ELEG-636: Statistical Signal Processing

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Eigen Analysis

Objective: Utilize tools from linear algebra to characterize and analyze matrices, especially the correlation matrix

- The correlation matrix plays a large role in statistical characterization and processing.
- Previously result: $\mathbf{R}$ is Hermitian.
- Further insight into the correlation matrix is achieved through eigen analysis
  - Eigenvalues and vectors
  - Matrix diagonalization
  - Application: Optimum filtering problems

Objective: For a Hermitian matrix $\mathbf{R}$, find a vector $\mathbf{q}$ satisfying

$$\mathbf{R}\mathbf{q} = \lambda \mathbf{q}$$

- Interpretation: Linear transformation by $\mathbf{R}$ changes the scale, but not the direction of $\mathbf{q}$
- Fact: A $M \times M$ matrix $\mathbf{R}$ has M eigenvectors and eigenvalues

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i \quad i = 1, 2, 3, \cdots, M$$

To see this, note

$$(\mathbf{R} - \lambda \mathbf{I})\mathbf{q} = \mathbf{0}$$

For this to be true, the row/columns of $(\mathbf{R} - \lambda \mathbf{I})$ must be linearly dependent,

$$\Rightarrow \det(\mathbf{R} - \lambda \mathbf{I}) = 0$$

Note: $\det(\mathbf{R} - \lambda \mathbf{I})$ is a Mth order polynomial in λ

- The roots of the polynomial are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_M$

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i$$

- Each eigenvector $\mathbf{q}_i$ is associated with one eigenvalue λ_i
- The eigenvectors are not unique

$$\begin{aligned} \mathbf{R}\mathbf{q}_i &= \lambda_i \mathbf{q}_i \\ \Rightarrow \mathbf{R}(a\mathbf{q}_i) &= \lambda_i (a\mathbf{q}_i) \end{aligned}$$

Consequence: eigenvectors are generally normalized, e.g., $|\mathbf{q}_i| = 1$ for $i = 1, 2, \ldots, M$

Example (General two dimensional case)

Let $M = 2$ and

$$\mathbf{R} = \begin{bmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{bmatrix}$$

Determine the eigenvalues and eigenvectors.

Thus

$$\begin{aligned} \det(\mathbf{R} - \lambda\mathbf{I}) &= 0 \\ \Rightarrow \begin{vmatrix} R_{1,1} - \lambda & R_{1,2} \\ R_{2,1} & R_{2,2} - \lambda \end{vmatrix} &= 0 \\ \Rightarrow \lambda^2 - \lambda(R_{1,1} + R_{2,2}) + (R_{1,1}R_{2,2} - R_{1,2}R_{2,1}) &= 0 \end{aligned}$$

$$\Rightarrow \lambda_{1,2} = \frac{1}{2}\left[(R_{1,1} + R_{2,2}) \pm \sqrt{4R_{1,2}R_{2,1} + (R_{1,1} - R_{2,2})}\right]$$

Back substitution yields the eigenvectors:

$$\begin{bmatrix} R_{1,1}-\lambda & R_{1,2} \\ R_{2,1} & R_{2,2}-\lambda \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In general, this yields a set of linear equations. In the $M = 2$ case:

$$\begin{aligned} (R_{1,1}-\lambda)q_1 + R_{1,2}q_2 =& 0 \\ R_{2,1}q_1 + (R_{2,2}-\lambda)q_2 =& 0 \end{aligned}$$

- Solving the set of linear equations for a specific eigenvalue λ_i yields the corresponding eigenvector, $\mathbf{q}_i$

Example (Two–dimensional white noise)

Let **R** be the correlation matrix of a two–sample vector of zero mean white noise

$$\mathbf{R} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

Determine the eigenvalues and eigenvectors.

Carrying out the analysis yields eigenvalues

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left[(R_{1,1}+R_{2,2}) \pm \sqrt{4R_{1,2}R_{2,1}+(R_{1,1}-R_{2,2})}\right] \\ &= \frac{1}{2}\left[(\sigma^2+\sigma^2) \pm \sqrt{0+(\sigma^2-\sigma^2)}\right] = \sigma^2 \end{aligned}$$

and eigenvectors

$$\mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: The eigenvectors are unit length (and orthogonal)

Eigen Properties

Property (eigenvalues of $\mathbf{R}^k$)

If $\lambda_1, \lambda_2, \cdots, \lambda_M$ are the eigenvalues of $\mathbf{R}$, then $\lambda_1^k, \lambda_2^k, \cdots, \lambda_M^k$ are the eigenvalues of $\mathbf{R}^k$.

Proof: Note $\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i$. Multiplying both sides by $\mathbf{R}$ $k-1$ times,

$$\mathbf{R}^k \mathbf{q}_i = \lambda_i \mathbf{R}^{k-1} \mathbf{q}_i = \lambda_i^k \mathbf{q}_i$$

Property (linear independence of eigenvectors)

The eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M$, of $\mathbf{R}$ are linearly independent, i.e.,

$$\sum_{i=1}^{M} a_i \mathbf{q}_i \neq \mathbf{0}$$

for all nonzero scalars $a_1, a_2, \cdots, a_M$.

Property (Correlation matrix eigenvalues are real & nonnegative)

The eigenvalues of **R** are real and nonnegative.

Proof:

$$\begin{aligned} \mathbf{R}\mathbf{q}_i &= \lambda_i \mathbf{q}_i \\ \Rightarrow \mathbf{q}_i^H \mathbf{R}\mathbf{q}_i &= \lambda_i \mathbf{q}_i^H \mathbf{q}_i \qquad \text{[pre–multiply by } \mathbf{q}_i^H \text{]} \\ \Rightarrow \lambda_i &= \frac{\mathbf{q}_i^H \mathbf{R}\mathbf{q}_i}{\mathbf{q}_i^H \mathbf{q}_i} \geq 0 \end{aligned}$$

Follows from the facts: **R** is positive semi-definite and $\mathbf{q}_i^H \mathbf{q}_i = |\mathbf{q_i}|^2 > 0$

Note: In most cases, **R** is positive definite and

$$\lambda_i > 0, \qquad i = 1, 2, \cdots, M$$

Property (Unique eigenvalues $\Rightarrow$ orthogonal eigenvectors)

If $\lambda_1, \lambda_2, \cdots, \lambda_M$ are unique eigenvalues of **R**, then the corresponding eigenvectors, $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M$, are orthogonal.

Proof:

$$\begin{aligned} \mathbf{R}\mathbf{q}_i &= \lambda_i \mathbf{q}_i \\ \Rightarrow \mathbf{q}_j^H \mathbf{R}\mathbf{q}_i &= \lambda_i \mathbf{q}_j^H \mathbf{q}_i \qquad (*) \end{aligned}$$

Also, since λ_j is real and **R** is Hermitian

$$\begin{aligned} \mathbf{R}\mathbf{q}_j &= \lambda_j \mathbf{q}_j \\ \Rightarrow \mathbf{q}_j^H \mathbf{R} &= \lambda_j \mathbf{q}_j^H \\ \Rightarrow \mathbf{q}_j^H \mathbf{R}\mathbf{q}_i &= \lambda_j \mathbf{q}_j^H \mathbf{q}_i \end{aligned}$$

Substituting the LHS from $(*)$

$$\Rightarrow \lambda_i \mathbf{q}_j^H \mathbf{q}_i = \lambda_j \mathbf{q}_j^H \mathbf{q}_i$$

Thus

$$\begin{aligned} \lambda_i \mathbf{q}_j^H \mathbf{q}_i &= \lambda_j \mathbf{q}_j^H \mathbf{q}_i \\ \Rightarrow (\lambda_i - \lambda_j)\mathbf{q}_j^H \mathbf{q}_i &= 0 \end{aligned}$$

Since $\lambda_1, \lambda_2, \cdots, \lambda_M$ are unique

$$\mathbf{q}_j^H \mathbf{q}_i = 0 \qquad i \neq j$$

$\Rightarrow \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M$ are orthogonal.

QED

Diagonalization of R

Objective: Find a transformation that transforms the correlation matrix into a diagonal matrix.

Let $\lambda_1, \lambda_2, \cdots, \lambda_M$ be unique eigenvalues of **R** and take $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M$ to be the M orthonormal eigenvectors

$$\mathbf{q}_i^H \mathbf{q}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Define $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]$ and $\boldsymbol{\Omega} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_M)$. Then consider

$$\mathbf{Q}^H \mathbf{R} \mathbf{Q} = \begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_M^H \end{bmatrix} \mathbf{R} [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]$$

$$
\begin{aligned}
\mathbf{Q}^H\mathbf{R}\mathbf{Q} &= \begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_M^H \end{bmatrix} \mathbf{R}[\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M] \\
&= \begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_M^H \end{bmatrix} [\lambda_1\mathbf{q}_1, \lambda_2\mathbf{q}_2, \cdots, \lambda_N\mathbf{q}_M] \\
&= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M \end{bmatrix} \\
\Rightarrow \mathbf{Q}^H\mathbf{R}\mathbf{Q} &= \Omega \quad \text{(eigenvector diagonalization of } \mathbf{R}\text{)}
\end{aligned}
$$

Property ($\mathbf{Q}$ is unitary)

$\mathbf{Q}$ is unitary, i.e., $\mathbf{Q}^{-1} = \mathbf{Q}^H$

Proof: Since the $\mathbf{q}_i$ eigenvectors are orthonormal

$$\begin{aligned}\mathbf{Q}^H\mathbf{Q} &= \begin{bmatrix}\mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_M^H\end{bmatrix}[\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M] = \mathbf{I} \\ \Rightarrow \mathbf{Q}^{-1} &= \mathbf{Q}^H\end{aligned}$$

Property (Eigen decomposition of $\mathbf{R}$)

The correlation matrix can be expressed as

$$\mathbf{R} = \sum_{i=1}^{M} \lambda_i \mathbf{q}_i \mathbf{q}_i^H$$

Proof: The correlation diagonalization result states

$$\mathbf{Q}^H\mathbf{R}\mathbf{Q} = \boldsymbol{\Omega}$$

Isolating **R** and expanding,

$$\begin{aligned}
\mathbf{R} &= \mathbf{Q}\boldsymbol{\Omega}\boldsymbol{Q}^H = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]\boldsymbol{\Omega}\begin{bmatrix} \mathbf{q}_1^H \\ \mathbf{q}_2^H \\ \vdots \\ \mathbf{q}_M^H \end{bmatrix} \\
&= [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]\begin{bmatrix} \lambda_1\mathbf{q}_1^H \\ \lambda_2\mathbf{q}_2^H \\ \vdots \\ \lambda_M\mathbf{q}_M^H \end{bmatrix} = \sum_{i=1}^{M}\lambda_i\mathbf{q}_i\mathbf{q}_i^H
\end{aligned}$$

Note: This also gives

$$\mathbf{R}^{-1} = (\mathbf{Q}^H)^{-1}\boldsymbol{\Omega}^{-1}\mathbf{Q}^{-1} = \mathbf{Q}\boldsymbol{\Omega}^{-1}\mathbf{Q}^H$$

where $\boldsymbol{\Omega}^{-1} = \text{diag}(1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_M)$

Aside (trace & determinant for matrix products)

Note $\text{trace}(\boldsymbol{A}) \triangleq \sum_i A_{i,i}$. Also,

$$\text{trace}(\boldsymbol{AB}) = \text{trace}(\boldsymbol{BA}) \qquad \text{similarly} \qquad \det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$$

Property (Determinant–Eigenvalue Relation)

The determinant of the correlation matrix is related to the eigenvalues as follows:

$$\det(\mathbf{R}) = \prod_{i=1}^{M} \lambda_i$$

Proof: Using $\mathbf{R} = \boldsymbol{Q\Omega Q}^H$ and the above,

$$\begin{aligned}\det(\mathbf{R}) &= \det(\boldsymbol{Q\Omega Q}^H) \\ &= \det(\mathbf{Q})\det(\mathbf{Q}^H)\det(\boldsymbol{\Omega}) = \det(\boldsymbol{\Omega}) = \prod^{M} \lambda_i\end{aligned}$$

Property (Trace–Eigenvalue Relation)

The trace of the correlation matrix is related to the eigenvalues as follows:

$$\text{trace}(\mathbf{R}) = \sum_{i=1}^{M} \lambda_i$$

Proof: Note

$$\begin{aligned}
\text{trace}(\mathbf{R}) &= \text{trace}(\mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^H) \\
&= \text{trace}(\mathbf{Q}^H\mathbf{Q}\boldsymbol{\Omega}) \\
&= \text{trace}(\boldsymbol{\Omega}) \\
&= \sum_{i=1}^{M} \lambda_i
\end{aligned}$$

QED

Definition (Normal Matrix)

A complex square matrix **A** is a normal matrix if

$$\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$$

That is, a matrix is normal if it commutes with its conjugate transpose.

Note

- All Hermitian symmetric matrices are normal
- Every matrix that can be diagonalized by the unitary transform is normal

Definition (Condition Number)

The condition number reflects how numerically well–conditioned a problem is, i.e, a low condition number $\Rightarrow$ well–conditioned; a high condition number $\Rightarrow$ ill–conditioned.

Definition (Condition Number for Linear Systems)

For a linear system

$$\mathbf{Ax} = \mathbf{b}$$

defined by a normal matrix **A**, the condition number is

$$\chi(\mathbf{A}) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum/minimum eigenvalues of **A**

Observations:

- Large eigenvalue spread $\Rightarrow$ ill–conditioned
- Small eigenvalue spread $\Rightarrow$ well–conditioned

Sensitivity Analysis: Suppose a system (filter) is related as:

$$\mathbf{R}\mathbf{w} = \mathbf{d}$$

where $\mathbf{w}$ defines the filter parameters; $\mathbf{R}$ and $\mathbf{d}$ are signal statistic matrices

$\Rightarrow$ Introduce small signal statistic perturbations

Let $\mathbf{R}$ and $\mathbf{d}$ are perturbed such that $||\delta\mathbf{R}||/||\mathbf{R}||$ and $||\delta\mathbf{d}||/||\mathbf{d}||$ are on the order $\epsilon \ll 1$

Result: A bound on the resulting parameter perturbations is given by

$$\frac{||\delta\mathbf{w}||}{||\mathbf{w}||} \leq \epsilon\chi(\mathbf{R}) = \epsilon\frac{\lambda_{\max}}{\lambda_{\min}}$$

Consequence: If $\mathbf{R}$ is ill conditioned, small changes in $\mathbf{R}$ or $\mathbf{d}$ can lead to big changes in $\mathbf{w}$

$\Rightarrow$ System sensitivity is related to eigenvalue spread

The discrete Karhmen-Loeve Transform (KLT)

Definition (The discrete Karhmen-Loeve Transform (KLT))

A M sample vector $\mathbf{x}(n)$ from the process $\{x(n)\}$ can be expressed as

$$\mathbf{x}(n) = \sum_{i=1}^{M} c_i(n)\mathbf{q}_i$$

where $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M$ are the orthonormal eigenvectors of the process correlation matrix, $\mathbf{R}$, and $c_1(n), c_2(n), \cdots, c_M(n)$ are a set of KLT coefficients.

- Signal is represented as a weighted sum of eigenvectors
- Need to determine the coefficients

Determining Coefficients: Write the expression in matrix form

$$\begin{aligned} \mathbf{x}(n) &= \sum_{i=1}^{M} c_i(n)\mathbf{q}_i \\ &= \mathbf{Q}\mathbf{c}(n) \qquad\qquad (*) \end{aligned}$$

where

$$\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_M]$$

and

$$\mathbf{c}(n) = [c_1(n), c_2(n), \cdots, c_M(n)]^T.$$

Solving $(*)$ for $\mathbf{c}(n)$:

$$\mathbf{c}(n) = \mathbf{Q}^{-1}\mathbf{x}(n) = \mathbf{Q}^H\mathbf{x}(n)$$

or

$$c_i(n) = \mathbf{q}_i^H\mathbf{x}(n)$$

Note: $c_i(n)$ is the projection of $\mathbf{x}(n)$ onto $\mathbf{q}_i$

Question: How related are the coefficients to reach other?

Answer: Consider the correlation between $c_i(n)$ terms

$$\begin{aligned}
\mathbf{R}_{\mathbf{c}(n)} &= E\{\mathbf{c}(n)\mathbf{c}^H(n)\} \\
&= E\{(\mathbf{Q}^H\mathbf{x}(n))(\mathbf{Q}^H\mathbf{x}(n))^H\} \\
&= E\{(\mathbf{Q}^H\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{Q}\} \\
&= \mathbf{Q}^H\mathbf{R}_x\mathbf{Q} \\
&= \boldsymbol{\Omega}
\end{aligned}$$

Result:

$$E\{c_i^*(n)c_j(n)\} = \begin{cases} \lambda_i & i=j \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ KLT transform coefficients are uncorrelated

- A desirable property – Why?

Question: Can we represent $\mathbf{x}(n)$ with fewer terms? If so, how do we minimize the representation error?

Approach: Use fewer terms in the KLT transform

$$\begin{aligned} \mathbf{x}(n) &= \sum_{i=1}^{M} c_i(n)\mathbf{q}_i \\ \Rightarrow \hat{\mathbf{x}}(n) &= \sum_{i=1}^{N} c_i(n)\mathbf{q}_i \quad N < M \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{x}(n) &= \hat{\mathbf{x}}(n) + \epsilon(n) \\ &= \sum_{i=1}^{N} c_i(n)\mathbf{q}_i + \sum_{i=N+1}^{M} c_i(n)\mathbf{q}_i \end{aligned}$$

Question: How do we minimize the representation error?

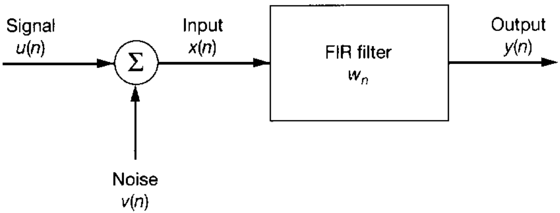
Approach: Analyzed and minimize the error power

The error power is given by

$$\begin{aligned}
\epsilon &= E\{\epsilon^H(n)\epsilon(n)\} \\
&= E\left\{\sum_{i=N+1}^{M} c_i^*(n)\mathbf{q}_i^H \sum_{j=N+1}^{M} c_j(n)\mathbf{q}_j\right\} \\
&= \sum_{i=N+1}^{M} E\{c_i^*(n)c_i(n)\} \qquad \text{[result of orthogonality]} \\
&= \sum_{i=N+1}^{M} \lambda_i \qquad \text{[from prior result]}
\end{aligned}$$

Result: To minimize the error select the $\mathbf{q}_i$ eigenvectors associated with M largest eigenvalues.

The Matched Filter



Objective: Find the optimal filter coefficients for two cases: (1) deterministic signals and (2) stochastic signals

Signal model:

$$x(n) = \underbrace{u(n)}_{\text{signal}} + \underbrace{v(n)}_{\text{noise}}$$

Definitions:

Filter parameters: $\mathbf{w} = [w_1, w_2, \cdots, w_M]^T$

Observation vector: $\mathbf{x}(n) = [x(n), x(n-1), \cdots, x(n-M+1)]^T$

Then using $x(n) = u(n) + v(n)$ and linearity,

$$\begin{aligned} y(n) &= \mathbf{w}^T\mathbf{x}(n) \\ &= \mathbf{w}^T\mathbf{u}(n) + \mathbf{w}^T\mathbf{v}(n) \\ &= y_s(n) + y_n(n) \end{aligned}$$

where

$$\begin{aligned} y_s(n) &= \mathbf{w}^T\mathbf{u}(n) \\ y_n(n) &= \mathbf{w}^T\mathbf{v}(n) \end{aligned}$$

- Consider deterministic and stochastic cases separately

Case 1: Matched Filters for Deterministic Signals

Approach:

Analyzed output SNR

Set filter coefficients to maximize SNR

The SNR at time n can be defined as

$$\text{SNR} = \frac{|y_s(n)|^2}{E\{|y_n(n)|^2\}}$$

where

$$\begin{aligned} y_s(n) &= \mathbf{w}^T\mathbf{u}(n) \qquad \text{[Deterministic]} \\ y_n(n) &= \mathbf{w}^T\mathbf{v}(n) \qquad \text{[Stochastic]} \end{aligned}$$

Note: Case assumes $u(n)$ is deterministic

Using $y_s(n) = \mathbf{w}^T\mathbf{u}(n)$ and $y_n(n) = \mathbf{w}^T\mathbf{v}(n)$,

$$
\begin{aligned}
\text{SNR} &= \frac{|y_s(n)|^2}{E\{|y_n(n)|^2\}} \\
&= \frac{(\mathbf{w}^T\mathbf{u}(n))^*(\mathbf{u}^T(n)\mathbf{w})}{E\{(\mathbf{w}^T\mathbf{v}(n))^*(\mathbf{v}^T(n)\mathbf{w})\}} \\
&= \frac{\mathbf{w}^H\mathbf{u}^*(n)\mathbf{u}^T(n)\mathbf{w}}{E\{\mathbf{w}^H\mathbf{v}^*(n)\mathbf{v}^T(n)\mathbf{w}\}} \\
&= \frac{\mathbf{w}^H\mathbf{u}^*(n)\mathbf{u}^T(n)\mathbf{w}}{\mathbf{w}^H\mathbf{R}_v\mathbf{w}}
\end{aligned}
$$

- To analyze further, restrict the noise statistics

Suppose $v(n)$ is zero mean and uncorrelated, $\mathbf{R_v} = \sigma_v^2 \mathbf{I}$. Then

$$\begin{aligned} \text{SNR} &= \frac{\mathbf{w}^H \mathbf{u}^*(n) \mathbf{u}^T(n) \mathbf{w}}{\mathbf{w}^H \mathbf{R}_v \mathbf{w}} \\ &= \frac{|\mathbf{w}^H \mathbf{u}^*(n)|^2}{\sigma_v^2 \mathbf{w}^H \mathbf{w}} \end{aligned}$$

If we restrict $|\mathbf{w}|^2 = 1$, then the final result is

$$\text{SNR} = \frac{|\mathbf{w}^H \mathbf{u}^*(n)|^2}{\sigma_v^2}$$

Definition (Cauchy–Schwarz Inequality (1821 disc.; 1859 cont.))

Cauchy-Schwarz's inequality states (matrix case)

$$|\mathbf{A}^H\mathbf{B}|^2 \leq (\mathbf{A}^H\mathbf{A})(\mathbf{B}^H\mathbf{B})$$

with equality only if $\mathbf{A} = k\mathbf{B}$, where k is a constant

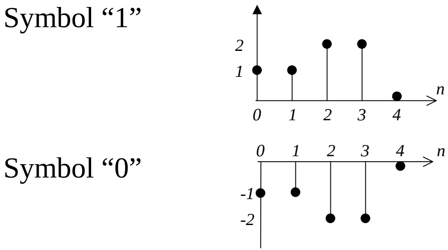
Thus

$$\begin{aligned}
\text{SNR} &= \frac{|\mathbf{w}^H\mathbf{u}^*(n)|^2}{\sigma_v^2} \\
&\leq \frac{(\mathbf{w}^H\mathbf{w})(\mathbf{u}^T(n)\mathbf{u}^*(n))}{\sigma_v^2} \\
&= \frac{\mathbf{u}^H(n)\mathbf{u}(n)}{\sigma_v^2} = \frac{|\mathbf{u}(n)|^2}{\sigma_v^2}
\end{aligned}$$

Result: The SNR is maximized when $\mathbf{w} = k\mathbf{u}^*(n)$

Example

A communications system sends out one of two signals



We receive

$$y(n) = u_i(n) + v(n)$$

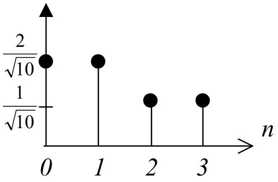
where $v(n)$ is i.i.d. Gaussian.

Objective: Find the optimal $\mathbf{w}$ (assume $n = 3$)

Set $\mathbf{w} = k\mathbf{u}_1(n)$ where k is such that $|\mathbf{w}|^2 = 1$. Thus,

$$\mathbf{w} = \mathbf{u}_1(n)/|\mathbf{u}_1(n)| = \frac{1}{\sqrt{10}}[2, 2, 1, 1]^T$$

The resulting impulse response of the filter is



Observation: Optimal filter impulse response is sent (desired) signal time–reversed & normalized

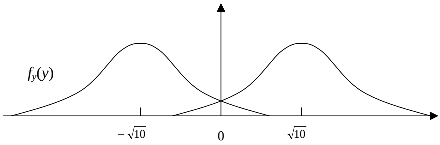
Determine $y_s(n)$ for signal "1" sent

$$y_s(n) = \mathbf{w}^T\mathbf{u}_1(n) = \frac{|\mathbf{u}_1(n)|^2}{|\mathbf{u}_1(n)|} = \sqrt{10}$$

For signal "0"

$$y_s(n) = \mathbf{w}^T \mathbf{u}_0(n) = \frac{-|\mathbf{u}_0(n)|^2}{|\mathbf{u}_1(n)|} = -\sqrt{10}$$

Received signal distributions:



The SNR for the optimized system is

$$\text{SNR} = \frac{|\mathbf{u}_1(n)|^2}{\sigma_v^2} = \frac{10}{\sigma_v^2}$$

Case 2: Matched Filter for Stochastic Signals

As before

$$\begin{aligned} y(n) &= \mathbf{w}^T \mathbf{x}(n) \\ &= \mathbf{w}^T \mathbf{u}(n) + \mathbf{w}^T \mathbf{v}(n) \\ &= y_s(n) + y_n(n) \end{aligned}$$

Now define the SNR as

$$\text{SNR} = \frac{E\{|y_s(n)|^2\}}{E\{|y_n(n)|^2\}} = \frac{\mathbf{w}^H \mathbf{R}_u \mathbf{w}}{\mathbf{w}^H \mathbf{R}_v \mathbf{w}}$$

As before, if $\mathbf{R}_v = \sigma_v^2 \mathbf{I}$ and $|\mathbf{w}|^2 = 1$

$$\text{SNR} = \frac{\mathbf{w}^H \mathbf{R}_u \mathbf{w}}{\sigma_v^2}$$

Since

$$\text{SNR} = \frac{\mathbf{w}^H \mathbf{R}_u \mathbf{w}}{\sigma_v^2},$$

maximizing the SNR is equivalent to maximizing $|\mathbf{w}^H \mathbf{R}_u \mathbf{w}|$

Recall from Cauchy–Schwarz,

$$|\mathbf{A}^H \mathbf{B}|^2 \leq (\mathbf{A}^H \mathbf{A})(\mathbf{B}^H \mathbf{B})$$

with equality only if $\mathbf{A} = k\mathbf{B}$

Let $\boldsymbol{A}^H = \mathbf{w}^H \mathbf{R}_u$ and $\mathbf{B} = \mathbf{w}$ and apply Cauchy–Schwarz

Result: $|\mathbf{w}^H \mathbf{R}_u \mathbf{w}|$ is maximized for $\mathbf{A} = k\mathbf{B}$

$$\Rightarrow \mathbf{R}_u \mathbf{w} = k\mathbf{w}$$

$\Rightarrow$ k is an eigenvalue and $\mathbf{w}$ is an eigenvector!

- $\mathbf{w} = \mathbf{q}_i$ and $k = \lambda_i$, where $\mathbf{R}_u \mathbf{q}_i = \lambda_i \mathbf{q}_i$

Question: How do we choose which eigenvector?

Using $\mathbf{w} = \mathbf{q}_i$, or $\mathbf{w}^H = \mathbf{q}_i^H$, and $\mathbf{R}_u\mathbf{w} = \mathbf{R}_u\mathbf{q}_i = \lambda_i\mathbf{q}_i$ gives

$$\begin{aligned} \text{SNR} &= \frac{\mathbf{w}^H\mathbf{R}_u\mathbf{w}}{\sigma_v^2} \\ &= \frac{\mathbf{q}_i^H\lambda_i\mathbf{q}_i}{\sigma_v^2} \qquad \text{[orthonormal eigenvectors]} \\ &= \frac{\lambda_i}{\sigma_v^2} \end{aligned}$$

Result: The SNR is maximized when $\mathbf{w} = \mathbf{q}_{\max}$, the eigenvector associated with the largest eigenvalue of $\mathbf{R}_u$.

- When using the optimal filter weights ($\mathbf{w} = \mathbf{q}_{\max}$), the resulting (maximized) SNR is

$$\text{SNR} = \frac{\lambda_{\max}}{\sigma_v^2}$$