e^{-j2\pi f\tau} = \underbrace{\cos(2\pi f\tau)}_{\text{real part}} - j\,\underbrace{\sin(2\pi f\tau)}_{\text{imag part}}$$



- shifting a signal in time does not change spectral magnitude
- the shift appears as a phase term in frequency
- phase encodes alignment information

Time shift in t $\longleftrightarrow$ Linear phase in f

Fourier Transform Properties: Duality

$$\boxed{G(t) \longleftrightarrow g(-f)}$$

Duality swaps the roles of time and frequency.

Key Takeaways

- duality swaps the roles of signal and transform
- many transform identities come in paired forms
- properties accelerate analysis and system design

Summary: Fourier Transform Properties

(1)	*Scaling*	$g\left(\frac{t}{T}\right)$	$\longleftrightarrow$	$T\,G(Tf)$
(2)	*Shift*	$g(t-\tau)$	$\longleftrightarrow$	$G(f)e^{-j2\pi f\tau}$
(3)	*Duality*	$G(t)$	$\longleftrightarrow$	$g(-f)$

Fourier Transform Pair Properties

(a)	*Scaling*	$g(t/T) \leftrightarrow TG(Tu)$
(b)	*Shift*	$g(t-T) \leftrightarrow G(u)e^{-j2\pi uT}$
(c)	*Duality*	$G(t) \leftrightarrow g(-u)$
(d)	*Linearity*	$ag(t) + bh(t) \leftrightarrow aG(u) + bH(u)$

Fourier Transform Pair Properties

(a) Scaling: $g(t/T) \leftrightarrow TG(Tu)$

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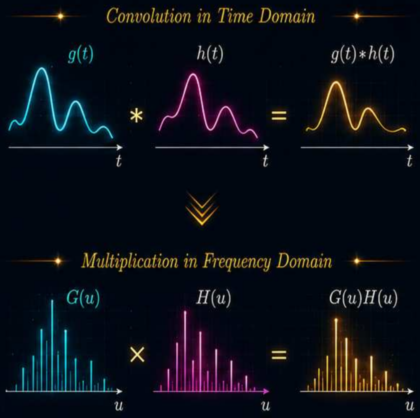
(c) Duality: $G(t) \leftrightarrow g(-u)$

(d) Linearity: $ag(t) + bh(t) \leftrightarrow aG(u) + bH(u)$

(e) Convolution: $g(t) * h(t) \leftrightarrow G(u)H(u)$

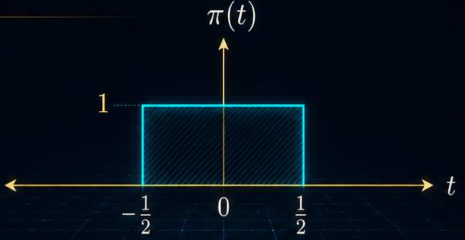


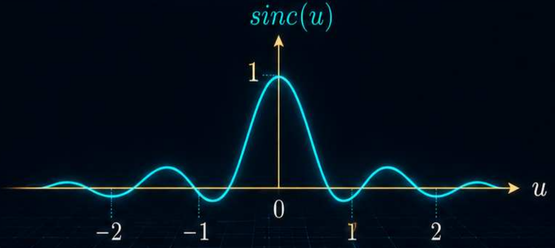
Common 1-D Transforms

Common 1-Dimensional transforms:

$$\pi(t) = rect(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$rect(t) \leftrightarrow sinc(u) = \frac{sin\pi u}{\pi u}$$





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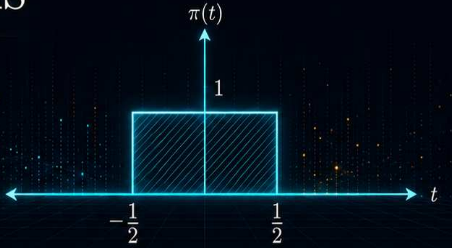

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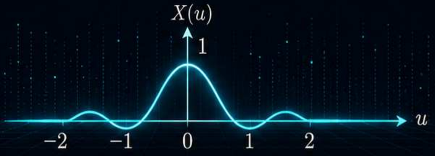
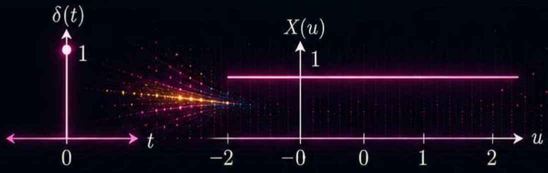


$$\delta(t) \longleftrightarrow 1$$



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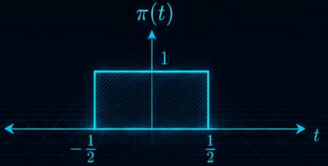
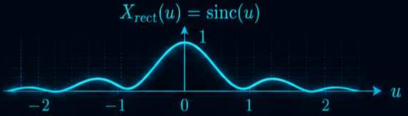
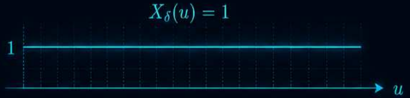
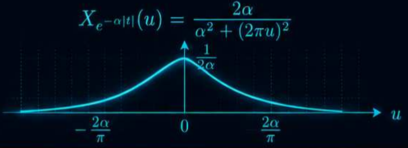
$$e^{-\alpha|t|},\ \alpha > 0 \quad \longleftrightarrow \quad \frac{2\alpha}{\alpha^2 + (2\pi u)^2}$$









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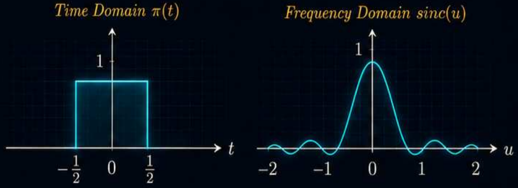
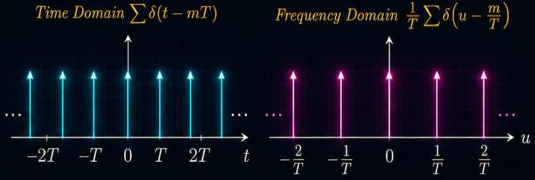
$$\sum_{m=-\infty}^{\infty} \delta(t - mT) \quad \leftrightarrow \quad \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta\left(u - \frac{m}{T}\right)$$

Rectangular Pulse



Comb Sampling



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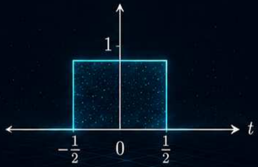
$$\cos(2\pi u_0 t) \longleftrightarrow \frac{1}{2}\left[\delta(u - u_0) + \delta(u + u_0)\right]$$

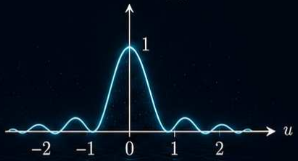
Time Domain

$\pi(t)$



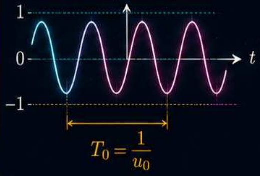
Frequency Domain

$sinc(u) = \frac{\sin \pi u}{\pi u}$



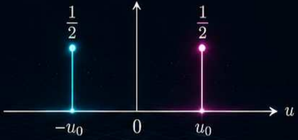
Time Domain

$\cos(2\pi u_0 t)$



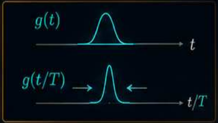
Frequency Domain

$\frac{1}{2}\left[\delta(u - u_0) + \delta(u + u_0)\right]$

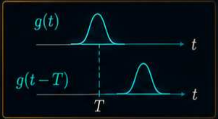


Fourier Transform Pair: Fundamental Properties

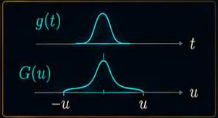
The Fourier transform pair satisfies several useful properties:



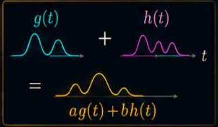
(a) Scaling: $$g(t/T) \leftrightarrow TG(uT)$$



(b) Shift: $$g(t-T) \leftrightarrow G(u)e^{-j2\pi uT}$$



(c) Duality: $$G(t) \leftrightarrow g(-u)$$



(d) Linearity: $$ag(t)+bh(t) \leftrightarrow aG(u)+bH(u)$$

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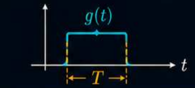
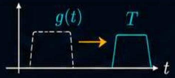
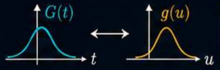
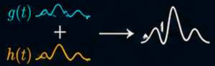
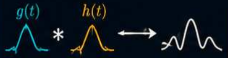
	(a) *Scaling*	$g(t/T) \leftrightarrow TG(uT)$
	(b) *Shift*	$g(t-T) \leftrightarrow G(u)e^{-j2\pi uT}$
	(c) *Duality*	$G(t) \leftrightarrow g(-u)$
	(d) *Linearity*	$ag(t) + bh(t) \leftrightarrow aG(u) + bH(u)$
	(e) *Convolution*	$g(t) * h(t) \leftrightarrow G(u)H(u)$

Common 1-D Transforms

- *Common 1-dimensional transforms:*

$$\pi(t) = rect(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

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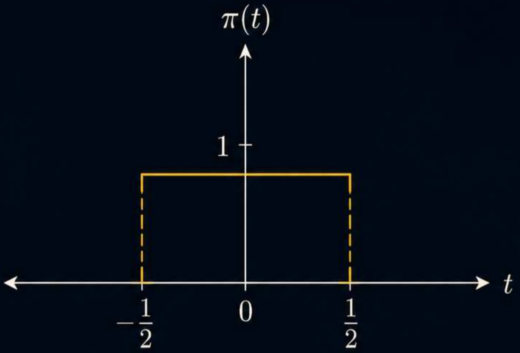
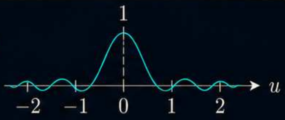


Transform is sinc(u)



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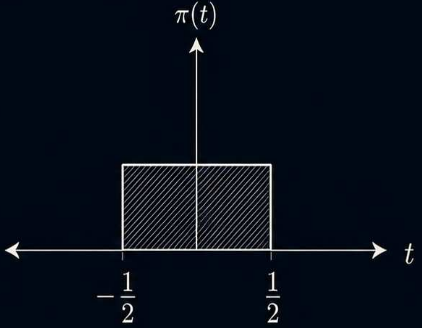

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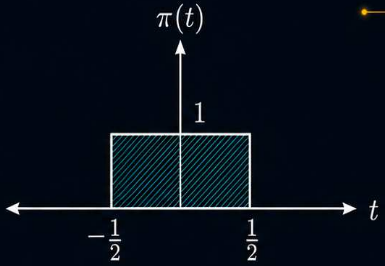

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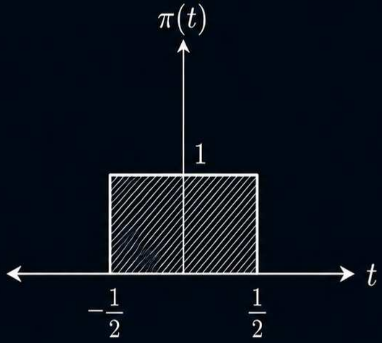

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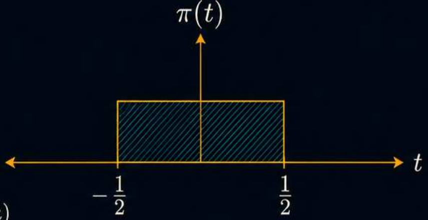

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Fourier Scaling Property

Compression in one domain produces expansion in the other

1D Scaling Theorem

If $x(t) \xleftrightarrow{\mathcal{F}} X(f)$, then

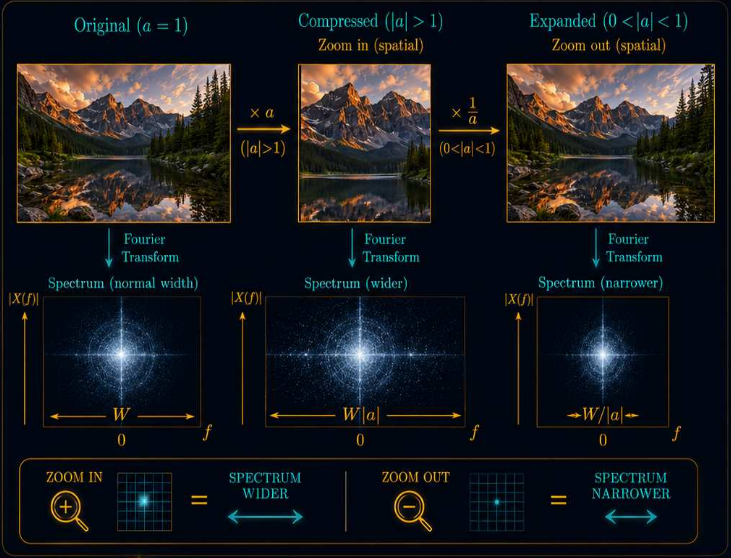
$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right), \quad a \neq 0.$$

2D Scaling Theorem (Image)

If $g(x, y) \xleftrightarrow{\mathcal{F}} G(u, v)$, then

$$g(ax, by) \xleftrightarrow{\mathcal{F}} \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right).$$

- Spatial compression ($|a| > 1$) *stretches the spectrum.*
- Spatial expansion ($0 < |a| < 1$) *compresses the spectrum.*
- The factor $1/|a|$ preserves total energy under the transform convention.



Proof of the Scaling Property

A direct change-of-variables derivation

$$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right).$$

1. $$\mathcal{F}\{x(a\,t)\}(f) = \int_{-\infty}^{\infty} x(a\,t)\,e^{-j2\pi f t}\,dt$$

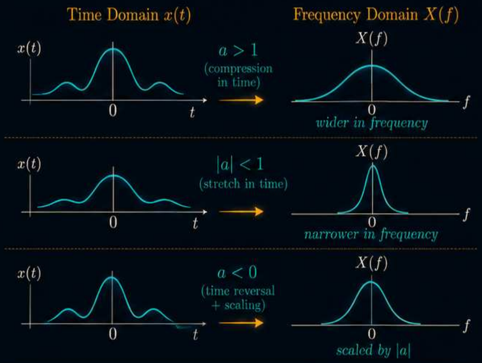
2. $$\text{Let}\quad u = a\,t, \qquad t = \frac{u}{a}, \qquad dt = \frac{du}{a}.$$

3. For $a > 0$: $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{a}\int_{-\infty}^{\infty} x(u)\,e^{-j2\pi (f/a) u}\,du$$

4. $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{a}\,X\left(\frac{f}{a}\right), \qquad a > 0$$

5. For $a < 0$, limits reverse, giving: $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right).$$

6. $$\boxed{\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right)}$$



Change of variables rescales both the frequency axis and the amplitude.

Fourier Convolution Theorem

Filtering in one domain becomes multiplication in the other

Convolution Theorem (1D)

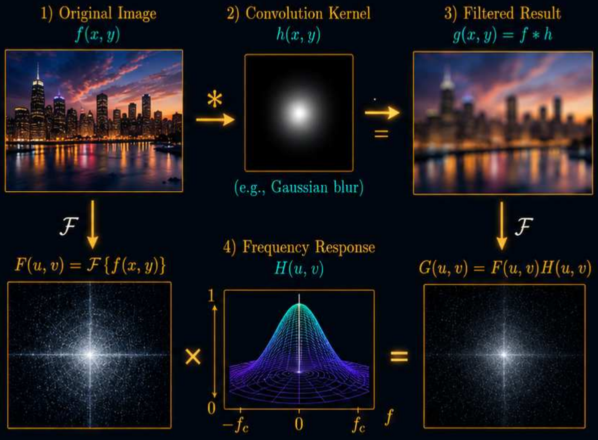
$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$\mathcal{F}\{x * h\}(f) = X(f)H(f).$$

Dual (Multiplication) Property

$$\mathcal{F}\{x(t)h(t)\}(f) = X(f) * H(f).$$

- *Convolution* in time or space corresponds to *multiplication* in frequency.
- A *smoothing* kernel *attenuates* high frequencies.
- A *sharpening* or *edge* kernel *emphasizes* high frequencies.



2D Convolution Theorem

$$g(x, y) * h(x, y) \xleftrightarrow{\mathcal{F}} G(u, v)H(u, v).$$

Proof of the Convolution Theorem

Deriving multiplication in the frequency domain

Definitions: $(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ $\qquad$ $\mathcal{F}\{x * h\}(f) = \int_{-\infty}^{\infty} (x * h)(t) e^{-j2\pi f t}\, dt$

1 $$\mathcal{F}\{x * h\}(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j2\pi f t}\, dt$$

2 $$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi f t}\, dt \right] d\tau$$

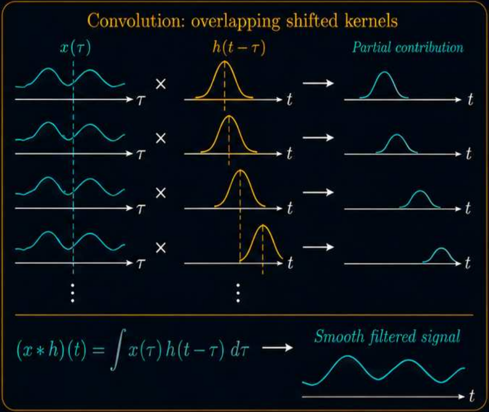
3 Let $u = t - \tau, \qquad t = u + \tau, \qquad dt = du.$

4 $$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f (u+\tau)}\, du \right] d\tau$$

5 $$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau \int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f u}\, du$$

6 $$= X(f)\, H(f)$$

7 $$\boxed{\mathcal{F}\{x * h\}(f) = X(f) H(f)}$$



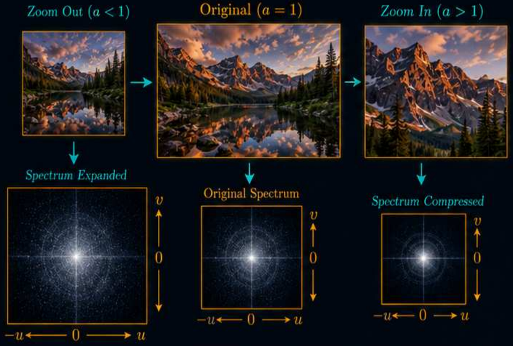
★ *Note: Interchanging the order of integration is justified under standard integrability assumptions.*

Why Scaling and Convolution Matter in Imaging

From zooming to filtering

Scaling / Zooming

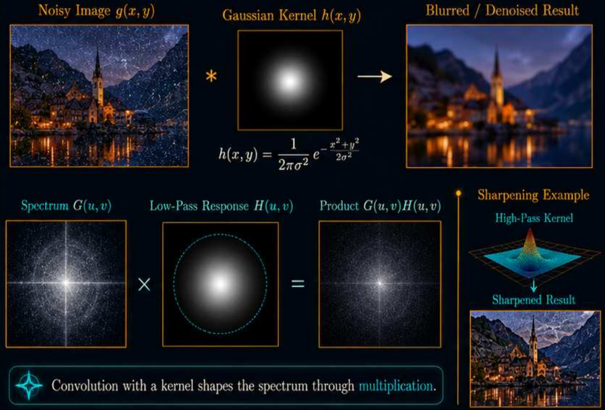
$$g(ax, ay) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|^2} G\left(\frac{u}{a}, \frac{v}{a}\right).$$



Magnification in space compresses the spectrum; minification expands it.

Filtering / Convolution

$$g(x, y) * h(x, y) \xleftrightarrow{\mathcal{F}} G(u, v)H(u, v).$$



Convolution with a kernel shapes the spectrum through multiplication.

1. Scaling explains how zoom affects frequency content.
2. Convolution explains filtering, blur, and sharpening.
3. These two properties are central to Fourier-based image analysis.

Fourier Scaling Property

Compression in one domain produces expansion in the other

1D Scaling Theorem

If $x(t) \xleftrightarrow{\mathcal{F}} X(f)$, then

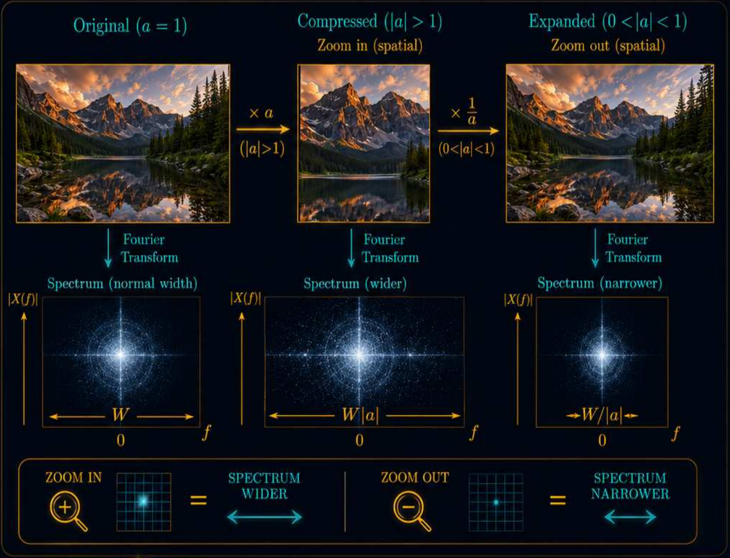
$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right), \quad a \neq 0.$$

2D Scaling Theorem (Image)

If $g(x, y) \xleftrightarrow{\mathcal{F}} G(u, v)$, then

$$g(ax, by) \xleftrightarrow{\mathcal{F}} \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right).$$

- Spatial compression ($|a| > 1$) *stretches the spectrum.*
- Spatial expansion ($0 < |a| < 1$) *compresses the spectrum.*
- The factor $1/|a|$ preserves total energy under the transform convention.



Proof of the Scaling Property

A direct change-of-variables derivation

$$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right).$$

1. $$\mathcal{F}\{x(a\,t)\}(f) = \int_{-\infty}^{\infty} x(a\,t)\,e^{-j2\pi f t}\,dt$$

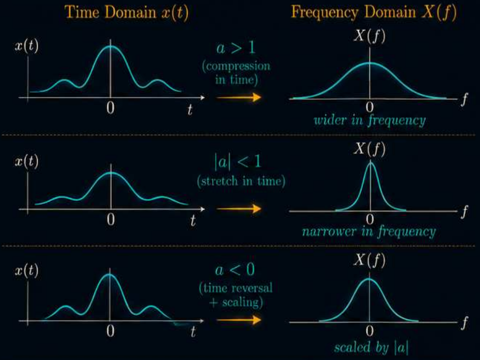
2. Let $u = a\,t, \quad t = \dfrac{u}{a}, \quad dt = \dfrac{du}{a}.$

3. For $a > 0$: $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{a}\int_{-\infty}^{\infty} x(u)\,e^{-j2\pi(f/a)u}\,du$$

4. $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{a}\,X\left(\frac{f}{a}\right), \qquad a > 0$$

5. For $a < 0$, limits reverse, giving: $$\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right).$$

6. $$\boxed{\mathcal{F}\{x(a\,t)\}(f) = \frac{1}{|a|}\,X\left(\frac{f}{a}\right)}$$



Change of variables rescales both the frequency axis and the amplitude.

Fourier Convolution Theorem

Filtering in one domain becomes multiplication in the other

Convolution Theorem (1D)

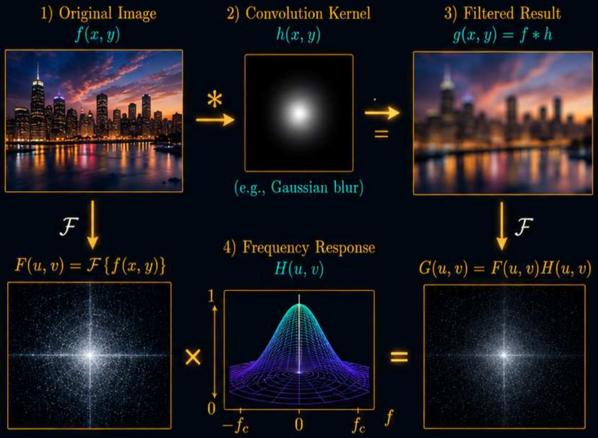
$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$\mathcal{F}\{x * h\}(f) = X(f)H(f).$$

Dual (Multiplication) Property

$$\mathcal{F}\{x(t)h(t)\}(f) = X(f) * H(f).$$

- *Convolution* in time or space corresponds to *multiplication* in frequency.
- A *smoothing* kernel *attenuates* high frequencies.
- A *sharpening* or *edge* kernel *emphasizes* high frequencies.



2D Convolution Theorem

$$g(x, y) * h(x, y) \xleftrightarrow{\mathcal{F}} G(u, v)H(u, v).$$

Proof of the Convolution Theorem

Deriving multiplication in the frequency domain

Definitions: $(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ $\qquad \mathcal{F}\{x * h\}(f) = \int_{-\infty}^{\infty} (x * h)(t) e^{-j2\pi f t}\, dt$

1. $\mathcal{F}\{x * h\}(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j2\pi f t}\, dt$

2. $= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi f t}\, dt \right] d\tau$

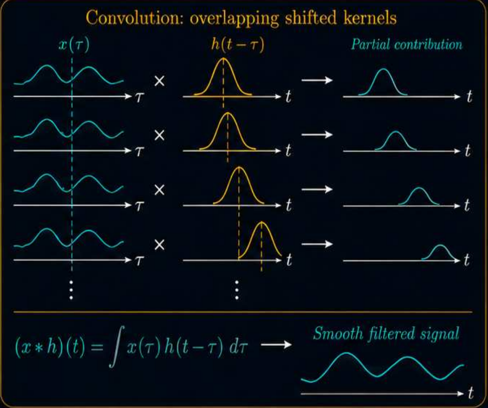
3. Let $u = t - \tau, \quad t = u + \tau, \quad dt = du.$

4. $= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f (u+\tau)}\, du \right] d\tau$

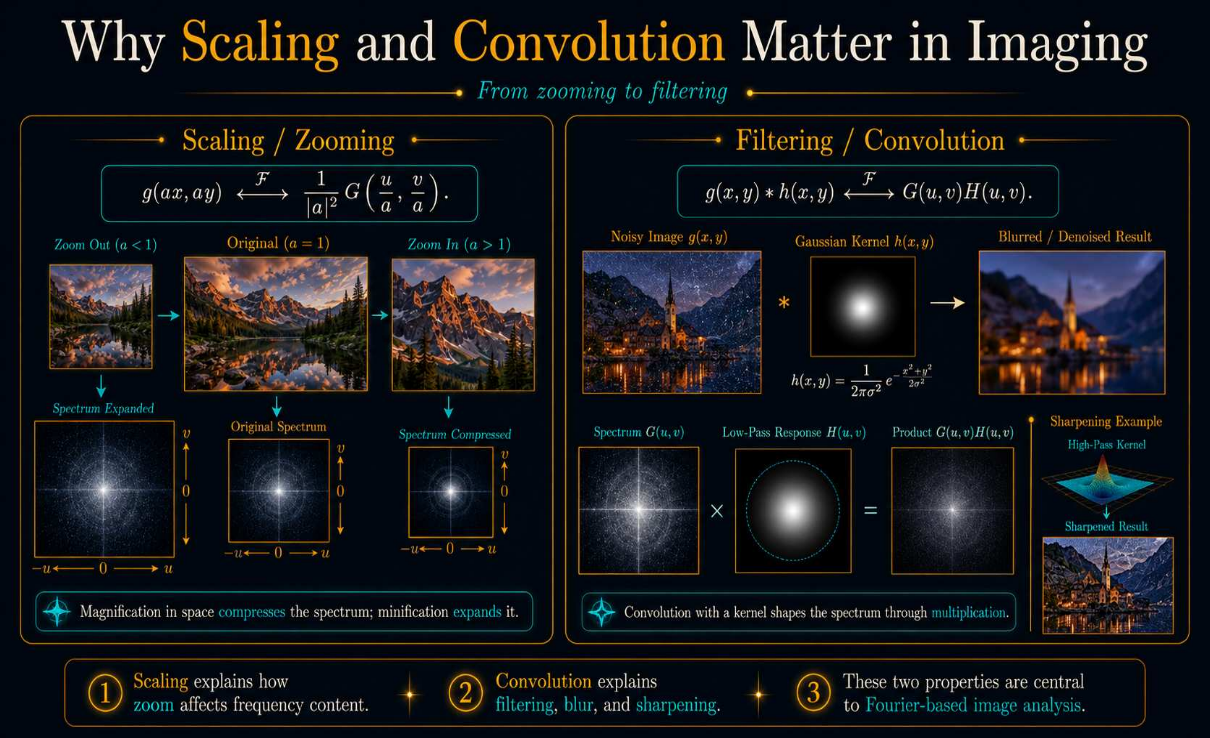
5. $= \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau \int_{-\infty}^{\infty} h(u)\, e^{-j2\pi f u}\, du$

6. $= X(f)\, H(f)$

7. $$\boxed{\mathcal{F}\{x * h\}(f) = X(f) H(f)}$$



★ *Note: Interchanging the order of integration is justified under standard integrability assumptions.*



Why Scaling and Convolution Matter in Imaging
From zooming to filtering
Scaling / Zooming
$g(ax, ay) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|^2} G\left(\frac{u}{a}, \frac{v}{a}\right).$
Zoom Out ($a < 1$)
Original ($a = 1$)
Zoom In ($a > 1$)
Spectrum Expanded
Original Spectrum
Spectrum Compressed
Magnification in space compresses the spectrum; minification expands it.
Filtering / Convolution
$g(x, y) * h(x, y) \overset{\mathcal{F}}{\longleftrightarrow} G(u, v)H(u, v).$
Noisy Image $g(x, y)$
Gaussian Kernel $h(x, y)$
Blurred / Denoised Result
$h(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$
Spectrum $G(u, v)$
Low-Pass Response $H(u, v)$
Product $G(u, v)H(u, v)$
Sharpening Example
High-Pass Kernel
Sharpened Result
Convolution with a kernel shapes the spectrum through multiplication.
1 Scaling explains how zoom affects frequency content.
2 Convolution explains filtering, blur, and sharpening.
3 These two properties are central to Fourier-based image analysis.

Example: Periodic Replication

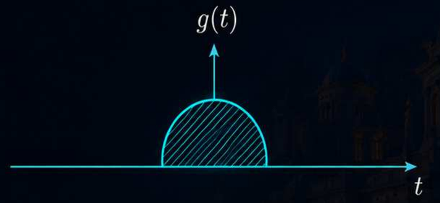
- We know:

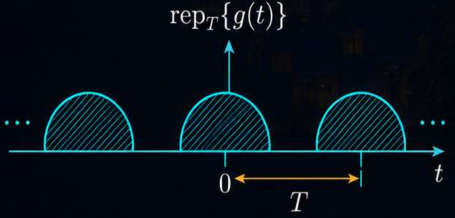
$$\sum_{n=-\infty}^{\infty} \delta(t-n) \longleftrightarrow \frac{1}{T}\sum_{m} \delta\left(u - \frac{m}{T}\right)$$

- Now consider:

$$\mathrm{rep}_T\{g(t)\} = \sum_{m} g(t - mT)$$

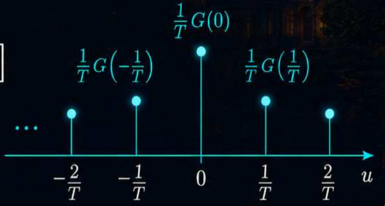
$$\mathrm{rep}_T\{g(t)\} = g(t) * \sum_{m} \delta(t - mT)$$





Fourier Transform of the Repeated Signal

$$
\begin{aligned}
F\{\mathrm{rep}_T(g(t))\} &= F\Big\{g(t) * \sum_m \delta(t - mT)\Big\} \\
&= G(u)\frac{1}{T}\sum_m \delta\left(u - \frac{m}{T}\right) \\
&= \frac{1}{T}\sum_m G\left(\frac{m}{T}\right)\delta\left(u - \frac{m}{T}\right) \\
&= \frac{1}{T}\,\mathrm{comb}_{\frac{1}{T}}[G(u)]
\end{aligned}
$$



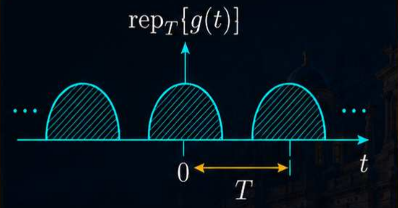
Periodic Replication and Spectral Sampling

$$\mathcal{F}\{\mathrm{rep}_T\{g(t)\}\} = \frac{1}{T}\sum_{m=-\infty}^{\infty} G\left(\frac{m}{T}\right)\delta\left(u-\frac{m}{T}\right)$$

Fourier-Series Interpretation

Hence:

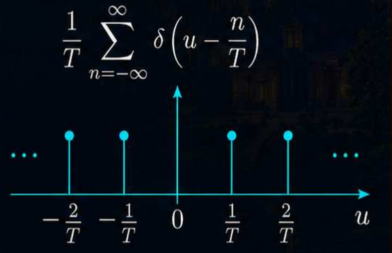
$$\mathrm{rep}_T[g(t)] = g(t) * \frac{1}{T}\sum_n e^{j2\pi n u_0 t}$$



Taking Fourier Transforms:

$$F\{\mathrm{rep}_T[g(t)]\} = G(u)\cdot\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(u-\frac{n}{T}\right)$$

$$= \frac{1}{T}\sum_n G(u)\,\delta\left(u-\frac{n}{T}\right)$$



Final Form of the Replicated-Signal Spectrum

- Hence:

$$\mathrm{rep}_T[g(t)] = g(t) * \frac{1}{T}\sum_n e^{j2\pi n u_0 t}$$

- Taking Fourier Transforms:

$$\begin{aligned}
F\{\mathrm{rep}_T[g(t)]\} &= G(u)\cdot\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(u-\frac{n}{T}\right)\\
&= \frac{1}{T}\sum_n G(u)\delta\left(u-\frac{n}{T}\right)\\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{T}\right)\delta\left(u-\frac{n}{T}\right)\\
&= \boxed{\frac{1}{T}\,comb_{\frac{1}{T}}[G(u)]}
\end{aligned}$$

Replicated-Signal Spectrum

$F\{\mathrm{rep}_T[g(t)]\}$

A line spectrum with spacing $\frac{1}{T}$, with weights $\frac{1}{T}G\left(\frac{n}{T}\right)$ at $u = \frac{n}{T}$.

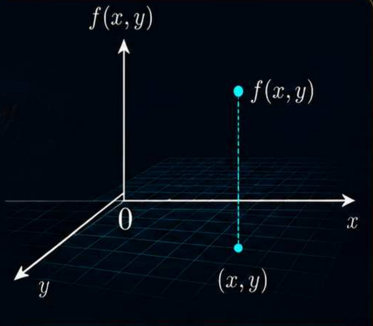
2-Dimensional Systems

In 2 dimensions, the input and output of a system are functions of two independent variables. In image processing, the variables are the spatial coordinates (x, y) and the value of the function is the intensity of the image at that point.

- A 2-D system is described by a function of two variables.

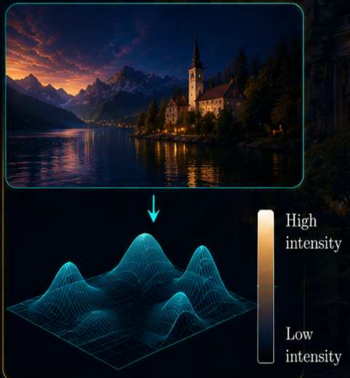
$$f : \mathbb{R}^2 \to \mathbb{R}$$

- For images, $f(x, y)$ gives the intensity (brightness) at the spatial location (x, y).
- Here, x and y are spatial coordinates (horizontal and vertical positions).



The height $f(x, y)$ represents the intensity of the image at spatial location (x, y).

Example: Image Intensity Function



Thus, a 2-D system maps each point in the plane (x, y) to a value $f(x, y)$ — the image intensity at that point.

Impulse Response and 2-D Convolution

✧ Impulse response characterizes a linear shift-invariant 2D system ✧

$$\delta(x,y) \longrightarrow \boxed{\text{System}} \longrightarrow h(x,y)$$

$$\ell(x,y) \longrightarrow \boxed{\text{System}} \longrightarrow g(x,y)$$

✧ Input–output relation

$$\ell(x,y) \rightarrow \boxed{\text{System}} \rightarrow g(x,y) = h(x,y) * \ell(x,y)$$

✧ 2-D convolution

$$g(x,y) = \iint_{-\infty}^{\infty} h(x-\sigma, y-\beta)\,\ell(\sigma,\beta)\, d\sigma\, d\beta$$

Input image

$\ell(x,y)$

$*$

Kernel $h(x,y)$

1	1	1
1	–8	1
1	1	1

$=$

Output image $g(x,y)$

$g(x,y)$

2-D Sinusoidal Basis Functions

- Any 2-D image can be synthesized by superimposing infinitely many sinusoidal components of the form

$$\cos 2\pi(ux + vy)$$

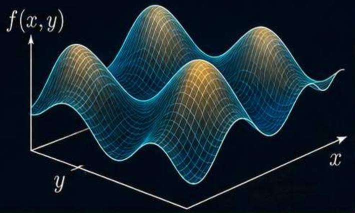
- Lines of constant amplitude (constant phase) satisfy

$$2\pi(ux + vy) = k$$

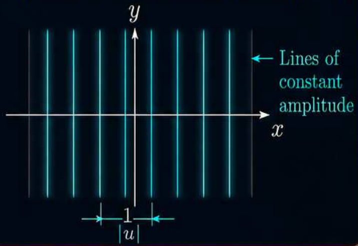
- Example

$$v = 0 \quad \text{(vertical frequency is zero)} \quad \Rightarrow \quad \cos 2\pi(ux + 0)$$

1) 2-D Sinusoidal Surface



2) Lines of Constant Amplitude ($v = 0$)



General Pattern of $\cos 2\pi(ux + vy)$

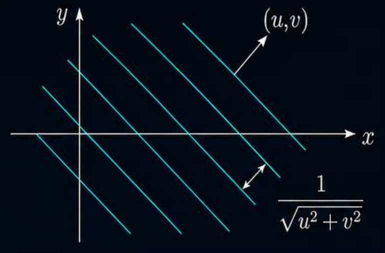
- In general, the basis function

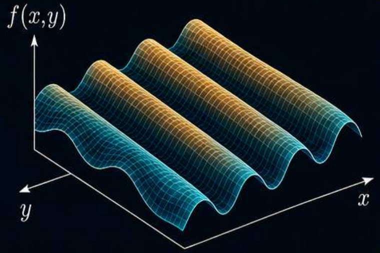
$$\cos 2\pi(ux + vy)$$

- creates oriented stripes whose constant-phase lines satisfy

$$ux + vy = c$$

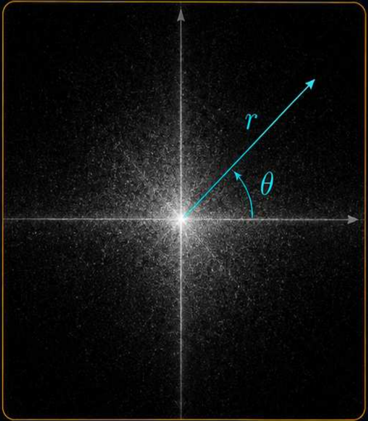
- Spacing $= \dfrac{1}{\sqrt{u^2 + v^2}}$





Fourier Transforms of Images

image



spectrum

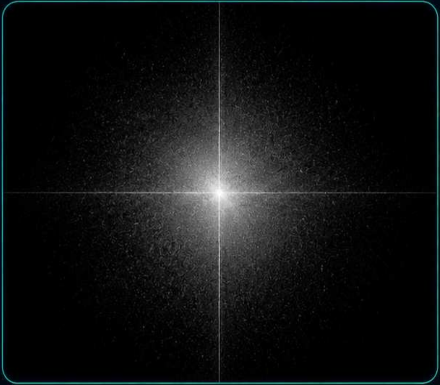
- θ gives angle of sinusoid
- r gives spatial frequency
- brightness gives amplitude of sinusoid present in image
- complete spectrum is two images — sines and cosines

```
fourier = fftshift(fft2(image))
```

A Typical Photograph

image

spectrum

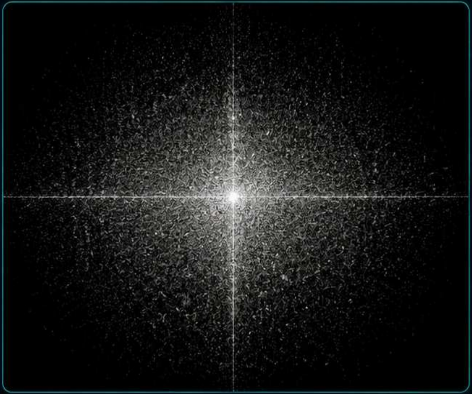


✧ Most image energy is concentrated near low spatial frequencies around the origin.

An Image with Higher Frequencies

image

spectrum

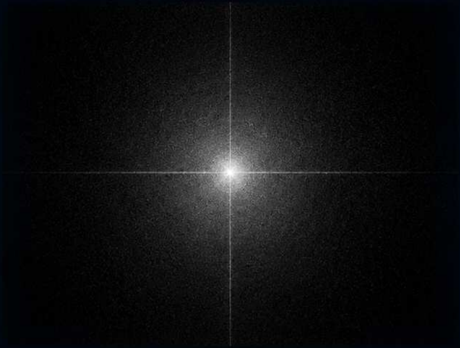


Fine detail and rapid intensity variations produce broader high-frequency content.

Blurring in the Fourier Domain

image

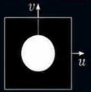
spectrum



Blurring suppresses high frequencies and preserves mainly low-frequency content.

Low-pass (ideal) filter

$H(u,v) =$

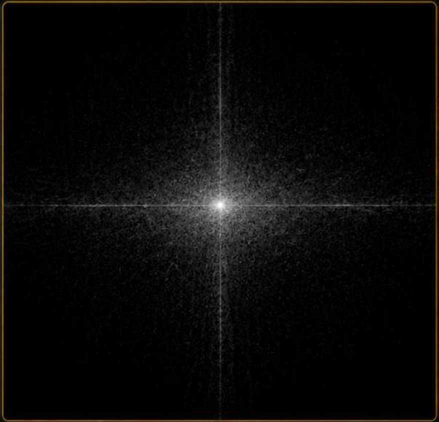


Passes low frequencies (center) and removes high frequencies (outside).

Original Flower

image

spectrum

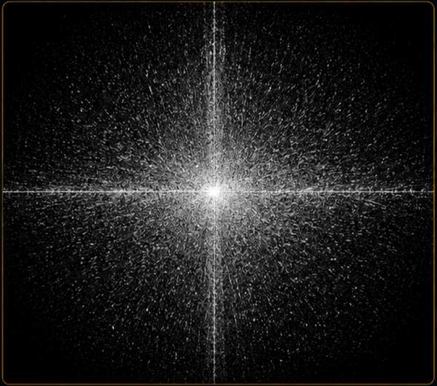


Reference image before frequency-domain filtering.

Sharpening in the Fourier Domain

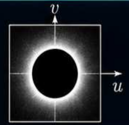
✧ image ✧

✧ spectrum ✧



Sharpening emphasizes high frequencies and enhances edges and detail.

$$H(u, v) =$$

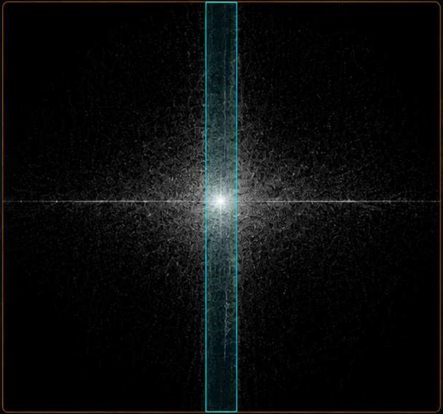


High-pass (ideal) or high-boost filter.
Passes high frequencies (outside); attenuates or boosts them.

What Does this Filtering Operation Do?

image

spectrum

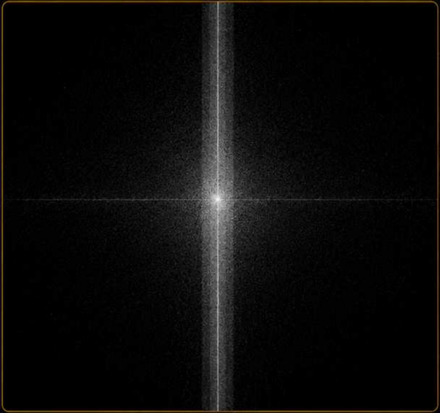


What visual effect results from retaining only a narrow vertical band of frequencies?

What Does this Filtering Operation Do?

image

spectrum

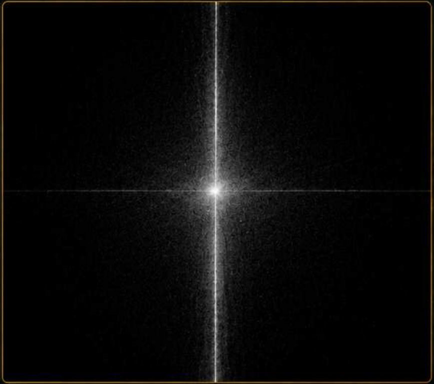


Predict the image produced by this directional spectral selection.

Blurring in x, sharpening in y

image

spectrum



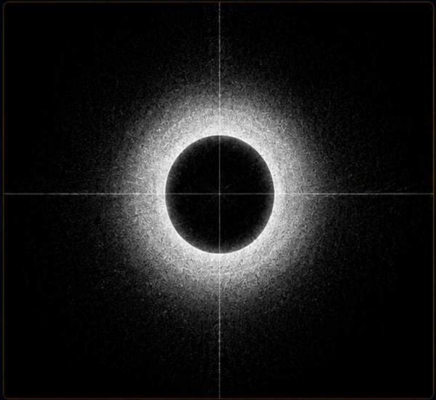
argh, astigmatism!

A vertical passband preserves variation in y while suppressing high frequencies in x.

Filtering – Band-pass Filter

Filtered Image (Band-pass Output)

Frequency-Domain Band-pass Mask

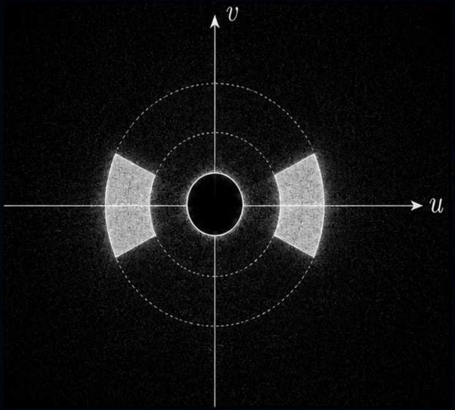


Band-pass filtering retains only a selected range of spatial frequencies.

Filtering – Oriented Band-pass Filter

Spatial-domain result (vertical edges emphasized)

Fourier-domain mask (oriented annular sector)



Frequencies in a narrow angular band around the horizontal u-axis (left and right annular sectors) are passed. Consequently, vertical-oriented edges and structures are retained, while horizontal details are largely suppressed.

Example: Fourier Transform of rect$[x, y]$

- Consider the separable 2-D rectangle:

$$\text{rect}[x, y] = \text{rect}(x)\,\text{rect}(y)$$

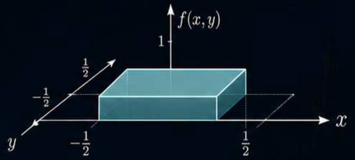
- Using separability of the Fourier transform:

$$G(u, v) = \int \text{rect}(x) e^{-j2\pi ux}\, dx \int \text{rect}(y) e^{-j2\pi vy}\, dy$$

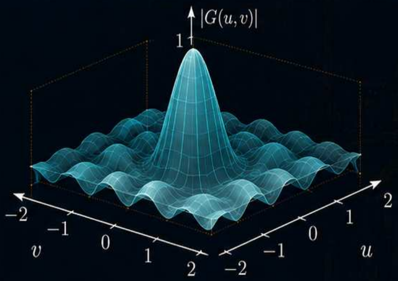
- Result:

$$\boxed{G(u, v) = \text{sinc}(u)\,\text{sinc}(v)}$$

Spatial domain: rect$[x, y]$



Frequency domain: $|G(u,v)| = |\text{sinc}(u)\,\text{sinc}(v)|$

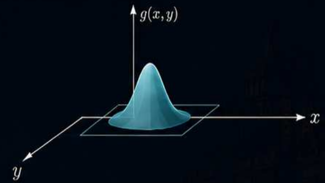


Sampling with $\mathrm{comb}_{XY}[g(x,y)]$

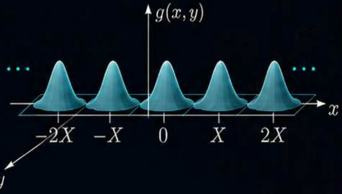
- $$\mathrm{comb}_{XY}[g(x,y)] = \sum_m \sum_n g(mX, nY)\,\delta(x-mX, y-nY)$$
$$= g(x,y)\sum_m \sum_n \delta(x-mX, y-nY)$$

- Some notation:
  - $$\mathrm{rep}_{X,\infty}[g(x,y)] = \sum_k g(x-kX, y)$$
  - $$\mathrm{rep}_{\infty,Y}[g(x,y)] = \sum_k g(x, y-kY)$$

Original $g(x,y)$



Replication along x (using $\mathrm{rep}_{X,\infty}$)



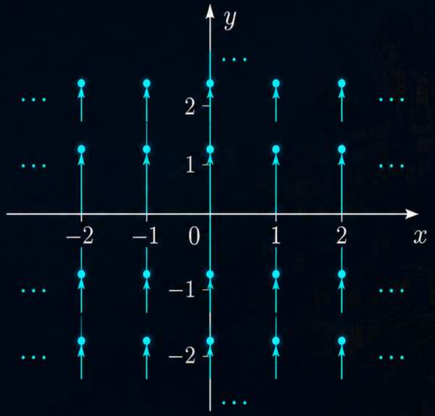
Example: Take F.T. of $comb[x,y]$

✧ $$comb[x,y] = \sum_m \sum_n \delta(x-m,\ y-n)$$

$$= \sum_m \sum_n \delta(x-m)\,\delta(y-n)$$

✧ 2-D impulse lattice in the spatial domain.

✧ Spatial domain: $comb[x,y]$ ✧

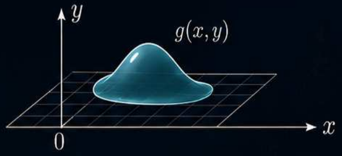


2-D Replication Operator

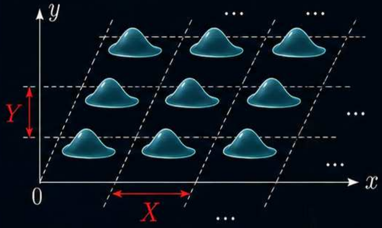
$$\mathrm{rep}_{XY}\left[g(x,y)\right] = \sum_m \sum_n g(x - mX, y - nY)$$

- Replicates a 2-D pattern on a lattice with spacings X and Y.
- Each copy is shifted by integer multiples of X and Y.

Single 2-D function $g(x,y)$



Replicated pattern $\mathrm{rep}_{XY}\left[g(x,y)\right]$



Common 2-Dimensional Functions

1) 2-D Rectangular Function (rect)

$$\mathrm{rect}[x, y] = \mathrm{rect}(x)\,\mathrm{rect}(y) = f(x, y) = \begin{cases} 1, & |x|, |y| < \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

2) 2-D Sinc Function (sinc)

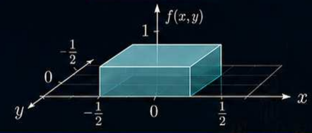
$$\mathrm{sinc}[x, y] = \mathrm{sinc}(x)\,\mathrm{sinc}(y) = \frac{\sin \pi x}{\pi x} \cdot \frac{\sin \pi y}{\pi y}$$

3) 2-D Comb Function (comb)

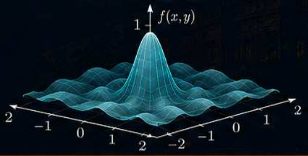
$$\mathrm{comb}[x, y] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)$$

$$\delta(x - m, y - n) = \delta(x - m)\,\delta(y - n)$$

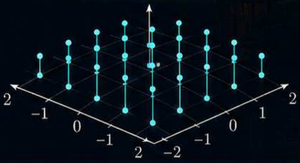
rect$[x, y]$



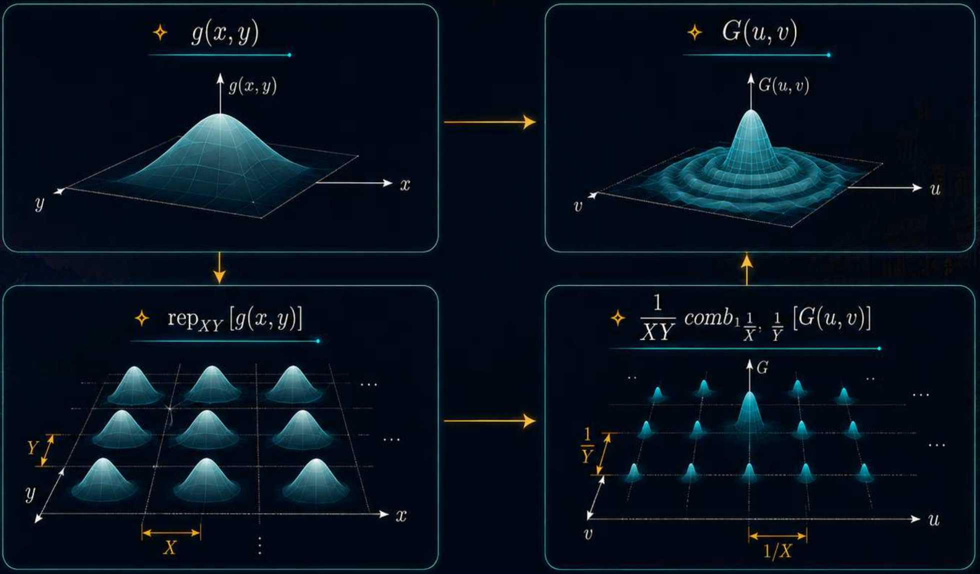
sinc$[x, y]$



comb$[x, y]$



Spatial Replication and Spectral Sampling



Fourier Transform of comb$[x, y]$

Derivation:

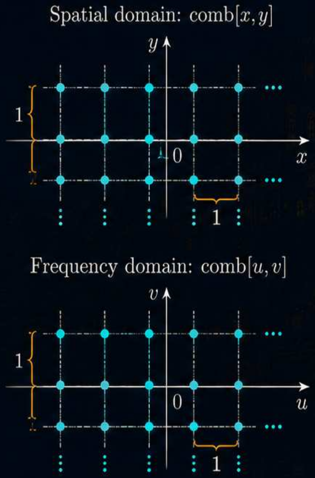
$$\mathcal{F}\{\mathrm{comb}[x,y]\} = \mathcal{F}\left\{\sum_m \delta(x-m)\sum_n \delta(y-n)\right\}$$
$$= \sum_m \delta(u-m)\sum_n \delta(v-n)$$
$$= comb\,[u,v]$$

Result:

$$comb\,[x,y] \longleftrightarrow comb\,[u,v]$$

$$comb_{XY}\,[x,y] \longleftrightarrow \frac{1}{XY}\,comb_{\frac{1}{X},\frac{1}{Y}}\,[u,v]$$

Dual lattices



Fourier Transform of a Replicated 2-D Function

- Replication in space:

$$\mathrm{rep}_{XY}[g(x,y)] = g(x,y) * comb_{XY}[x,y]$$

- Taking Fourier transform:

$$\therefore\ \mathcal{F}\{\mathrm{rep}\} = G(u,v) \cdot \frac{1}{XY} \sum_{m,n} \delta\left(u - \frac{m}{X}, v - \frac{n}{Y}\right)$$

- Using sifting property of impulse:

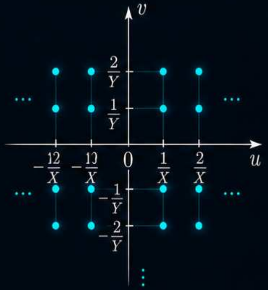
$$= \frac{1}{XY} \sum_{m,n} G\left(\frac{m}{X}, \frac{n}{Y}\right) \delta\left(u - \frac{m}{X}, v - \frac{n}{Y}\right)$$

- Final form:

$$\boxed{= \frac{1}{XY}\, comb_{\frac{1}{X}, \frac{1}{Y}}\, [G(u,v)]}$$

Spectral interpretation

Spatial replication corresponds to spectral sampling.

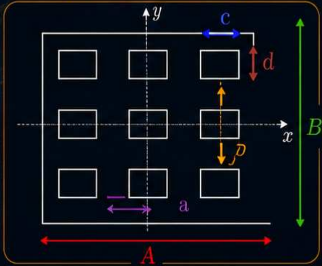


3-D Geometry of the Block Pattern

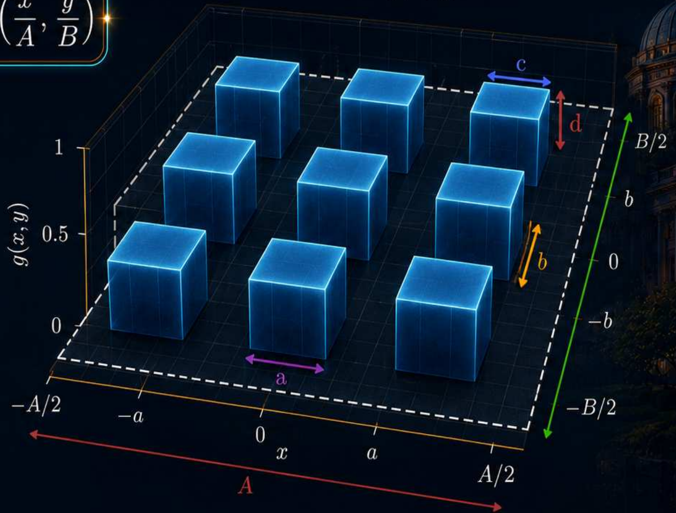
$$g(x,y) = \mathrm{rep}_{a,b}\left[\mathrm{rect}\left(\frac{x}{c},\frac{y}{d}\right)\right]\mathrm{rect}\left(\frac{x}{A},\frac{y}{B}\right)$$

- Outer support window: $A \times B$
- Repetition periods: a along x, b along y
- Individual block size: c along x, d along y

Top view (parameter map)



3-D view of $g(x,y)$



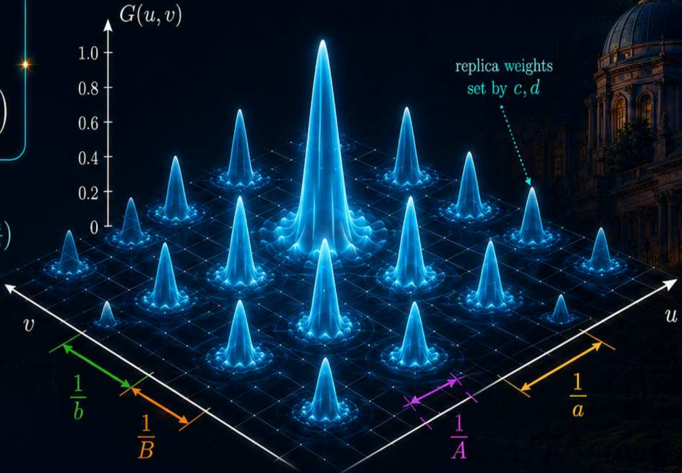
3-D Spectrum of the Block Pattern

$$G(u,v) = \frac{ABcd}{ab} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{cm}{a}, \frac{dn}{b}\right) \operatorname{sinc}\left(A\left(u - \frac{m}{a}\right), B\left(v - \frac{n}{b}\right)\right)$$

overall amplitude scale $\frac{ABcd}{ab}$

replica weights set by $\operatorname{sinc}\left(\frac{cm}{a}, \frac{dn}{b}\right)$

2-D sinc lobe centered at $\left(\frac{m}{a}, \frac{n}{b}\right)$



Parameter Interpretation

- Replica spacing: $\frac{1}{a}$ in u, $\frac{1}{b}$ in v
- Lobe width (main-lobe): $\sim\frac{1}{A}$ in u, $\sim\frac{1}{B}$ in v
- Replica weights: $\operatorname{sinc}\left(\frac{cm}{a}, \frac{dn}{b}\right)$
- Overall scale factor: $\frac{ABcd}{ab}$

- Spectrum replicas are located at $(m/a, n/b)$, shaped by sinc lobes and weighted by $\operatorname{sinc}(cm/a,\, dn/b)$.

Two-Dimensional Convolution Example

- Let $f_1(x,y)$ and $f_2(x,y)$ be as shown below, where the shaded area represents a value of 1, else 0.

- a. In the first case,

$$f_1(x,y) = \mathrm{rect}\left(\frac{x}{A}, \frac{y}{A}\right)$$

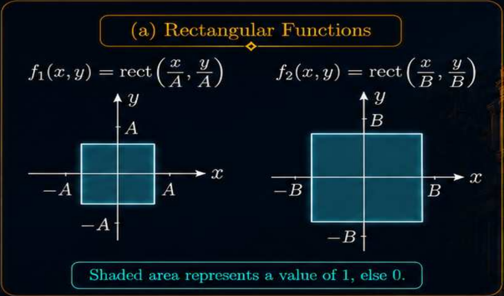
and $f_2(x,y) = \mathrm{rect}\left(\frac{x}{B}, \frac{y}{B}\right)$,

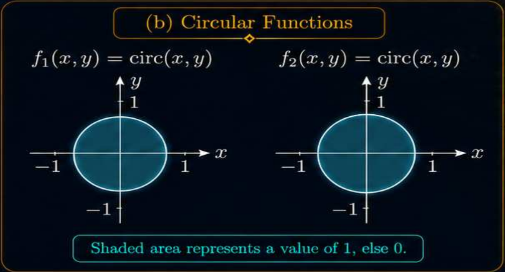
where $A < B$.

- b. In the second case,

$$f_1(x,y) = f_2(x,y) = \mathrm{circ}(x,y).$$

- Convolve the two sets of functions and plot the results.





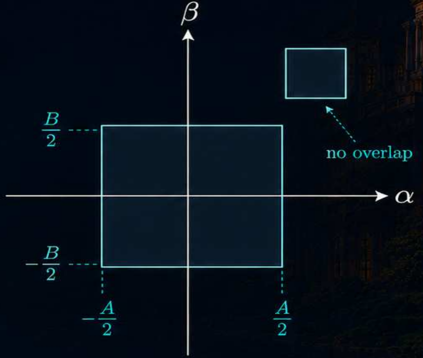
2-D Convolution Setup

- The $2-D$ convolution is given by

$$g(x,y)=\int\int f_1(x-\alpha,y-\beta)f_2(\alpha,\beta)d\alpha d\beta$$

- for $|y|,|x|>\frac{A+B}{2}, \quad g(x,y)=0.$

(α,β) – plane



Convolution of Two Rectangles — Central and Side Regions

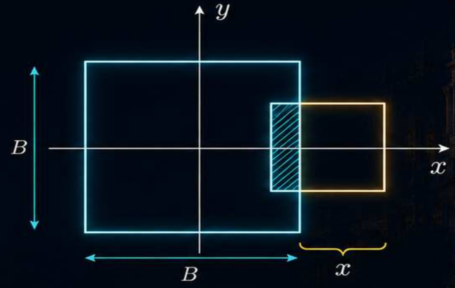
1. for $|y|, |x| < \frac{B-A}{2}$,

$$g(x,y) = A^2.$$

2. for $\frac{B-A}{2} < |x| < \frac{A+B}{2}$,

$|y| < \frac{B-A}{2}$,

$$g(x,y) = A\left[\frac{B}{2} - \left(x - \frac{A}{2}\right)\right].$$



Center region ($|x|, |y| < \frac{B-A}{2}$): full overlap gives constant A^2.

Side regions ($\frac{B-A}{2} < |x| < \frac{A+B}{2}$, $|y| < \frac{B-A}{2}$): overlap decreases linearly with $|x|$.

Convolution of Two Rectangles — Top and Corner Regions

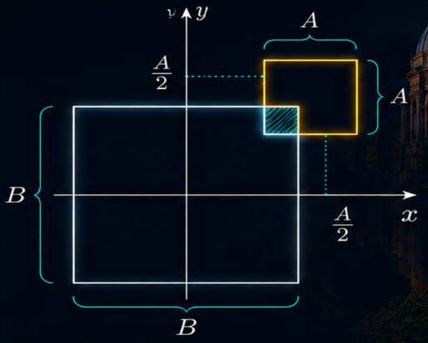
1 for $\dfrac{B-A}{2} < |y| < \dfrac{A+B}{2}$,

$$|x| < \frac{B-A}{2},$$

$$g(x,y) = A\left[\frac{B}{2} - \left(y - \frac{A}{2}\right)\right].$$

2 for $\dfrac{B-A}{2} < |x|,\ |y| < \dfrac{A+B}{2}$,

$$g(x,y) = \left[\frac{B}{2} - \left(x - \frac{A}{2}\right)\right]\left[\frac{B}{2} - \left(y - \frac{A}{2}\right)\right].$$



- Top-edge case: the overlap length in y decreases linearly with y.
- Corner region: overlap equals the product of two linear factors in x and y.

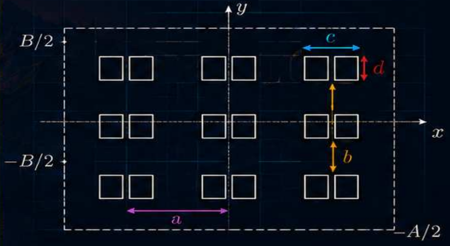
Example

Final worked example: a replicated pair of shifted rectangles convolved with a large support window.

✧ We derive $G(u,v)$ and then visualize its 3-D spectrum.

1) Spatial-domain view: $g(x,y)$

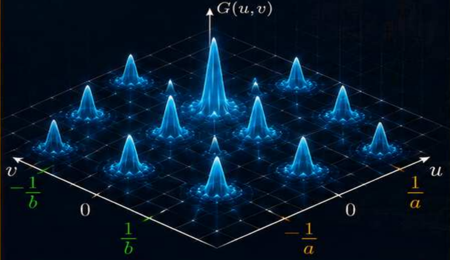
Replicated pair of shifted rectangles inside a large support window $A \times B$.



Offsets (c,d) inside each block; replication periods (a,b); window $A \times B$.

2) Spectral preview: $G(u,v)$

3-D spectrum of the replicated pattern: a lattice of weighted sinc-like lobes.



Replica spacing: $\frac{1}{a}$ in u, $\frac{1}{b}$ in v; shaped by sinc lobes $\mathrm{sinc}(cm/a,\ dn/b)$.

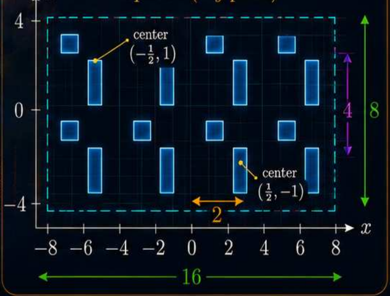
3-D Spatial Plot of the Final Example

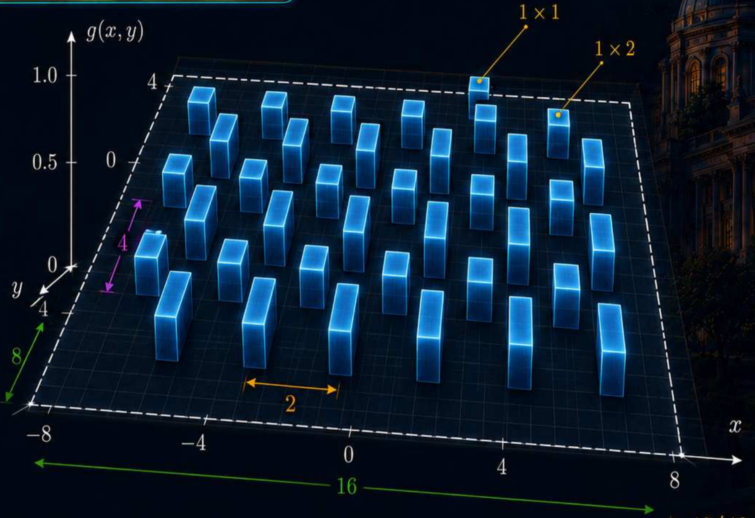
$$g(x,y) = \mathrm{rep}_{2,4}\left[\mathrm{rect}\left(\frac{x+1/2}{1}, \frac{y-1}{1}\right) + \mathrm{rect}\left(\frac{x-1/2}{1}, \frac{y+1}{2}\right)\right] * \mathrm{rect}\left(\frac{x}{16}, \frac{y}{8}\right)$$

Parameter Interpretation

- periods: 2 in x, 4 in y
- support window: 16×8
- heights: 1 on the blocks, 0 elsewhere

Top view (x-y plane)





Final Example — Analytical Spectrum

1. Spatial-domain definition:

$$g(x,y) = \mathrm{rep}_{2,4}\left[\mathrm{rect}\left(\frac{x+1/2}{1}, \frac{y-1}{1}\right) + \mathrm{rect}\left(\frac{x-1/2}{1}, \frac{y+1}{2}\right)\right] * \mathrm{rect}\left(\frac{x}{16}, \frac{y}{8}\right)$$

replication lattice 1/2 by 1/4

window envelope 16 by 8

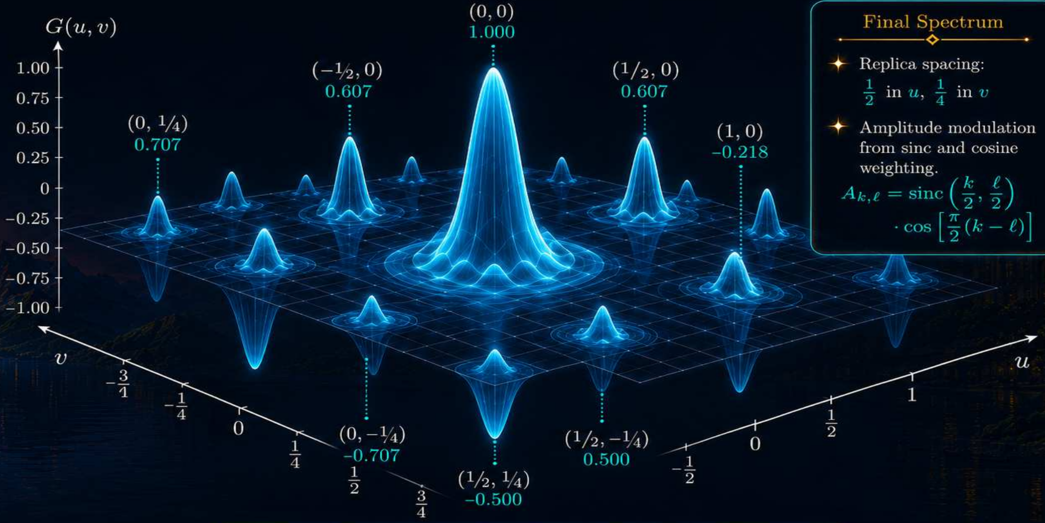
2. Frequency-domain expression:

$$G(u,v) = \frac{1}{8}\,\mathrm{comb}_{\frac{1}{2},\frac{1}{4}}\left[2\mathrm{sinc}(u, 2v)\left(e^{-j2\pi(-\frac{u}{2}+v)} + e^{j2\pi(-\frac{u}{2}+v)}\right)\right] * 2\mathrm{sinc}(16u, 8v)$$

3. Expanded form:

$$G(u,v) = 24\sum_k \sum_\ell \mathrm{sinc}\left(\frac{k}{2}, \frac{\ell}{2}\right)\cos\left[\pi\left(\frac{k-\ell}{2}\right)\right]\mathrm{sinc}\left[16\left(u-\frac{k}{2}\right), 8\left(v-\frac{\ell}{4}\right)\right]$$

Spectrum of the Final Example



Final Spectrum

- Replica spacing: $\frac{1}{2}$ in u, $\frac{1}{4}$ in v
- Amplitude modulation from sinc and cosine weighting.

$$A_{k,\ell} = \operatorname{sinc}\left(\frac{k}{2}, \frac{\ell}{2}\right) \cdot \cos\left[\frac{\pi}{2}(k - \ell)\right]$$

Replicas centered at $u = \frac{k}{2}$, $v = \frac{\ell}{4}$ for integers k, ℓ.