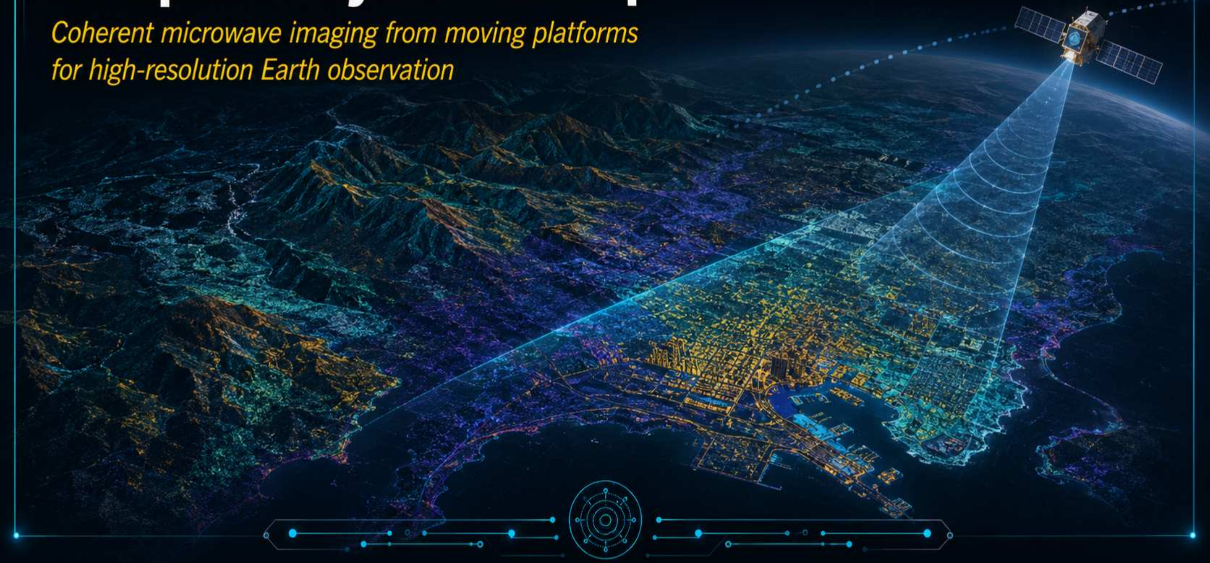
ELEG404/604 Imaging & Deep Learning

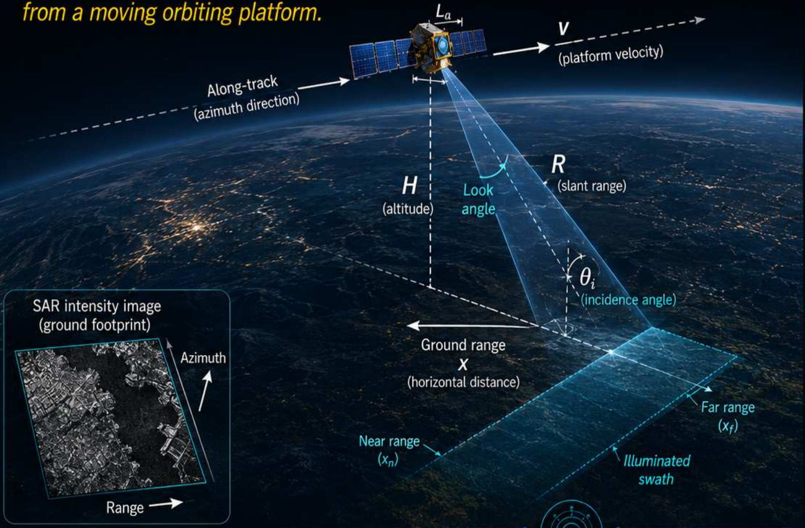
Chapter: Synthetic Aperture Radar

Coherent microwave imaging from moving platforms for high-resolution Earth observation



1. Satellite SAR Sensing Geometry

Side-looking coherent radar measures distance, Doppler, and phase from a moving orbiting platform.



GEOMETRY & SIGNAL MODEL

$$R(\eta) = \sqrt{R_0^2 + (v\eta)^2}$$

η = slow time (azimuth time)

τ = fast time (range time)

$t_d = \dfrac{2R}{c}$ (round-trip delay)

$z(\tau, \eta)$ = received complex echo

KEY PROPERTIES

Side-looking geometry
Antenna illuminates the Earth to the side of the flight path.

Coherent measurement
Measures amplitude and phase of the scattered signal.

Day/Night, All-Weather Imaging
Microwaves penetrate clouds and operate independent of sunlight.

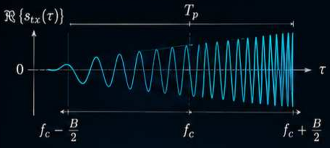
INTUITIVE NOTES

- Range is measured from round-trip delay.
- Azimuth is synthesized from platform motion.
- Phase preserves sub-wavelength path information.

From Transmitted Chirp to Received Echo

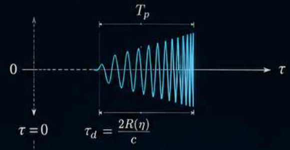
A linear-FM pulse and its delayed return encode range and reflectivity.

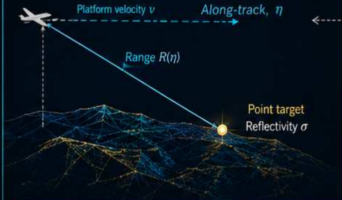
1) TRANSMITTED CHIRP $s_{tx}(\tau)$



f_c Carrier frequency
T_p Pulse duration
B Chirp bandwidth
$K_r = \frac{B}{T_p}$ Chirp rate [Hz/s]

2) RECEIVED ECHO $s_{rx}(\tau, \eta)$

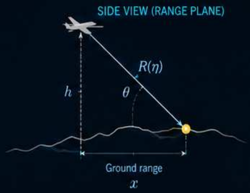




Round-trip delay

$$\tau_d = \frac{2R(\eta)}{c}$$

c	Speed of light
$R(\eta)$	Slant range
τ	Fast time (range time)
η	Slow time (azimuth time)



MATHEMATICAL MODEL

Transmitted linear-FM (LFM) chirp

$$s_{tx}(\tau) = \mathrm{rect}\left(\frac{\tau}{T_p}\right) \exp\left\{j2\pi\left(f_c\tau + \frac{K_r}{2}\tau^2\right)\right\}$$

Received echo from a point target at range $R(\eta)$

$$s_{rx}(\tau,\eta) = \sigma\, w_a(\eta)\, \mathrm{rect}\left(\frac{\tau - \frac{2R(\eta)}{c}}{T_p}\right) \times \exp\left\{j\pi K_r\left(\tau - \frac{2R(\eta)}{c}\right)^2\right\} \exp\left(-j\frac{4\pi R(\eta)}{\lambda}\right)$$

After demodulation (to baseband)

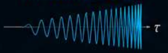
$$s_{bb}(\tau,\eta) = A(\eta)\exp\{j\phi(\tau,\eta)\}$$

Definitions

σ Target reflectivity (complex)
$\lambda = \frac{c}{f_c}$ Wavelength
$w_a(\eta)$ Antenna (azimuth) weighting / pattern
c Speed of light
$\mathrm{rect}(x) = 1$ for $|x| \le \frac{1}{2}$, 0 otherwise

VISUAL INTUITION: CHIRP → DELAY → ECHO → MATCHED FILTER

TRANSMITTED CHIRP

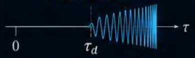


Linear-FM pulse sweeps band B in T_p.

PROPAGATION & DELAY

Round-trip delay $\tau_d = \frac{2R}{c}$ encodes range $R(\eta)$.

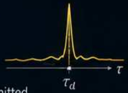
RECEIVED ECHO



A delayed and phase-shifted copy carries range and reflectivity.

MATCHED FILTER (PULSE COMPRESSION)

$$s_{rx}(\tau,\eta) \rightarrow \otimes \rightarrow h(\tau) = s_{tx}^*(-\tau)$$



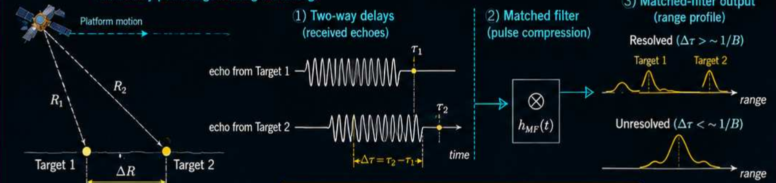
Correlate the received chirp with the transmitted chirp to compress energy and localize targets in range.

Chapter: Range Resolution and Bandwidth

Wide bandwidth, not short pulse duration alone, determines how finely SAR separates targets in range.

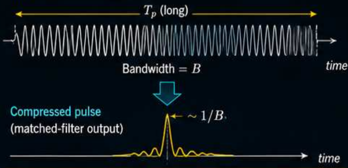
Key Idea
Bigger bandwidth (B)
→ shorter compressed pulse
→ better range resolution

Two nearby point targets along slant range



Two targets are resolvable in range if $\Delta\tau \gtrsim \frac{1}{B}$ (approximately)

Pulse Compression: Long Energy, Fine Resolution



Pulse compression lets us transmit long (high-energy) pulses while achieving fine range resolution.

Key Equations

Range resolution (slant range) $$\rho_r = \frac{c}{2B}$$

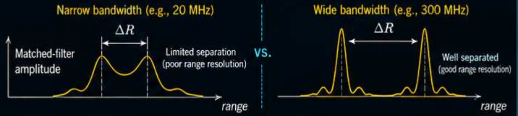
Delay resolution $$\Delta\tau \approx \frac{1}{B}$$

Ground-range resolution $$\rho_{gr} = \frac{c}{2B\sin(\theta_i)}$$

Chirp rate (linear FM) $$K_r = \frac{B}{T_p}$$

- c = speed of light $\approx 3 \times 10^8$ m/s
- B = transmitted bandwidth (Hz)
- T_p = chirp (pulse) duration (s)
- K_r = chirp rate (Hz/s)
- $\Delta\tau$ = two-way delay difference (s)
- ρ_r = slant-range resolution (m)
- ρ_{gr} = ground-range resolution (m)
- θ_i = incidence angle from vertical

Effect of Bandwidth on Compressed Peaks



Numerical Example (Slant-Range Resolution)

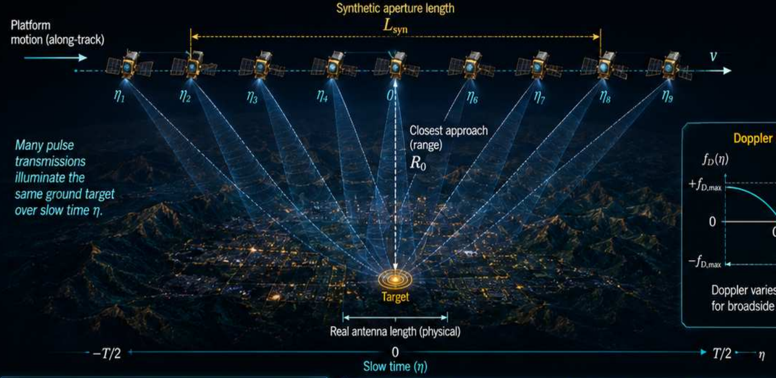
Bandwidth B	Delay Resolution $\Delta\tau \approx 1/B$	Slant-Range Resolution $\rho_r = c/(2B)$	Assume $\theta_i = 30°$ Ground-Range Resolution $\rho_{gr} = c/(2B\sin\theta_i)$
20 MHz	50 ns	7.50 m	15.0 m
100 MHz	10 ns	1.50 m	3.00 m
300 MHz	3.33 ns	0.50 m	1.00 m

Takeaway
Increasing bandwidth B
→ decreases $\Delta\tau \approx 1/B$
→ narrows compressed pulse
→ improves range resolution
Linearly with $1/B$.

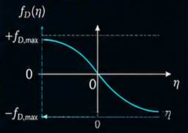
Practical Note
In real systems, factors like windowing, sidelobes, and SNR influence the usable resolution, but bandwidth remains the dominant driver.

Synthetic Aperture and Azimuth Resolution

Repeated pulses along the flight path synthesize a long aperture and sharpen along-track detail.



Doppler History



Doppler varies linearly with η for broadside stripmap SAR.

Geometry & Signal Model

- Slant range history (broadside, small angle)

$$R(\eta) \approx R_0 + \frac{v^2\eta^2}{2R_0}$$

- Phase history

$$\phi(\eta) = -\frac{4\pi}{\lambda} R(\eta)$$

- Doppler frequency (instantaneous)

$$f_D(\eta) = \frac{1}{2\pi}\frac{d\phi(\eta)}{d\eta} \approx -\frac{2v^2}{\lambda R_0}\eta$$

- Synthetic aperture length

$$L_{\mathrm{syn}} \approx \frac{\lambda R_0}{L_a}$$

- Azimuth (along-track) resolution (stripmap, broadside)

$$\rho_{az} \approx \frac{\lambda R_0}{2L_{\mathrm{syn}}} \approx \frac{L_a}{2}$$

Why SAR Achieves Finer Azimuth Resolution

- Each pulse sees the target from a slightly different position.
- The target's phase history vs. slow time (η) contains Doppler information that encodes along-track position.
- Coherent processing of the full aperture compresses energy into a narrow peak, improving resolution by $\sim L_{\mathrm{syn}}/L_a$.
- Typically, $L_{\mathrm{syn}} \gg L_a \Rightarrow \rho_{az} \ll$ antenna beam footprint.

Real Aperture Blur vs. Synthetic Aperture Focus

Real Aperture (single pulse)

Beamwidth footprint (large along-track blur)

Synthetic Aperture (coherent processing)

Azimuth resolution $\sim \frac{L_a}{2}$ (sharp along-track detail)

Key Takeaways

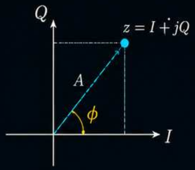
- Synthetic aperture length L_{syn} is much longer than the real antenna L_a.
- Doppler history provides the "beat note" that lets us localize targets along-track.
- Azimuth resolution: $\rho_{az} \approx \frac{L_a}{2}$ for stripmap at broadside.
- SAR forms a narrow focus from many looks — turning motion into resolution.

Slide 5: Complex Phase Measurements in SAR

SAR records both amplitude and phase, enabling coherent imaging and exquisite sensitivity to path length.

1) Complex Sample and Phasors

I/Q (Complex) Plane



$$z = I + jQ = Ae^{j\phi}$$

$$A = |z|$$

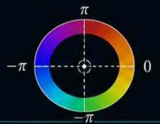
$$\phi = \angle z$$

$$\phi = -\frac{4\pi R}{\lambda} + \phi_s$$

$$\Delta\phi = -\frac{4\pi \Delta R}{\lambda}$$

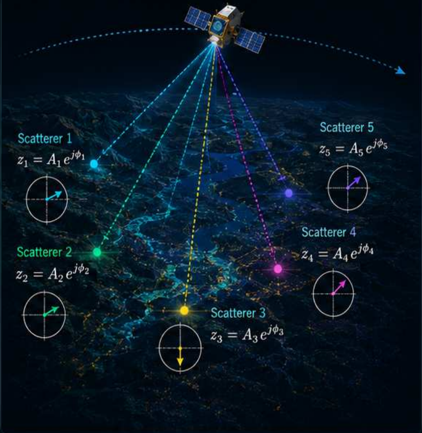
$$\Delta R = -\frac{\lambda\,\Delta\phi}{4\pi}$$

Wrapped Phase ($-\pi$ to π)



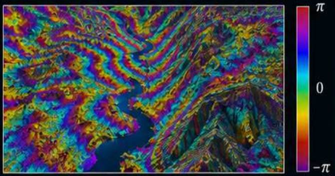
- A: backscatter amplitude (magnitude of the complex return)
- ϕ_s: scattering phase (depends on target properties and geometry)
- R: slant range (two-way path length / 2)
- λ: radar wavelength

2) Returns from Scene Scatterers

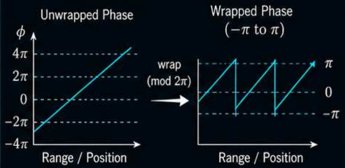


3) Phase Images and Wrapping

Wrapped Phase Image ($-\pi$ to π)



Phase Wrapping (modulo 2π)



Amplitude forms the intensity image.
Bright pixels = strong backscatter.
Used for detection and visualization.

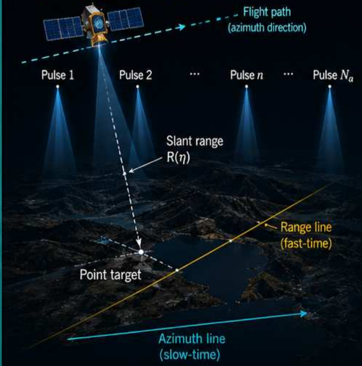
Phase captures sub-wavelength path changes.
Sensitive to millimeter-level motion and deformation.
Encodes geometry through the term $-\frac{4\pi R}{\lambda}$.

Key to interferometry and precise motion sensing.
Differencing phase reveals tiny range changes:
$\Delta R = -\frac{\lambda\,\Delta\phi}{4\pi}$ enables high-precision measurement.

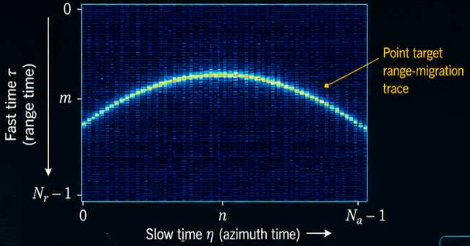
Slide 6: Fast-Time, Slow-Time, and the Raw SAR Data Matrix

SAR data are sampled over range time within each pulse and over azimuth time across many transmitted pulses.

1) Sampling Geometry



2) Raw SAR Data Matrix (Echo Matrix)



$\tau_m = m/F_s$

$\eta_n = n/\mathrm{PRF}$

$s[m,n] = s(\tau_m, \eta_n)$

N_r = number of range samples

N_a = number of azimuth pulses

3) Key Equations

$$\tau_m = \frac{m}{F_s}, \qquad m = 0, 1, \ldots, N_r - 1$$

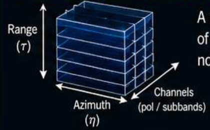
$$\eta_n = \frac{n}{PRF}, \qquad n = 0, 1, \ldots, N_a - 1$$

$$s[\mathrm{m}, n] = s(\tau_m, \eta_n)$$

Point-target echo (phase history):

$$s(\tau, \eta) \propto \exp\left(-j\frac{4\pi}{\lambda}R(\eta)\right)$$

4) What the Sensor Stores (Before Imaging)



A matrix (or data cube) of complex echoes, not an image.

Size: $N_r \times N_a \times N_c$

(N_c = number of channels or subbands)

5) Definitions

- PRF (Pulse Repetition Frequency): pulses per second along-track (Hz).
- F_s (Sampling Rate): ADC samples per second within each pulse (Hz).
- Range line (fast-time): dimension of samples inside a pulse.
- Azimuth line (slow-time): dimension across successive pulses.
- Raw data cube: complex-valued samples before any focusing.

6) From Raw Echoes to Image (High-Level Processing Flow)

Slide 7: Radar Bands for Satellite SAR

Different microwave bands trade resolution, penetration, coherence, and scattering behavior.

Band	Frequency Range (approx.)	Wavelength λ (approx.)	Key Sensing Properties	Typical Earth Observation Uses
X BAND	8 – 12 GHz	2.5 – 3.75 cm	Short wavelength → high spatial resolution; high sensitivity to small roughness; more sensitive to surface moisture and noise.	Fine urban detail, buildings, roads, critical infrastructure, ship detection.
C BAND	4 – 8 GHz	3.75 – 7.5 cm	Good balance of resolution and penetration; good coherence; less sensitive to rain than X band.	Agriculture monitoring, crop mapping, sea ice, coastal zones, wide-area monitoring.
S BAND	2 – 4 GHz	7.5 – 15 cm	Stronger penetration than C/X; more robust to weather; highlights larger structures.	Soil moisture, surface deformation, geology, disaster monitoring.
L BAND	1 – 2 GHz	15 – 30 cm	Penetrates vegetation canopy; high coherence over vegetated areas; sensitive to structure/volume scattering.	Forests, deformation (InSAR), coherence through vegetation, landslide and flood monitoring.
P BAND	0.3 – 1 GHz	30 – 100 cm	Deep penetration into vegetation and soil; sensitive to trunks and biomass; coarser spatial resolution.	Biomass estimation, forest structure, trunks, deeper canopy penetration, carbon stock assessment.

Rule of Thumb: *Longer wavelengths (L, P) generally penetrate vegetation and soil better and maintain coherence over time.*
Shorter wavelengths (X, C) emphasize surface detail and small-scale features with higher spatial resolution.

What Different SAR Bands Reveal

The same landscape can look very different at X, C, L, and P band because wavelength controls interaction with surface and vegetation structure.

X BAND (8–12 GHz)

- Sharp surface detail
- Strong urban returns
- Sensitive to small features

C BAND (4–8 GHz)

- Moderate vegetation sensitivity
- Good field texture
- Balanced surface & vegetation

L BAND (1–2 GHz)

- Stronger forest-structure response
- Better penetration through canopy
- Sensitive to large branches

P BAND (0.3–1 GHz)

- Strongest canopy penetration
- Highlights trunks & biomass
- Insensitive to leaves & moisture

SHORTER WAVELENGTH — Higher Frequency → **INCREASING WAVELENGTH →** (Larger Wavelength (cm)) → **LONGER WAVELENGTH** — Lower Frequency

SHALLOWER PENETRATION — Surface Emphasis → **INCREASING PENETRATION →** → **DEEPER PENETRATION** — Deeper / Through Canopy

APPLICATION HIGHLIGHTS

URBAN MAPPING
- Building outlines
- Infrastructure detection
- Change detection

CROP MONITORING
- Crop type & extent
- Growth stage
- Soil moisture (indirect)

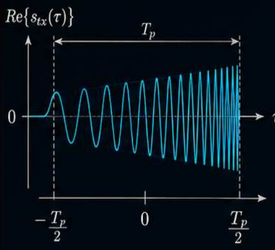
FOREST BIOMASS
- Forest structure
- Biomass estimation
- Disturbance monitoring

SUBSURFACE / UNDER-CANOPY
- Soil & geology mapping
- Flood mapping under canopy
- Trunk & biomass sensitivity

SAR Pulse Transmission and Chirp Signal

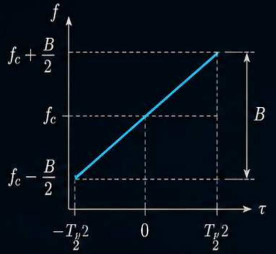
A linear-FM pulse sweeps frequency during the pulse and carries the energy used for later range compression.

① TRANSMITTED LINEAR–FM CHIRP
(time domain)



Linear-FM pulse: frequency increases linearly during the pulse duration T_p.

② FREQUENCY SWEEP DURING T_p
(instantaneous frequency)

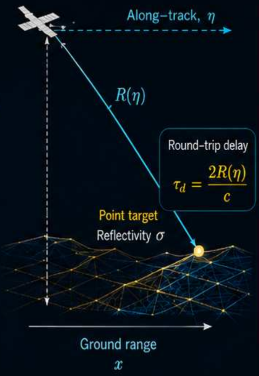


Bandwidth: B

Pulse duration: T_p

Chirp rate: $K_r = \frac{B}{T_p}$ [Hz/s]

③ SIDE-LOOKING GEOMETRY
(range vs. slow time)



TRANSMITTED LINEAR–FM CHIRP

$$s_{tx}(\tau) = \mathrm{rect}\left(\frac{\tau}{T_p}\right) \exp\left\{ j2\pi \left(f_c \tau + \frac{K_r}{2}\tau^2 \right) \right\}$$

valid for $|\tau| \leq \frac{T_p}{2}$ (zero otherwise)

DEFINITIONS

f_c	Carrier frequency [Hz]
B	Chirp bandwidth [Hz]
T_p	Pulse duration [s]
K_r	Chirp rate $= \frac{B}{T_p}$ [Hz/s]
c	Speed of light ($\approx 3 \times 10^8$ m/s)
τ	Fast time (range time) [s]
η	Slow time (azimuth time) [s]

REMARK

The transmitted chirp contains a wide bandwidth B over the duration T_p to enable high range resolution after matched filtering (range compression).

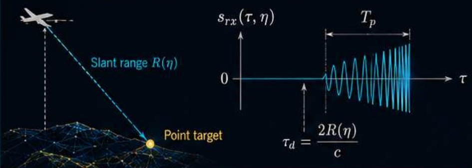
Transmit linear–FM pulse → Energy illuminates scene → Echo received after delay → Range compression in processing

From Received Echo to Range Compression

The delayed echo carries reflectivity and phase; matched filtering compresses the chirp into a sharp range peak.

1) RECEIVED ECHO FROM A POINT TARGET

A delayed and phase-shifted copy of the transmitted chirp.



Delay encodes range: $R(\eta) = \dfrac{c}{2}\,\tau_d$

2) RECEIVED ECHO MODEL (BASEBAND)

Passband (after propagation):

$$s_{rx}(\tau,\eta) = \underbrace{\sigma\, w_a(\eta)}_{\text{Reflectivity \& azimuth weighting}}\ \underbrace{\mathrm{rect}\left(\frac{\tau - \frac{2R(\eta)}{c}}{T_p}\right)}_{\text{Window of duration } T_p}\ \underbrace{\exp\left\{j\pi K_r\left(\tau - \frac{2R(\eta)}{c}\right)^2\right\}}_{\text{Chirp term (rate } K_r)}\ \underbrace{\exp\left\{-j\,\frac{4\pi R(\eta)}{\lambda}\right\}}_{\text{Phase term (path length)}}$$

After demodulation (to baseband):

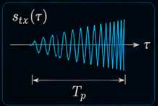
$$s_{bb}(\tau,\eta) = A(\eta)\exp\{j\phi(\tau,\eta)\}$$

- Delay encodes range $\tau_d = \dfrac{2R(\eta)}{c}$
- Phase encodes path length $\phi = -\dfrac{4\pi R(\eta)}{\lambda}$
- Amplitude reflects target reflectivity σ and antenna pattern $w_a(\eta)$

3) MATCHED FILTER / PULSE COMPRESSION

Correlate the received signal with the transmitted chirp to maximize SNR and localize targets in range.

Transmitted chirp



Matched filter

$$h(\tau) = s_{tx}^{*}(-\tau)$$

Time-reversed complex conjugate

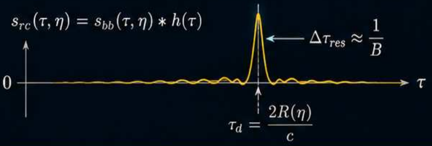
Correlation

$$s_{bb}(\tau,\eta) \circledast h(\tau)$$

Correlation with the transmitted chirp coherently sums energy when the delay matches—compressing the pulse and improving range localization.

4) RANGE-COMPRESSED OUTPUT

A sharp peak at the correct range with high SNR.



Range resolution

$$\Delta R \approx \frac{c}{2B}$$

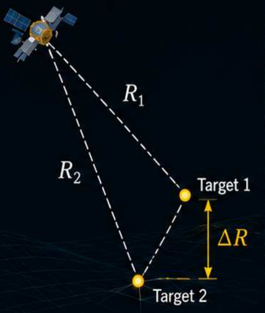
B : chirp bandwidth
ΔR : range resolution

Matched filtering compresses the chirp into a narrow peak—accurate range, higher SNR, and better target separation.

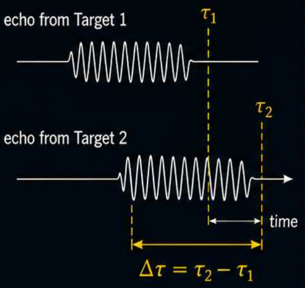
Range Resolution: Two Targets in Slant Range

SAR separates targets in range by differences in round-trip delay, then uses pulse compression to resolve them.

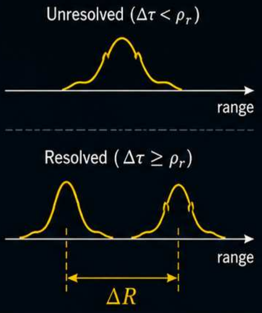
1) Geometry: Two nearby point targets in slant range



2) Received echoes (versus fast time)



3) Matched-filter output (range profile)



4) Key relationships

$$\Delta\tau = \frac{2\,\Delta R}{c}$$

$$\rho_r = \frac{c}{2B}$$

Larger bandwidth B
→ finer range resolution.

Interpretation

Two targets are separated in range by a difference in round-trip delay $\Delta\tau$ (or slant range ΔR). Pulse compression concentrates the energy of each echo into a narrow peak of width $\approx \rho_r$. Targets are resolved when $\Delta\tau \geq \rho_r$ (equivalently, $\Delta R \geq \rho_r$).

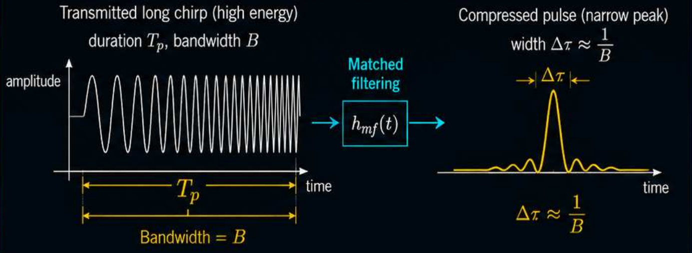
Why pulse duration alone does not set resolution

A long transmitted pulse increases energy, improving SNR. Pulse compression uses the known waveform to compress that energy in time, achieving a resolution determined by bandwidth B, not by the transmitted duration T_p.

Pulse Compression and the Role of Bandwidth

A long chirp can carry high energy while matched filtering compresses it to a narrow peak whose width is controlled by bandwidth.

Pulse Compression: Long Chirp → Narrow Peak



Key Equations

$$K_r = \frac{B}{T_p}$$ Chirp rate (Hz/s)

$$\Delta\tau \approx \frac{1}{B}$$ Compressed pulse width

$$\rho_r = \frac{c}{2B}$$ Slant-range resolution

$$\rho_{gr} = \frac{c}{2B\sin(\theta_i)}$$ Ground-range resolution

Numerical Comparison (Slant-Range Resolution)

Bandwidth B	Chirp Duration T_p	Approx. Slant-Range Resolution $\rho_r = \frac{c}{2B}$
20 MHz	7.50 μs	$\approx$ 7.5 m
100 MHz	7.50 μs	$\approx$ 1.5 m
300 MHz	7.50 μs	$\approx$ 0.5 m

Assuming $c = 3 \times 10^8$ m/s and matched filtering.

☆ Intuition: Why Larger B Improves Range Resolution

→ A wider bandwidth contains more different frequencies.

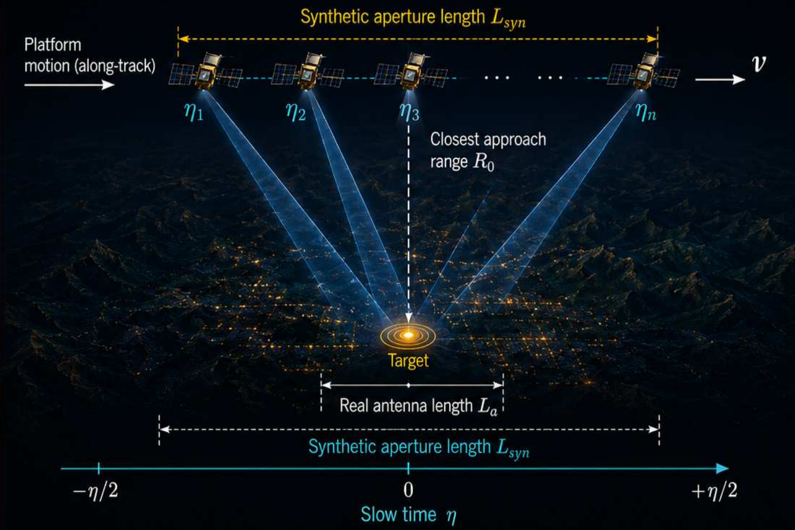
→ After matched filtering, those frequencies reinforce each other only at the correct delay.

→ More frequencies → a narrower mainlobe → smaller $\Delta\tau$ → better range resolution.

Double the bandwidth → half the resolution (approximately).

Synthetic Aperture Geometry

As the satellite moves, many pulses illuminate the same target from different positions, synthesizing a long aperture.



Geometry & Signal Model

$$R(\eta) \approx R_0 + \frac{v^2\eta^2}{2R_0}$$

$$\phi(\eta) = -\frac{4\pi}{\lambda} R(\eta)$$

$R(\eta)$: slant-range history $\phi(\eta)$: phase history

Intuition

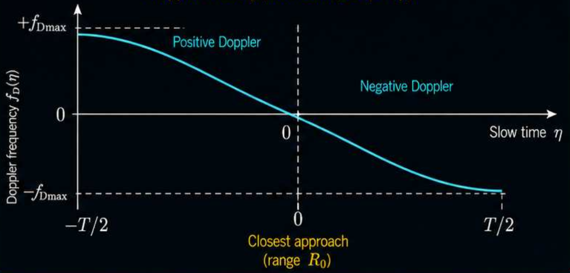
Each pulse views the target from a slightly different angle.

→ Coherent processing aligns the phase history to focus energy and turn motion into resolution.

Doppler History and Azimuth Resolution

The target's phase history creates a Doppler signature that allows SAR to focus along-track detail.

Doppler History (Broadside, Stripmap)



Key Relationships (Stripmap, Broadside)

Doppler frequency (instantaneous)

$$f_D(\eta) = \frac{1}{2\pi}\frac{d\phi(\eta)}{d\eta} \approx -\frac{2v^2}{\lambda R_0}\eta$$

Synthetic aperture length

$$L_{syn} \approx \frac{\lambda R_0}{L_a}$$

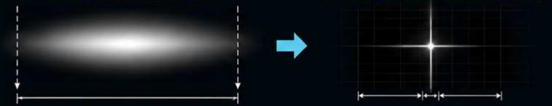
Azimuth (along-track) resolution (stripmap, broadside)

$$\rho_{az} \approx \frac{\lambda R_0}{2L_{syn}} \approx \frac{L_a}{2}$$

(for stripmap broadside SAR)

Real Aperture (Single Pulse) vs. Synthetic Aperture (After Focusing)

Real Aperture (Single Pulse) | Synthetic Aperture (After Focusing)



Beamwidth footprint (large along-track blur)

Azimuth resolution $\approx \rho_{az}$ (sharp along-track detail)

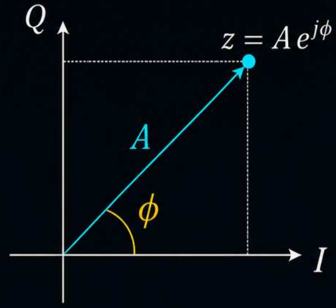
Key Takeaway

- The synthetic aperture length $L_{syn} = \frac{\lambda R_0}{L_a}$ is much longer than the physical antenna length L_a.
- This long effective aperture produces a large Doppler bandwidth and a narrow impulse response in azimuth.
- Therefore, azimuth resolution $\rho_{az} \approx \frac{\lambda R_0}{2L_{syn}} \approx \frac{L_a}{2}$ is much finer than the real-aperture (beamwidth) resolution.

Complex SAR Samples: Amplitude and Phase

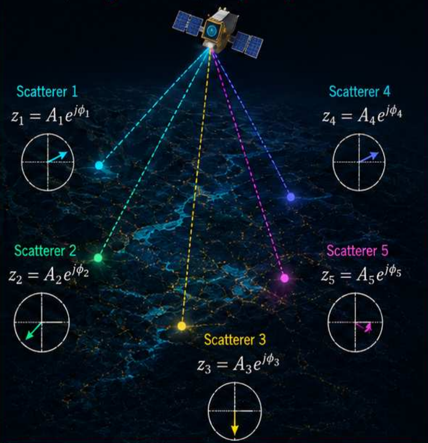
SAR records a complex-valued return $z = I + jQ$, whose magnitude gives brightness and whose angle gives phase.

1) I/Q Plane: A Single Sample



$$A = |z|$$
$$\phi = \angle z$$

2) Many Scatterers, Many Complex Returns



Each ground **scatterer** k returns a complex value $z_k = A_k e^{j\phi_k}$ with its own amplitude A_k and phase ϕ_k.

3) Key Equations and Meaning

$$z = I + jQ = A e^{j\phi}$$
$$A = |z|$$
$$\phi = \angle z$$
$$\phi = -\frac{4\pi R}{\lambda} + \phi_s$$

R = slant range (two-way path length / 2)
λ = radar wavelength
ϕ_s = scattering phase (target properties and geometry)

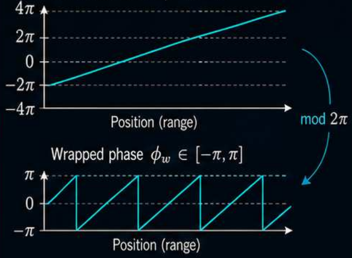
Amplitude forms the intensity image.
Bright pixels = strong backscatter.

Phase preserves path-length information.
Enables interferometry, deformation mapping, and precise measurements.

Phase Unwrapping in SAR and InSAR

Recovering continuous phase from modulo-2π measurements to reveal topography and deformation

1 Why phase wraps

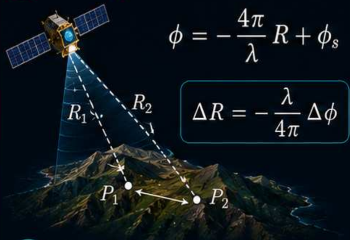


$$\phi_w = \mathrm{wrap}(\phi) = ((\phi + \pi) \bmod 2\pi) - \pi$$

$$\phi = \phi_w + 2\pi k \qquad k \in \mathbb{Z}$$

(integer ambiguity)

2 Geometry intuition



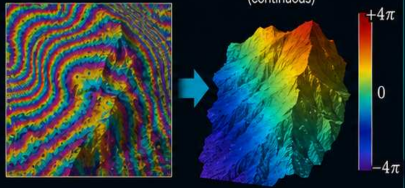
$$\phi = -\frac{4\pi}{\lambda} R + \phi_s$$

$$\Delta R = -\frac{\lambda}{4\pi} \Delta\phi$$

- A small change in path length R produces a phase change.
- 2π But we only measure phase modulo 2π, so the true phase is known only up to an unknown integer k.

$$\phi_{\text{true}} = \phi_w + 2\pi k \quad , \quad k \in \mathbb{Z}$$

3 What unwrapping does



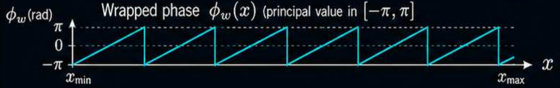
- 2π Estimate integer cycle counts k (missing multiples of 2π).
- Enforce consistency between neighbors to obtain a smooth, continuous phase.
- Use phase quality / coherence to guide reliable unwrapping.

Unwrapping estimates the **missing multiples of 2π** so that phase becomes a **smooth, physically meaningful field.**

Mathematical Model of Phase Wrapping

Wrapped phase is the principal value of the true phase, leaving an unknown integer multiple of 2π.

1 Phase and Wrapped Phase



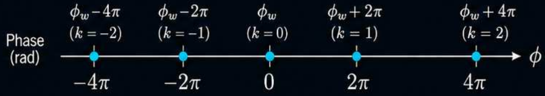
$$\phi_w(x) = \mathrm{Arg}\{\exp(j\phi(x))\} = ((\phi(x)+\pi) \bmod 2\pi) - \pi$$

2 Integer Ambiguity Model

Wrapped phase leaves an unknown integer multiple of 2π.

$$\phi(x) = \phi_w(x) + 2\pi k(x), \quad k(x) \in \mathbb{Z}$$

At any location x, the true phase can differ by integer multiples of 2π.



3 Interferometric Phase Difference

$$\Delta\phi_w = \mathrm{wrap}(\phi_2 - \phi_1)$$

$$\Delta R = -\frac{\lambda}{4\pi}\Delta\phi$$

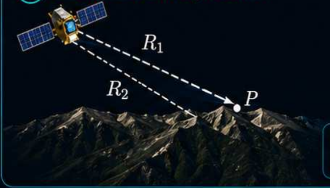
Unwrapping estimates the correct integer cycles so that the true phase difference is

$$\Delta\phi = \Delta\phi_w + 2\pi k, \quad k \in \mathbb{Z}$$

Symbols

ϕ	true phase (rad)
ϕ_w	wrapped phase (rad) in $[-\pi, \pi)$
k	integer ambiguity, $k \in \mathbb{Z}$
λ	radar wavelength (m)
ΔR	path-length change (m)

4 Geometric Interpretation



Path-length change:

$$\Delta R = R_2 - R_1$$

Phase change:

$$\Delta\phi = -\frac{4\pi}{\lambda}\Delta R$$

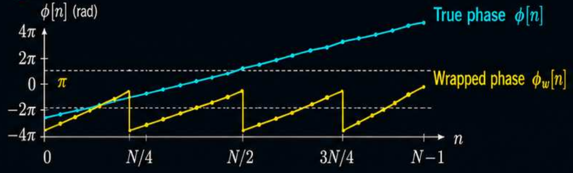
Tiny range changes produce measurable phase changes.

The core problem is to infer the integer field $k(x)$ that makes phase **spatially consistent** and **physically meaningful**.

1D Phase Unwrapping by Neighbor Differences

When adjacent phase changes stay below π in magnitude, wrapped differences recover the true increments.

① True vs. wrapped phase over samples



② Wrapped increments and the Itoh condition

$$d_w[n] = \mathrm{wrap}\left(\phi_w[n] - \phi_w[n-1]\right)$$

Under the **Itoh condition** $|\phi[n] - \phi[n-1]| < \pi$, we have

$$\boxed{d_w[n] = \phi[n] - \phi[n-1]}$$

☆ **Itoh condition:** Adjacent true phase changes must stay below π in magnitude so wrapped differences equal the true increments.

③ Recursive unwrapping algorithm

Recursive form

$$\hat{\phi}[0] = \phi_w[0]$$

$$\hat{\phi}[n] = \hat{\phi}[n-1] + d_w[n]$$

Practical jump-correction form

$\Delta = \phi_w[n] - \phi_w[n-1]$

$$d_w[n] = \begin{cases} \Delta - 2\pi, & \text{if } \Delta > \pi \\ \Delta + 2\pi, & \text{if } \Delta < -\pi \\ \Delta, & \text{otherwise} \end{cases}$$

Worked example (radians)

$\phi_w[n]$: 0.2, 1.0, 1.8, -2.8, -2.0, -1.2

$\Delta = \phi_w[n] - \phi_w[n-1]$: 0.8, 0.8, -4.6, 0.8, 0.8

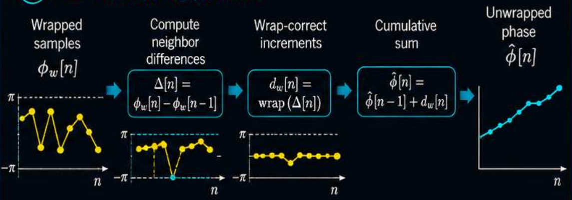
$d_w[n]$ (corrected): 0.8, 0.8, 1.683, 0.8, 0.8

$\hat{\phi}[n]$ (unwrapped): 0.2, 1.0, 1.8, 3.483, 4.283, 5.083

↑

Corrected jump: -4.6 → +1.683

④ 1D unwrapping pipeline



☆ 1D unwrapping is **simple cumulative integration of corrected phase differences**, but it **fails** if adjacent jumps **exceed π** due to noise or undersampling.

2D Phase Unwrapping: Residues and Path Dependence

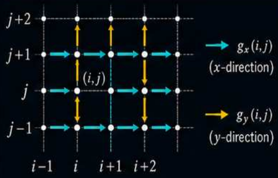
In two dimensions, wrapped gradients may be inconsistent around loops, creating residues that must be handled carefully.

1 Wrapped gradients on a grid

Define horizontal and vertical wrapped gradients at pixel (i, j):

$$g_x(i,j) = \mathrm{wrap}(\phi_w(i+1,j) - \phi_w(i,j))$$

$$g_y(i,j) = \mathrm{wrap}(\phi_w(i,j+1) - \phi_w(i,j))$$



$\mathrm{wrap}(\cdot)$ maps values to $(-\pi, \pi]$

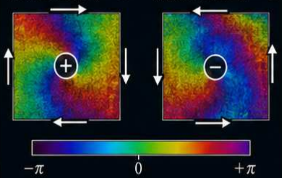
2 Residues from a 2×2 plaquette

Consider the 2×2 plaquette with top-left corner at (i, j). The discrete residue (charge) around the loop is

$$q(i,j) = \frac{1}{2\pi}\Big[g_x(i,j) + g_y(i+1,j) - g_x(i,j+1) - g_y(i,j)\Big]$$

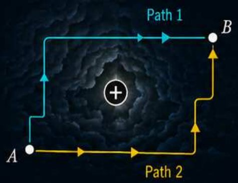
$$q \in \{\ldots, -1, 0, +1, \ldots\}$$

In practice, ±1 residues dominate.



3 Path dependence

If the sum of wrapped gradients around a closed loop is not zero, integrating along different paths gives different results.



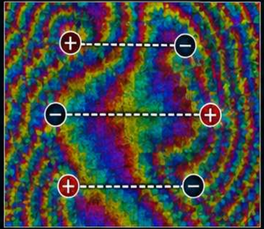
$$\sum_{\text{loop}} g_x, g_y = 2\pi q \neq 0 \;\Rightarrow$$

integrals along different paths differ.

Mismatch in recovered phase $= 2\pi q$ (here $q = +1$)

4 Branch cuts to resolve residues

Connect positive and negative residues with branch cuts to prevent inconsistent integration loops.



- positive residue ($q = +1$)
- negative residue ($q = -1$)
- ---- branch cut (discontinuity line)

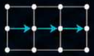
2D unwrapping is harder than 1D because **inconsistent local phase gradients** create **topological defects** called **residues.**

Least-Squares Phase Unwrapping

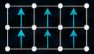
Recover a smooth phase field whose gradients best match the wrapped phase differences.

① Measured wrapped gradients and objective

Given wrapped phase differences on a 2D grid:



$$g_x(i,j) = \mathrm{wrap}(\phi_{i+1,j} - \phi_{i,j})$$



$$g_y(i,j) = \mathrm{wrap}(\phi_{i,j+1} - \phi_{i,j})$$

Estimate phase ϕ by minimizing the weighted sum of squared gradient mismatches:

$$\min_{\phi} \; J(\phi) = \sum_{i,j} w_x(i,j)\left(\phi_{i+1,j} - \phi_{i,j} - g_x(i,j)\right)^2 + \sum_{i,j} w_y(i,j)\left(\phi_{i,j+1} - \phi_{i,j} - g_y(i,j)\right)^2$$

$w_x(i,j) \ge 0,\ w_y(i,j) \ge 0$ are confidence (coherence) weights. Larger weights enforce consistency more strongly.

Typical choice: $w_x(i,j) = c_x(i,j),\ \ w_y(i,j) = c_y(i,j)$ with $c \in [0,1]$ coherence/quality maps.

② Normal equation (conceptual)

Taking derivatives of $J(\phi)$ and setting to zero yields the weighted normal equation:

$$\underbrace{\nabla\cdot(W\nabla\phi)}_{\text{weighted Laplacian of } \phi} = \underbrace{\nabla\cdot(Wg)}_{\text{divergence of weighted data}}$$

where

$$\nabla\phi = \begin{bmatrix}\phi_x\\ \phi_y\end{bmatrix}, \quad \nabla\cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

$$g = \begin{bmatrix} g_x \\ g_y \end{bmatrix}, \quad W = \begin{bmatrix} w_x & 0 \\ 0 & w_y \end{bmatrix}$$

Unweighted (all $w_x = w_y = 1$):

$$\underbrace{\Delta\phi}_{\text{Laplacian of } \phi} = \underbrace{\mathrm{div}(g)}_{\text{divergence of measured gradients}}$$

Minimizing gradient mismatch $\Longrightarrow$ solving a (weighted) Poisson equation.

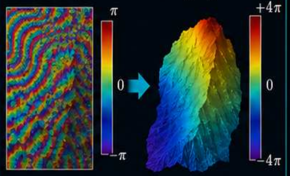
③ Numerical solution

Typical pipeline:

1. Build wrapped gradients $g_x(i,j),\ g_y(i,j)$
2. Form divergence $b = \nabla\cdot(Wg)$
3. Solve linear system $\nabla\cdot(W\nabla\phi) = b$ using:
   - FFT / DCT (periodic or Neumann BC)
   - Conjugate Gradient (CG)
   - Multigrid (MG)
4. Obtain unwrapped phase $\phi(i,j)$ (unique up to an additive constant)

Fix the gauge by setting $\mathrm{mean}(\phi) = 0$ or by pinning one pixel.

④ Example result



- Least-squares is global: uses all data simultaneously for a smooth, consistent solution.
- Robust to noise and residues; naturally handles missing data via weights.
- May oversmooth true discontinuities (e.g., object edges or phase jumps). Incorporate edge-aware weights or model discontinuities if needed.

Least-squares unwrapping converts phase recovery into a **global optimization problem**, often solved as a **Poisson equation**.

Quality-Guided and Minimum-Cost-Flow Unwrapping

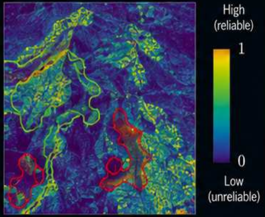
Reliable pixels should unwrap first, and residue charges can be connected by graph optimization.

① Define a quality map $q(i,j)$

Quality reflects reliability / coherence.

$$q(i,j) = |\gamma(i,j)| \quad \text{or} \quad q(i,j) = \frac{1}{\sigma_\phi^2(i,j) + \varepsilon}$$

Example coherence / quality map



High-quality regions unwrap early

Low-quality regions unwrap late

② Quality-guided unwrapping

Seed high-quality pixels
Start from the most reliable pixels.

Unwrap neighbors in descending reliability
Expand the front to neighboring pixels in order of decreasing $q(i,j)$.

Avoid low-quality regions until later
Delay traversal through uncertain areas to reduce error propagation.

Local update using wrapped differences

$$\hat{\phi}(i,j) = \hat{\phi}(p) + \mathrm{wrap}(\phi_w(i,j) - \phi_w(p))$$

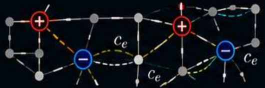
$p \in \mathcal{N}(i,j)$ (already unwrapped neighbor)

Choose the neighbor p with the highest $q(p)$ first.

③ Residues, charges & minimum-cost-flow

Residues are phase unbalance (charges).

- Positive residue $(+1)$
- Negative residue (-1)



Connect + and − residues with edges of cost c_e. Edges represent branch cuts (where we allow phase discontinuities).

Minimum-cost-flow formulation

$$\min_{f_e} \sum_e c_e f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} s_v f_e = s_v, \quad \forall v$$

$s_v \in \{+1, -1\}$ is residue charge at node v (0 otherwise).
$f_e \geq 0$ is flow on edge e, $\delta(v)$ is edges incident to v.

The optimal flow pairs residues while minimizing **total cut cost** = integration inconsistency.

④ Method comparison

Method	Strengths	Weaknesses
Path-following (quality-guided)	Simple, fast Uses reliability Good in high coherence	Error may propagate Sensitive to low-quality barriers
Branch-cut (heuristic)	Breaks unresolvable cycles Effective with residues	Heuristic cuts may be non-optimal May leave inconsistency
Least-squares (LS)	Global optimization Robust to noise No need to detect residues	Needs boundary conditions Smoothing bias, blur
Minimum-cost-flow (branch-cut optimal)	Optimal pairing of residues Minimizes inconsistency Very robust	Higher computational cost Requires good cost design

Quality information and graph optimization greatly improve **robustness in noisy or low-coherence interferograms.**

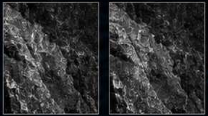
Practical InSAR Phase Unwrapping Workflow

From complex SAR images to wrapped interferograms, coherence masks, unwrapped phase, and geophysical products.

① Form Interferogram

Two co-registered complex SAR images

s_1 s_2



Interferogram (complex)

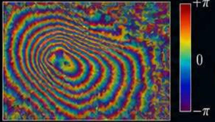
$$\boxed{z = s_1 s_2^*}$$

$$s_1,\ s_2 \in \mathbb{C}$$

② Wrapped Phase & Coherence

Wrapped interferometric phase

$$\phi_w = \arg(z)$$



Coherence (magnitude consistency)

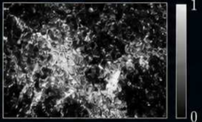
$$\gamma = \frac{|\mathbb{E}[z]|}{\sqrt{\mathbb{E}[|s_1|^2]\ \mathbb{E}[|s_2|^2]}}$$

$$0 \le \gamma \le 1$$

③ Coherence Masking / Filtering

Low-coherence pixels are masked or down-weighted before unwrapping.

Coherence map γ



Apply mask or weights:

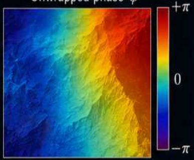
- Mask: *ignore pixels where* $\gamma < \gamma_{thr}$
- Weights: $w(x) = \gamma(x)^p,\quad p \ge 1$

Typical threshold: $\gamma_{thr} \approx 0.2 - 0.4$

④ Unwrap Phase

Phase unwrapping by a reliable method (e.g., *SNAPHU, MCF, REGION GROWING, STATISTICAL-COST*).

Unwrapped phase ϕ



$$\phi = \phi_w + 2\pi k,\quad k \in \mathbb{Z}$$

(continuous phase)

⑤ Convert to Physical Products

Convert unwrapped phase $\Delta\phi$ to geophysical quantities.

Line-of-sight displacement

$$\boxed{d_{\mathrm{LOS}} = -\frac{\lambda}{4\pi}\Delta\phi}$$

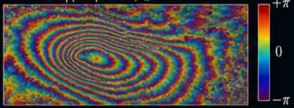
Topographic height sensitivity

$$\boxed{h = \frac{\lambda R \sin\theta}{4\pi B_\perp}\Delta\phi}$$

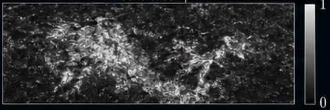
λ	*wavelength*
R	*slant range (satellite to target)*
θ	*incidence angle (from vertical)*
$B_\perp$	*perpendicular baseline*
$\Delta\phi$	*unwrapped phase (radians)*
d_{LOS}	*line-of-sight displacement*
h	*height change / topography*

What the Data Looks Like (Example)

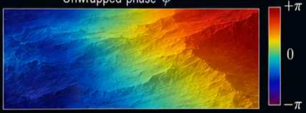
Wrapped phase ϕ_w



Coherence γ



Unwrapped phase ϕ

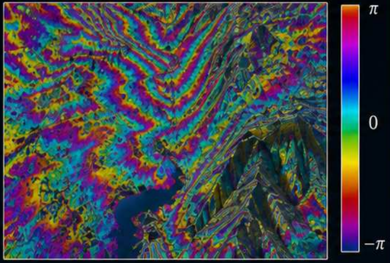


☆ Phase unwrapping is not the final goal; it is **the bridge** between interferometric fringes and physical measurements such as **deformation and topography.**

Phase Wrapping and Interferometric Sensitivity

Because phase is measured modulo 2 pi, unwrapping and phase differences reveal tiny path changes.

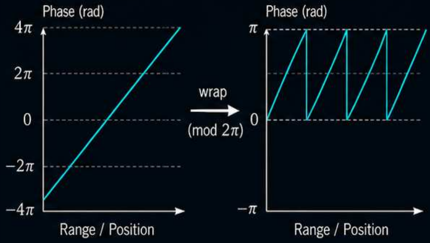
Wrapped Phase Image ($-\pi$ to π)



Phase is measured modulo 2π and wrapped into $[-\pi, \pi]$.

Phase vs. Range / Position

Unwrapped Phase (continuous) | **Wrapped Phase** (modulo 2π)



Phase–Range Relationship

$$\Delta\phi = -\frac{4\pi\,\Delta R}{\lambda}$$

$$\Delta R = -\frac{\lambda\,\Delta\phi}{4\pi}$$

$\Delta\phi$ = phase difference (radians)
ΔR = range change (meters)
λ = radar wavelength (meters)

Phase Wrapping

2π

Phase is measured modulo 2π and mapped into the interval $[-\pi, \pi]$.
All phases outside this range wrap around.

Interferometric Sensitivity

By taking phase differences between two observations, we can detect tiny path length changes.
Sensitivity is on the order of $\lambda/(4\pi)$, which is much smaller than λ.

Why It Matters

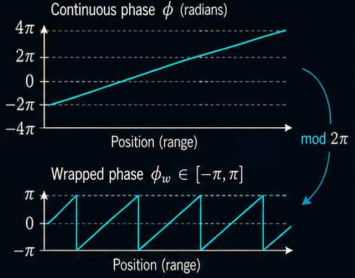
Phase differences enable sub-wavelength sensitivity to deformation or motion—revealing millimeter-level changes from kilometer-range measurements.

This is the basis for interferometric SAR: measuring phase differences to map deformation, displacement, and surface change with exceptional precision.

Phase Unwrapping in SAR and InSAR

Recovering continuous phase from modulo-2π measurements to reveal topography and deformation

1 Why phase wraps

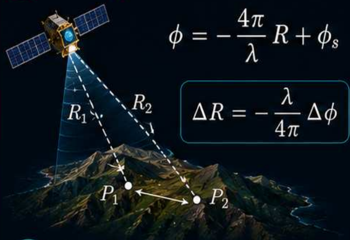


$$\phi_w = \mathrm{wrap}(\phi) = ((\phi + \pi) \bmod 2\pi) - \pi$$

$$\phi = \phi_w + 2\pi k \qquad k \in \mathbb{Z}$$

(integer ambiguity)

2 Geometry intuition



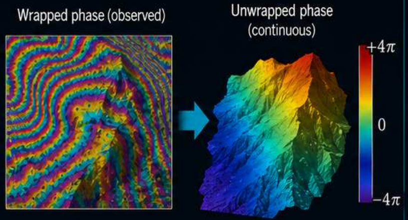
$$\phi = -\frac{4\pi}{\lambda} R + \phi_s$$

$$\Delta R = -\frac{\lambda}{4\pi} \Delta\phi$$

- A small change in path length R produces a phase change.
- 2π But we only measure phase modulo 2π, so the true phase is known only up to an unknown integer k.

$$\phi_{\text{true}} = \phi_w + 2\pi k \quad , \quad k \in \mathbb{Z}$$

3 What unwrapping does



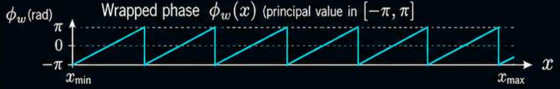
- 2π Estimate integer cycle counts k (missing multiples of 2π).
- Enforce consistency between neighbors to obtain a smooth, continuous phase.
- Use phase quality / coherence to guide reliable unwrapping.

☆ Unwrapping estimates the **missing multiples of 2π** so that phase becomes a **smooth, physically meaningful field.**

Mathematical Model of Phase Wrapping

Wrapped phase is the principal value of the true phase, leaving an unknown integer multiple of 2π.

1 Phase and Wrapped Phase



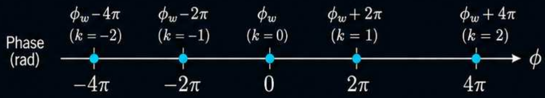
$$\phi_w(x) = \mathrm{Arg}\{\exp(j\phi(x))\} = ((\phi(x) + \pi) \bmod 2\pi) - \pi$$

2 Integer Ambiguity Model

Wrapped phase leaves an unknown integer multiple of 2π.

$$\phi(x) = \phi_w(x) + 2\pi k(x), \quad k(x) \in \mathbb{Z}$$

At any location x, the true phase can differ by integer multiples of 2π.



3 Interferometric Phase Difference

$$\Delta\phi_w = \mathrm{wrap}(\phi_2 - \phi_1)$$

$$\Delta R = -\frac{\lambda}{4\pi}\Delta\phi$$

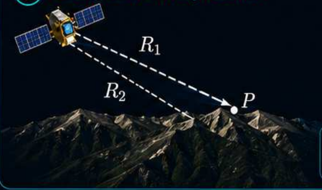
Unwrapping estimates the correct integer cycles so that the true phase difference is

$$\Delta\phi = \Delta\phi_w + 2\pi k, \quad k \in \mathbb{Z}$$

Symbols

ϕ	true phase (rad)
ϕ_w	wrapped phase (rad) in $[-\pi, \pi)$
k	integer ambiguity, $k \in \mathbb{Z}$
λ	radar wavelength (m)
ΔR	path-length change (m)

4 Geometric Interpretation



Path-length change:

$$\Delta R = R_2 - R_1$$

Phase change:

$$\Delta\phi = -\frac{4\pi}{\lambda}\Delta R$$

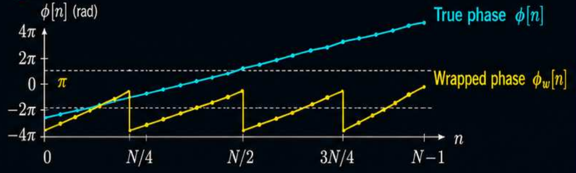
Tiny range changes produce measurable phase changes.

The core problem is to infer the integer field $k(x)$ that makes phase **spatially consistent** and **physically meaningful.**

1D Phase Unwrapping by Neighbor Differences

When adjacent phase changes stay below π in magnitude, wrapped differences recover the true increments.

1 True vs. wrapped phase over samples



2 Wrapped increments and the Itoh condition

$$d_w[n] = \mathrm{wrap}\left(\phi_w[n] - \phi_w[n-1]\right)$$

Under the **Itoh condition** $|\phi[n] - \phi[n-1]| < \pi$, we have

$$\boxed{d_w[n] = \phi[n] - \phi[n-1]}$$

Itoh condition: Adjacent true phase changes must stay below π in magnitude so wrapped differences equal the true increments.

3 Recursive unwrapping algorithm

Recursive form

$$\hat{\phi}[0] = \phi_w[0]$$

$$\hat{\phi}[n] = \hat{\phi}[n-1] + d_w[n]$$

Practical jump-correction form

$\Delta = \phi_w[n] - \phi_w[n-1]$

$$d_w[n] = \begin{cases} \Delta - 2\pi, & \text{if } \Delta > \pi \\ \Delta + 2\pi, & \text{if } \Delta < -\pi \\ \Delta, & \text{otherwise} \end{cases}$$

Worked example (radians)

$\phi_w[n]$: 0.2, 1.0, 1.8, -2.8, -2.0, -1.2

$\Delta = \phi_w[n] - \phi_w[n-1]$: 0.8, 0.8, -4.6, 0.8, 0.8

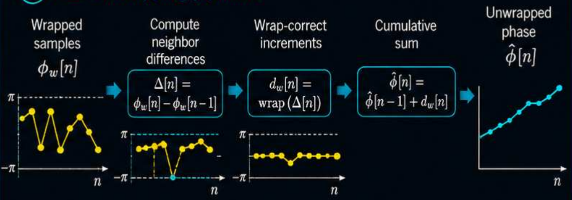
$d_w[n]$ (corrected): 0.8, 0.8, 1.683, 0.8, 0.8

$\hat{\phi}[n]$ (unwrapped): 0.2, 1.0, 1.8, 3.483, 4.283, 5.083

↑

Corrected jump: $-4.6 \rightarrow +1.683$

4 1D unwrapping pipeline



1D unwrapping is **simple cumulative integration of corrected phase differences**, but it **fails** if adjacent jumps **exceed π** due to noise or undersampling.

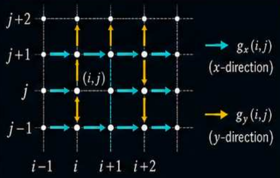
2D Phase Unwrapping: Residues and Path Dependence

In two dimensions, wrapped gradients may be inconsistent around loops, creating residues that must be handled carefully.

① Wrapped gradients on a grid

Define horizontal and vertical wrapped gradients at pixel (i, j):

$$g_x(i,j) = \mathrm{wrap}(\phi_w(i+1,j) - \phi_w(i,j))$$
$$g_y(i,j) = \mathrm{wrap}(\phi_w(i,j+1) - \phi_w(i,j))$$



$\mathrm{wrap}(\cdot)$ maps values to $(-\pi, \pi]$

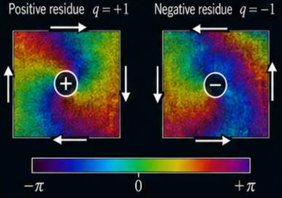
② Residues from a 2×2 plaquette

Consider the 2×2 plaquette with top-left corner at (i, j). The discrete residue (charge) around the loop is

$$q(i,j) = \frac{1}{2\pi}\left[g_x(i,j) + g_y(i+1,j) - g_x(i,j+1) - g_y(i,j)\right]$$

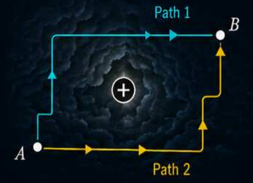
$$q \in \{\ldots, -1, 0, +1, \ldots\}$$

In practice, ±1 residues dominate.



③ Path dependence

If the sum of wrapped gradients around a closed loop is not zero, integrating along different paths gives different results.



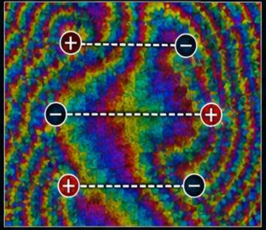
$$\sum_{\text{loop}} g_x, g_y = 2\pi q \neq 0 \;\Rightarrow$$

integrals along different paths differ.

Mismatch in recovered phase $= 2\pi q$ (here $q = +1$)

④ Branch cuts to resolve residues

Connect positive and negative residues with branch cuts to prevent inconsistent integration loops.



⊕ positive residue ($q = +1$)

⊖ negative residue ($q = -1$)

---- branch cut (discontinuity line)

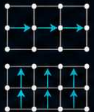
☆ 2D unwrapping is harder than 1D because **inconsistent local phase gradients** create **topological defects** called **residues.**

Least-Squares Phase Unwrapping

Recover a smooth phase field whose gradients best match the wrapped phase differences.

① Measured wrapped gradients and objective

Given wrapped phase differences on a 2D grid:



$$g_x(i,j) = \mathrm{wrap}\left(\phi_{i+1,j} - \phi_{i,j}\right)$$

$$g_y(i,j) = \mathrm{wrap}\left(\phi_{i,j+1} - \phi_{i,j}\right)$$

Estimate phase ϕ by minimizing the weighted sum of squared gradient mismatches:

$$\min_{\phi} \; J(\phi) = \sum_{i,j} w_x(i,j)\left(\phi_{i+1,j} - \phi_{i,j} - g_x(i,j)\right)^2 + \sum_{i,j} w_y(i,j)\left(\phi_{i,j+1} - \phi_{i,j} - g_y(i,j)\right)^2$$

$w_x(i,j) \geq 0$, $w_y(i,j) \geq 0$ are confidence (coherence) weights. Larger weights enforce consistency more strongly.

Typical choice: $w_x(i,j) = c_x(i,j)$, $w_y(i,j) = c_y(i,j)$ with $c \in [0,1]$ coherence/quality maps.

② Normal equation (conceptual)

Taking derivatives of $J(\phi)$ and setting to zero yields the weighted normal equation:

$$\underbrace{\nabla \cdot (W \nabla \phi)}_{\text{weighted Laplacian of } \phi} = \underbrace{\nabla \cdot (W g)}_{\text{divergence of weighted data}}$$

where

$$\nabla \phi = \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}, \quad \nabla \cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

$$g = \begin{bmatrix} g_x \\ g_y \end{bmatrix}, \quad W = \begin{bmatrix} w_x & 0 \\ 0 & w_y \end{bmatrix}$$

Unweighted (all $w_x = w_y = 1$):

$$\underbrace{\Delta \phi}_{\text{Laplacian of } \phi} = \underbrace{\mathrm{div}(g)}_{\text{divergence of measured gradients}}$$

Minimizing gradient mismatch $\Longrightarrow$ solving a (weighted) Poisson equation.

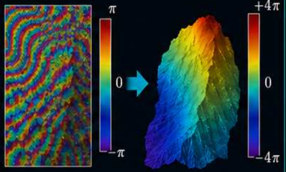
③ Numerical solution

Typical pipeline:

1. Build wrapped gradients $g_x(i,j)$, $g_y(i,j)$
2. Form divergence $b = \nabla \cdot (W g)$
3. Solve linear system $\nabla \cdot (W \nabla \phi) = b$ using:
   - FFT / DCT (periodic or Neumann BC)
   - Conjugate Gradient (CG)
   - Multigrid (MG)
4. Obtain unwrapped phase $\phi(i,j)$ (unique up to an additive constant)

Fix the gauge by setting mean$(\phi) = 0$ or by pinning one pixel.

④ Example result



- Least-squares is global: uses all data simultaneously for a smooth, consistent solution.
- Robust to noise and residues; naturally handles missing data via weights.
- May oversmooth true discontinuities (e.g., object edges or phase jumps). Incorporate edge-aware weights or model discontinuities if needed.

Least-squares unwrapping converts phase recovery into a **global optimization problem**, often solved as a **Poisson equation**.

Quality-Guided and Minimum-Cost-Flow Unwrapping

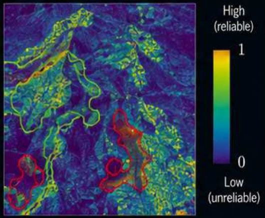
Reliable pixels should unwrap first, and residue charges can be connected by graph optimization.

1 Define a quality map $q(i,j)$

Quality reflects reliability / coherence.

$$q(i,j) = |\gamma(i,j)| \quad \text{or} \quad q(i,j) = \frac{1}{\sigma_\phi^2(i,j) + \varepsilon}$$

Example coherence / quality map



High-quality regions unwrap early

Low-quality regions unwrap late

2 Quality-guided unwrapping

Seed high-quality pixels
Start from the most reliable pixels.

Unwrap neighbors in descending reliability
Expand the front to neighboring pixels in order of decreasing $q(i,j)$.

Avoid low-quality regions until later
Delay traversal through uncertain areas to reduce error propagation.

Local update using wrapped differences

$$\hat{\phi}(i,j) = \hat{\phi}(p) + \mathrm{wrap}(\phi_w(i,j) - \phi_w(p))$$

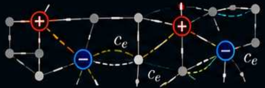
$p \in \mathcal{N}(i,j)$ (already unwrapped neighbor)

Choose the neighbor p with the highest $q(p)$ first.

3 Residues, charges & minimum-cost-flow

Residues are phase unbalance (charges).

- Positive residue $(+1)$
- Negative residue (-1)



Connect $+$ and $-$ residues with edges of cost c_e. Edges represent branch cuts (where we allow phase discontinuities).

Minimum-cost-flow formulation

$$\min_{f_e} \sum_e c_e f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} s_v f_e = s_v, \quad \forall v$$

$s_v \in \{+1, -1\}$ is residue charge at node v (0 otherwise).
$f_e \geq 0$ is flow on edge e, $\delta(v)$ is edges incident to v.

The optimal flow pairs residues while minimizing **total cut cost** = integration inconsistency.

4 Method comparison

Method	Strengths	Weaknesses
Path-following (quality-guided)	Simple, fast; Uses reliability; Good in high coherence	Error may propagate; Sensitive to low-quality barriers
Branch-cut (heuristic)	Breaks unresolvable cycles; Effective with residues	Heuristic cuts may be non-optimal; May leave inconsistency
Least-squares (LS)	Global optimization; Robust to noise; No need to detect residues	Needs boundary conditions; Smoothing bias, blur
Minimum-cost-flow (branch-cut optimal)	Optimal pairing of residues; Minimizes inconsistency; Very robust	Higher computational cost; Requires good cost design

Quality information and graph optimization greatly improve **robustness in noisy or low-coherence interferograms.**

Practical InSAR Phase Unwrapping Workflow

From complex SAR images to wrapped interferograms, coherence masks, unwrapped phase, and geophysical products.

① Form Interferogram

Two co-registered complex SAR images

s_1

s_2

Interferogram (complex)

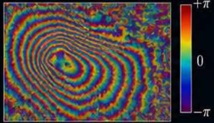
$$\boxed{z = s_1 s_2^*}$$

$$s_1,\ s_2 \in \mathbb{C}$$

② Wrapped Phase & Coherence

Wrapped interferometric phase

$$\phi_w = \arg(z)$$



Coherence (magnitude consistency)

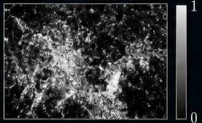
$$\gamma = \frac{|\mathbb{E}[z]|}{\sqrt{\mathbb{E}[|s_1|^2]\ \mathbb{E}[|s_2|^2]}}$$

$$0 \le \gamma \le 1$$

③ Coherence Masking / Filtering

Low-coherence pixels are masked or down-weighted before unwrapping.

Coherence map γ



Apply mask or weights:

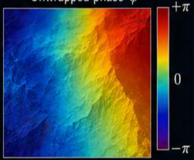
- Mask: *ignore pixels where* $\gamma < \gamma_{\text{thr}}$
- Weights: $w(x) = \gamma(x)^p, \quad p \ge 1$

Typical threshold: $\gamma_{\text{thr}} \approx 0.2 - 0.4$

④ Unwrap Phase

Phase unwrapping by a reliable method (e.g., *SNAPHU, MCF, REGION GROWING, STATISTICAL-COST).*

Unwrapped phase ϕ



$$\phi = \phi_w + 2\pi k, \quad k \in \mathbb{Z}$$

(continuous phase)

⑤ Convert to Physical Products

Convert unwrapped phase $\Delta\phi$ to geophysical quantities.

Line-of-sight displacement

$$\boxed{d_{\text{LOS}} = -\frac{\lambda}{4\pi}\Delta\phi}$$

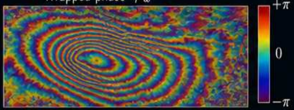
Topographic height sensitivity

$$\boxed{h = \frac{\lambda R \sin\theta}{4\pi B_\perp}\Delta\phi}$$

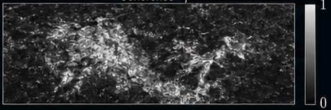
λ	*wavelength*
R	*slant range (satellite to target)*
θ	*incidence angle (from vertical)*
$B_\perp$	*perpendicular baseline*
$\Delta\phi$	*unwrapped phase (radians)*
d_{LOS}	*line-of-sight displacement*
h	*height change / topography*

What the Data Looks Like (Example)

Wrapped phase ϕ_w



Coherence γ



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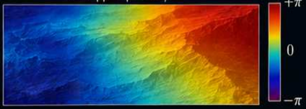

☆ Phase unwrapping is not the final goal; it is **the bridge** between interferometric fringes and physical measurements such as **deformation and topography.**

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Phase Unwrapping in SAR and InSAR

Recovering continuous phase from modulo-2π measurements to reveal topography and deformation

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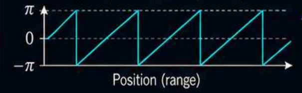

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(integer ambiguity)

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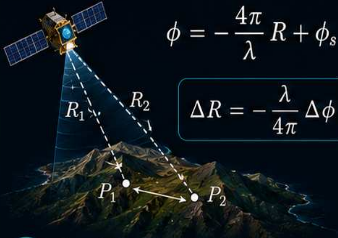

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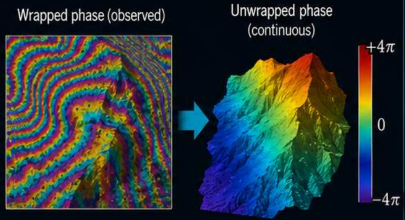

- 2π Estimate integer cycle counts k (missing multiples of 2π).
- Enforce consistency between neighbors to obtain a smooth, continuous phase.
- Use phase quality / coherence to guide reliable unwrapping.

☆ Unwrapping estimates the **missing multiples of 2π** so that phase becomes a **smooth, physically meaningful field.**

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Mathematical Model of Phase Wrapping

Wrapped phase is the principal value of the true phase, leaving an unknown integer multiple of 2π.

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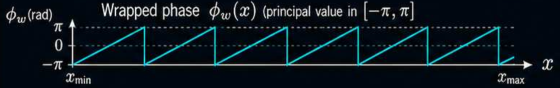

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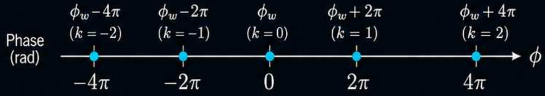
At any location x, the true phase can differ by integer multiples of 2π.



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$$\Delta\phi_w = \mathrm{wrap}(\phi_2 - \phi_1)$$

$$\Delta R = -\frac{\lambda}{4\pi}\Delta\phi$$

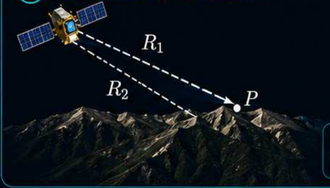
Unwrapping estimates the correct integer cycles so that the true phase difference is

$$\Delta\phi = \Delta\phi_w + 2\pi k, \quad k \in \mathbb{Z}$$

Symbols

ϕ	true phase (rad)
ϕ_w	wrapped phase (rad) in $[-\pi, \pi)$
k	integer ambiguity, $k \in \mathbb{Z}$
λ	radar wavelength (m)
ΔR	path-length change (m)

4 Geometric Interpretation



Path-length change:

$$\Delta R = R_2 - R_1$$

Phase change:

$$\Delta\phi = -\frac{4\pi}{\lambda}\Delta R$$

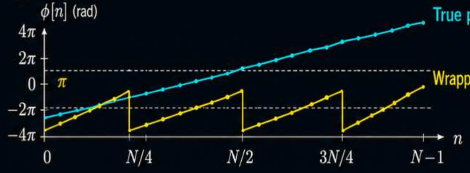
Tiny range changes produce measurable phase changes.

The core problem is to infer the integer field $k(x)$ that makes phase **spatially consistent** and **physically meaningful.**

1D Phase Unwrapping by Neighbor Differences

When adjacent phase changes stay below π in magnitude, wrapped differences recover the true increments.

① True vs. wrapped phase over samples



② Wrapped increments and the Itoh condition

$$d_w[n] = \mathrm{wrap}\left(\phi_w[n] - \phi_w[n-1]\right)$$

Under the **Itoh condition** $|\phi[n] - \phi[n-1]| < \pi$, we have

$$\boxed{d_w[n] = \phi[n] - \phi[n-1]}$$

Itoh condition: Adjacent true phase changes must stay below π in magnitude so wrapped differences equal the true increments.

③ Recursive unwrapping algorithm

Recursive form

$$\hat{\phi}[0] = \phi_w[0]$$

$$\hat{\phi}[n] = \hat{\phi}[n-1] + d_w[n]$$

Practical jump-correction form

$\Delta = \phi_w[n] - \phi_w[n-1]$

$$d_w[n] = \begin{cases} \Delta - 2\pi, & \text{if } \Delta > \pi \\ \Delta + 2\pi, & \text{if } \Delta < -\pi \\ \Delta, & \text{otherwise} \end{cases}$$

Worked example (radians)

$\phi_w[n]$: 0.2, 1.0, 1.8, -2.8, -2.0, -1.2

$\Delta = \phi_w[n] - \phi_w[n-1]$: 0.8, 0.8, -4.6, 0.8, 0.8

$d_w[n]$ (corrected): 0.8, 0.8, 1.683, 0.8, 0.8

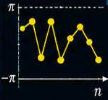
$\hat{\phi}[n]$ (unwrapped): 0.2, 1.0, 1.8, 3.483, 4.283, 5.083

↑

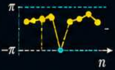
Corrected jump: -4.6 → +1.683

④ 1D unwrapping pipeline

Wrapped samples $\phi_w[n]$



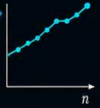
Compute neighbor differences: $\Delta[n] = \phi_w[n] - \phi_w[n-1]$



Wrap-correct increments: $d_w[n] = \mathrm{wrap}(\Delta[n])$

Cumulative sum: $\hat{\phi}[n] = \hat{\phi}[n-1] + d_w[n]$

Unwrapped phase $\hat{\phi}[n]$



1D unwrapping is **simple cumulative integration of corrected phase differences**, but it **fails** if adjacent jumps **exceed π** due to noise or undersampling.

2D Phase Unwrapping: Residues and Path Dependence

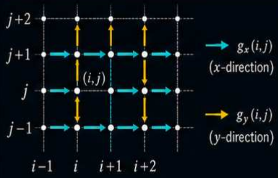
In two dimensions, wrapped gradients may be inconsistent around loops, creating residues that must be handled carefully.

1 Wrapped gradients on a grid

Define horizontal and vertical wrapped gradients at pixel (i,j):

$$g_x(i,j) = \mathrm{wrap}(\phi_w(i+1,j) - \phi_w(i,j))$$

$$g_y(i,j) = \mathrm{wrap}(\phi_w(i,j+1) - \phi_w(i,j))$$



$\mathrm{wrap}(\cdot)$ maps values to $(-\pi, \pi]$

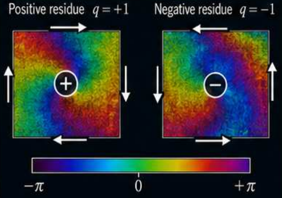
2 Residues from a 2×2 plaquette

Consider the 2×2 plaquette with top-left corner at (i,j). The discrete residue (charge) around the loop is

$$q(i,j) = \frac{1}{2\pi}\Big[g_x(i,j) + g_y(i+1,j) - g_x(i,j+1) - g_y(i,j)\Big]$$

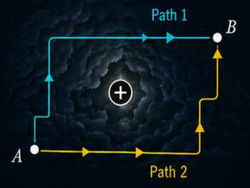
$$q \in \{\dots, -1, 0, +1, \dots\}$$

In practice, ± 1 residues dominate.



3 Path dependence

If the sum of wrapped gradients around a closed loop is not zero, integrating along different paths gives different results.



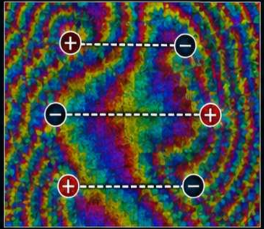
$$\sum_{\text{loop}} g_x, g_y = 2\pi q \neq 0 \;\Rightarrow$$

integrals along different paths differ.

Mismatch in recovered phase $= 2\pi q$ (here $q=+1$)

4 Branch cuts to resolve residues

Connect positive and negative residues with branch cuts to prevent inconsistent integration loops.



⊕ positive residue ($q=+1$)

⊖ negative residue ($q=-1$)

---- branch cut (discontinuity line)

☆ 2D unwrapping is harder than 1D because **inconsistent local phase gradients** create **topological defects** called **residues.**

ELEG404/604 Imaging & Deep Learning

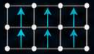
Least-Squares Phase Unwrapping

Recover a smooth phase field whose gradients best match the wrapped phase differences.

1 Measured wrapped gradients and objective

Given wrapped phase differences on a 2D grid:

$$g_x(i,j) = \mathrm{wrap}\left(\phi_{i+1,j} - \phi_{i,j}\right)$$



$$g_y(i,j) = \mathrm{wrap}\left(\phi_{i,j+1} - \phi_{i,j}\right)$$

Estimate phase ϕ by minimizing the weighted sum of squared gradient mismatches:

$$\min_{\phi} \; J(\phi) = \sum_{i,j} w_x(i,j)\left(\phi_{i+1,j} - \phi_{i,j} - g_x(i,j)\right)^2 + \sum_{i,j} w_y(i,j)\left(\phi_{i,j+1} - \phi_{i,j} - g_y(i,j)\right)^2$$

$w_x(i,j) \geq 0$, $w_y(i,j) \geq 0$ are confidence (coherence) weights. Larger weights enforce consistency more strongly.

Typical choice: $w_x(i,j) = c_x(i,j)$, $w_y(i,j) = c_y(i,j)$ with $c \in [0,1]$ coherence/quality maps.

2 Normal equation (conceptual)

Taking derivatives of $J(\phi)$ and setting to zero yields the weighted normal equation:

$$\underbrace{\nabla \cdot (W \nabla \phi)}_{\text{weighted Laplacian of } \phi} = \underbrace{\nabla \cdot (W g)}_{\text{divergence of weighted data}}$$

where

$$\nabla \phi = \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}, \quad \nabla \cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

$$g = \begin{bmatrix} g_x \\ g_y \end{bmatrix}, \quad W = \begin{bmatrix} w_x & 0 \\ 0 & w_y \end{bmatrix}$$

Unweighted (all $w_x = w_y = 1$):

$$\underbrace{\Delta \phi}_{\text{Laplacian of } \phi} = \underbrace{\mathrm{div}(g)}_{\text{divergence of measured gradients}}$$

Minimizing gradient mismatch $\Longrightarrow$ solving a (weighted) Poisson equation.

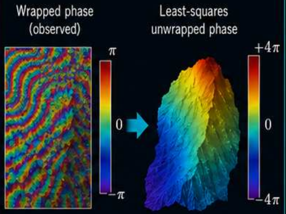
3 Numerical solution

Typical pipeline:

1. Build wrapped gradients $g_x(i,j)$, $g_y(i,j)$
2. Form divergence $b = \nabla \cdot (W g)$
3. Solve linear system $\nabla \cdot (W \nabla \phi) = b$ using:
   - FFT / DCT (periodic or Neumann BC)
   - Conjugate Gradient (CG)
   - Multigrid (MG)
4. Obtain unwrapped phase $\phi(i,j)$ (unique up to an additive constant)

Fix the gauge by setting $\mathrm{mean}(\phi) = 0$ or by pinning one pixel.

4 Example result



- Least-squares is global: uses all data simultaneously for a smooth, consistent solution.
- Robust to noise and residues; naturally handles missing data via weights.
- May oversmooth true discontinuities (e.g., object edges or phase jumps). Incorporate edge-aware weights or model discontinuities if needed.

Least-squares unwrapping converts phase recovery into a **global optimization problem**, often solved as a **Poisson equation**.

Quality-Guided and Minimum-Cost-Flow Unwrapping

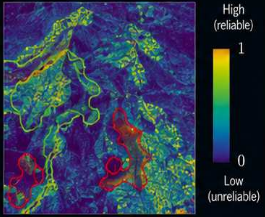
Reliable pixels should unwrap first, and residue charges can be connected by graph optimization.

1 Define a quality map $q(i,j)$

Quality reflects reliability / coherence.

$$q(i,j) = |\gamma(i,j)| \quad \text{or} \quad q(i,j) = \frac{1}{\sigma_\phi^2(i,j) + \varepsilon}$$

Example coherence / quality map



High-quality regions unwrap early

Low-quality regions unwrap late

2 Quality-guided unwrapping

Seed high-quality pixels
Start from the most reliable pixels.

Unwrap neighbors in descending reliability
Expand the front to neighboring pixels in order of decreasing $q(i,j)$.

Avoid low-quality regions until later
Delay traversal through uncertain areas to reduce error propagation.

Local update using wrapped differences

$$\hat{\phi}(i,j) = \hat{\phi}(p) + \mathrm{wrap}(\phi_w(i,j) - \phi_w(p))$$

$p \in \mathcal{N}(i,j)$ (already unwrapped neighbor)

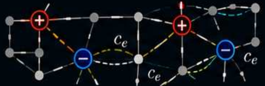
Choose the neighbor p with the highest $q(p)$ first.

3 Residues, charges & minimum-cost-flow

Residues are phase unbalance (charges).

Positive residue $(+1)$

Negative residue (-1)



Connect $+$ and $-$ residues with edges of cost c_e. Edges represent branch cuts (where we allow phase discontinuities).

Minimum-cost-flow formulation

$$\min_{f_e} \sum_e c_e f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} s_v f_e = s_v, \quad \forall v$$

$s_v \in \{+1, -1\}$ is residue charge at node v (0 otherwise).
$f_e \geq 0$ is flow on edge e, $\delta(v)$ is edges incident to v.

The optimal flow pairs residues while minimizing total cut cost = integration inconsistency.

4 Method comparison

Method	Strengths	Weaknesses
Path-following (quality-guided)	Simple, fast; Uses reliability; Good in high coherence	Error may propagate; Sensitive to low-quality barriers
Branch-cut (heuristic)	Breaks unresolvable cycles; Effective with residues	Heuristic cuts may be non-optimal; May leave inconsistency
Least-squares (LS)	Global optimization; Robust to noise; No need to detect residues	Needs boundary conditions; Smoothing bias, blur
Minimum-cost-flow (branch-cut optimal)	Optimal pairing of residues; Minimizes inconsistency; Very robust	Higher computational cost; Requires good cost design

Quality information and graph optimization greatly improve **robustness in noisy or low-coherence interferograms.**

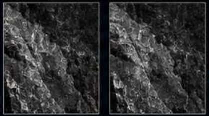
Practical InSAR Phase Unwrapping Workflow

From complex SAR images to wrapped interferograms, coherence masks, unwrapped phase, and geophysical products.

① Form Interferogram

Two co-registered complex SAR images

s_1 s_2



Interferogram (complex)

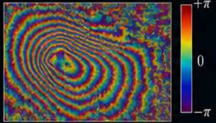
$$\boxed{z = s_1 s_2^*}$$

$$s_1,\ s_2 \in \mathbb{C}$$

② Wrapped Phase & Coherence

Wrapped interferometric phase

$$\phi_w = \arg(z)$$



Coherence (magnitude consistency)

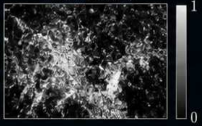
$$\gamma = \frac{|\mathbb{E}[z]|}{\sqrt{\mathbb{E}[|s_1|^2]\ \mathbb{E}[|s_2|^2]}}$$

$$0 \le \gamma \le 1$$

③ Coherence Masking / Filtering

Low-coherence pixels are masked or down-weighted before unwrapping.

Coherence map γ



Apply mask or weights:

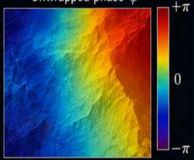
- Mask: *ignore pixels where* $\gamma < \gamma_{\text{thr}}$
- Weights: $w(x) = \gamma(x)^p, \quad p \ge 1$

Typical threshold: $\gamma_{\text{thr}} \approx 0.2 - 0.4$

④ Unwrap Phase

Phase unwrapping by a reliable method (e.g., *SNAPHU, MCF, REGION GROWING, STATISTICAL-COST*).

Unwrapped phase ϕ



$$\phi = \phi_w + 2\pi k, \quad k \in \mathbb{Z}$$

(continuous phase)

⑤ Convert to Physical Products

Convert unwrapped phase $\Delta\phi$ to geophysical quantities.

Line-of-sight displacement

$$\boxed{d_{\text{LOS}} = -\frac{\lambda}{4\pi}\Delta\phi}$$

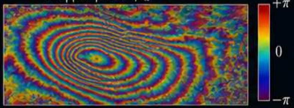
Topographic height sensitivity

$$\boxed{h = \frac{\lambda R \sin\theta}{4\pi B_\perp}\Delta\phi}$$

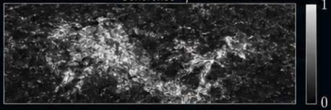
λ	*wavelength*
R	*slant range (satellite to target)*
θ	*incidence angle (from vertical)*
$B_\perp$	*perpendicular baseline*
$\Delta\phi$	*unwrapped phase (radians)*
d_{LOS}	*line-of-sight displacement*
h	*height change / topography*

What the Data Looks Like (Example)

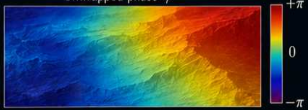
Wrapped phase ϕ_w



Coherence γ



Unwrapped phase ϕ

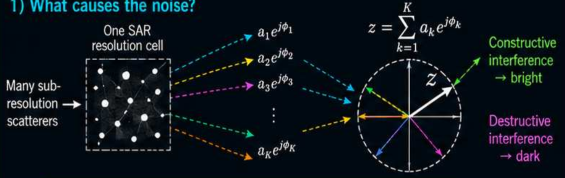


Phase unwrapping is not the final goal; it is **the bridge** between interferometric fringes and physical measurements such as **deformation and topography.**

Noise in SAR Imagery: Speckle, Thermal Noise, and Statistics

SAR is a coherent imaging system, so its dominant granular noise is speckle rather than ordinary Poisson shot noise.

1) What causes the noise?



Thermal receiver noise adds incoherently as an additive term n.

Observed complex return: $z = \sum_{k=1}^{K} a_k e^{j\phi_k} + n$

2) Image formation / noise model

$z = x + n$ (complex baseband)

$I = |z|^2$ (intensity image)

$z = \sqrt{x}\, s$ (multiplicative speckle view)

$I = x\, u$ (multilook intensity model)

$u \sim \mathrm{Gamma}(L, L)$

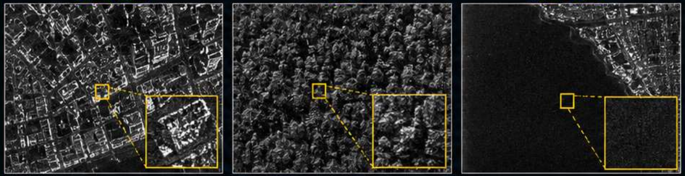
$\mathrm{E}[u] = 1,\ \mathrm{Var}[u] = \frac{1}{L}$

- x : true backscatter power (mean intensity)
- z : complex SAR return
- I : measured intensity
- u : speckle (unit-mean) random variable
- L : number of looks (independent averages)
- n : complex thermal noise (often $\sim \mathcal{CN}(0, \sigma_n^2)$)

Optical photon shot noise is often modeled as Poisson; SAR grain is mainly **coherent speckle**.

3) Examples of SAR amplitude images

Urban scene | Forest | Water / coast



Low — High (Amplitude)

4) Why it matters

- Speckle **reduces local contrast**, making bright and dark areas less visually distinct.
- It **obscures fine features** and small targets, especially in single-look images.
- It **complicates segmentation**, detection, and classification because adjacent pixels are highly correlated.
- It motivates **multilooking** (increases L) and **despeckling** filters to improve interpretability and measurement.

Key idea: in SAR, the image looks noisy because each pixel is a coherent sum of many scatterers.

Speckle Statistics and the Role of Looks

Single-look SAR intensity is highly variable; multilooking reduces variance and improves interpretability.

1 Single-look model

A single-look complex pixel is the coherent sum of many scatterers with random phases.

Amplitude is often Rayleigh-distributed; intensity is exponential.

$$z = \sum_k a_k e^{j\phi_k}$$

$$I = |z|^2$$

$$I = xu$$

$$u \sim \mathrm{Exp}(1) \quad \text{(single look)}$$

$$\mathrm{E}[u] = 1, \quad \mathrm{Var}[u] = 1$$

z : complex return

I : intensity

x : true backscatter power (mean intensity)

u : unit-mean speckle random variable

2 Multilook model

Average L independent looks of intensity to reduce variance.

$$I_L = \frac{1}{L}\sum_{\ell=1}^{L} I_\ell$$

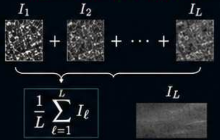
$$I_L = xu$$

$$u \sim \mathrm{Gamma}(L, L)$$

$$\mathrm{E}[u] = 1, \quad \mathrm{Var}[u] = \frac{1}{L}$$

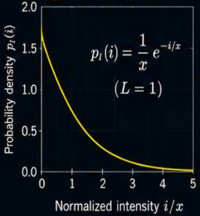
$$\mathrm{ENL} = \frac{\mu^2}{\sigma^2} \approx L$$

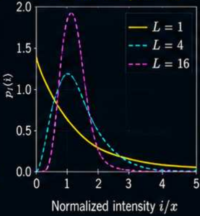
Intuition: average L noisy looks

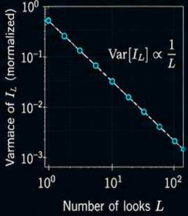


More looks → smoother image, lower variance (less speckle).

3 Speckle statistics







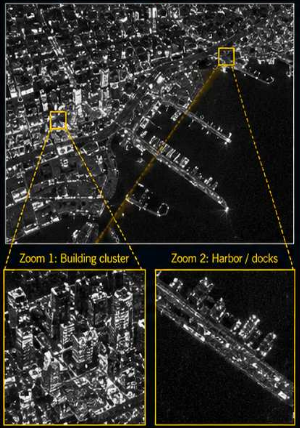
4 Effect of looks on SAR images (same scene)

Key takeaway: More looks -> lower speckle variance, but averaging trades spatial detail for smoother appearance.

Real SAR Images: Structure vs. Speckle

Granular noise is strongest in homogeneous regions, while bright man-made targets and edges remain visible but contaminated.

1) Urban downtown / port scene

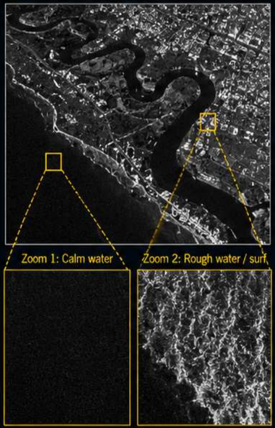


Bright, structured returns from buildings and ships.
Strong double-bounce and corner reflectors.

2) Agricultural / vegetation scene

Vegetation produces volume scattering.
Mottled texture (speckle) dominates homogeneous areas.

3) River / coastal scene

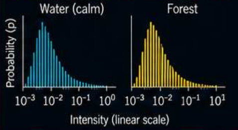


Calm water is dark but still speckled.
Rough water and wind-ruffled surfaces are brighter.

4) What the images tell us

- Urban areas have strong double-bounce and corner reflectors → bright, structured returns.
- Vegetation produces volume scattering → mottled appearance.
- Calm water is dark due to low backscatter, but still speckled.
- Rough water (wind, waves) increases backscatter → brighter texture.

5) Homogeneous-region histogram



Both water and forest are homogeneous, yet intensities are granular and follow a skewed distribution (speckle statistics).

Takeaway: Speckle is not just random nuisance; it is the *coherent fingerprint* of the scattering physics, but it makes *visual interpretation* and *measurement* harder.

Despeckling Strategies for SAR Imagery

The goal is to suppress speckle while preserving edges, point targets, and textural information.

1) The multiplicative model and log transform

Multiplicative model

$$I = x\,u$$

I : observed intensity
x : true backscatter (unknown)
u : speckle (unit mean)

Log transform →

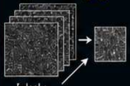
$$y = \log I = \log x + \log u$$

Why multiplicative noise is harder:

- *Signal-dependent variance:* $\mathrm{Var}(I) = x^2\mathrm{Var}(u) = x^2\,\sigma_u^2$
- *Not Gaussian (intensity is Gamma distributed), so linear filters are suboptimal.*

2) Classic despeckling methods

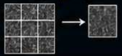
Multilooking (Spatial averaging)



L looks

Increases ENL by L. Simple and effective.

Lee Filter (Adaptive)



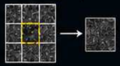
$$\hat{x} = \bar{x} + W\,(I - \bar{x})$$

$\bar{x}$: local mean

$$W = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_u^2} \quad (0 \le W \le 1)$$

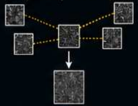
W depends on local statistics.

Frost Filter (Adaptive)



Exponential weighting based on distance and damping factor.
Better edge preservation than Lee.

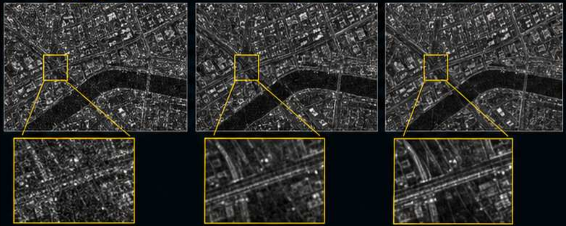
Nonlocal Methods (BM3D, NL-SAR)



Exploit self-similarity across the image.
Preserve textures and weak targets.

3) Despeckling results: same scene

Noisy (Single-look) | Classical (Lee Filter) | Modern (AI-based)



4) Trade-offs and evaluation

Too much smoothing

- *Blurs edges and boundaries*
- *Suppresses weak/point targets*
- *Loses textural details*

Too little smoothing

- *High variance remains*
- *Noisy textures*
- *Poor interpretability*

Metric	What it Measures	Better
ENL	Equivalent Number of Looks (higher → lower variance)	Higher
PSNR (dB)	Fidelity to reference (higher is better)	Higher
SSIM	Structural similarity (0–1, higher is better)	Higher
Edge Preservation (Edge Score / EPI)	How well edges are retained (higher is better)	Higher

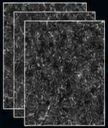
Takeaway: Good SAR denoising balances variance reduction with preservation of scattering structure.

From Speckle to Better SAR Products

Noise modeling matters for detection, classification, change analysis, and interferometric measurement.

1) From raw data to analysis-ready products

Raw complex SAR data



$z = x + n$

Amplitude / intensity (image with speckle)

$I = |z|^2$

Multilook / despeckle

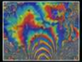
Variance $\downarrow$ as $1/L$

Segmentation / detection / classification / InSAR products

Segmentation

Detection

Classification



InSAR

2) Common noise and statistical models

$$z = x + n \quad \text{(complex additive thermal model)}$$
$$n \sim \mathcal{CN}(0, \sigma_n^2)$$
$$I = |z|^2 = x\,u \quad \text{(multiplicative intensity model)}$$
$$u \sim \mathrm{Gamma}(L, L)$$

Coherence:

$$\gamma = \frac{\mathrm{E}[z_1 z_2^*]}{\sqrt{\mathrm{E}[|z_1|^2]\,\mathrm{E}[|z_2|^2]}} \in [0, 1]$$

Equivalent Number of Looks (ENL):

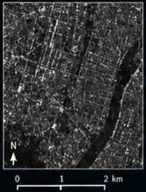
$$\mathrm{ENL} = \frac{\mu^2}{\sigma^2} \approx L \quad \text{(homogeneous areas)}$$

- z : complex SAR return
- x : true backscatter (complex)
- n : thermal noise (complex)
- I : intensity
- u : unit-mean gamma speckle
- L : number of looks
- μ : mean intensity
- σ^2 : variance of intensity

Optical photon shot noise is often modeled as Poisson, but SAR granular noise is mainly **coherent speckle (Gamma)**.

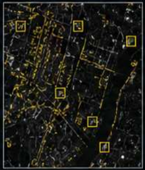
3) Real-world example: products from one SAR scene

Original (speckled)

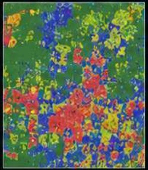


Despeckled (multilook)

Edge / target detection



Change detection (color)



Stable · Increase · Decrease · New · Water

4) Key takeaways for practice

- **Speckle** dominates over ordinary Poisson shot noise in coherent SAR imagery. *It is multiplicative, signal-dependent, and granular.*
- **Thermal noise** *matters in low-SNR regions and must be modeled when estimating noise floors or detecting weak targets.*
- **Multilooking** *(averaging L looks) reduces speckle variance as $1/L$, at the cost of resolution.*
- **Advanced filtering or AI** *can improve readability and downstream performance, but may hallucinate details or oversmooth if used poorly.*

For SAR, understanding noise is essential before doing reconstruction, interpretation, or InSAR phase analysis.

Real SAR Imagery and Optical Comparison

The same scene looks very different in visible light and coherent radar backscatter

Buildings / corner reflectors → bright in SAR

Calm water → dark in SAR

Vegetation → textured volume scattering

Roads and docks → linear scatterers

- Optical measures reflected sunlight and color.
- SAR measures microwave backscatter and phase.
- SAR works day/night and through clouds.
- Appearance depends on geometry, roughness, and dielectric properties.