ELEG-636: Statistical Signal Processing

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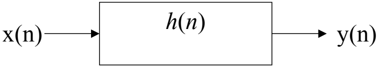
Transformation by a Linear System

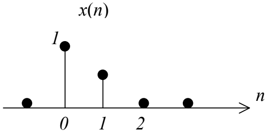
Consider a LTI system characterized by impulse response $h(n)$

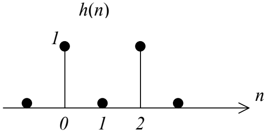


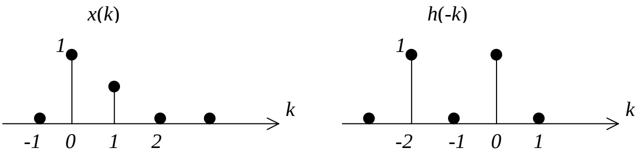
$$\Rightarrow y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

Example (Deterministic Case)

Suppose $x(n)$ and $h(n)$ are

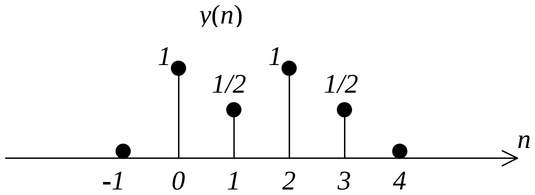






$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$$

$$\begin{aligned} \Rightarrow y(n) &= 0 \quad n < 0 \\ y(0) &= 1 \\ y(1) &= 1/2 \\ y(2) &= 1 \\ y(3) &= 1/2 \\ y(n) &= 0 \quad n \geq 4 \end{aligned}$$



Stochastic Case: If $x(n)$ is a stationary RV

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
\Rightarrow E\{y(n)\} &= \sum_{k=-\infty}^{\infty} h(k)E\{x(n-k)\} \\
\Rightarrow m_y &= m_x \sum_{k=-\infty}^{\infty} h(k)
\end{aligned}
$$

Next, evaluate the cross-correlation

$$
\begin{aligned}
y(n) &= \sum_k h(n-k)x(k) \\
\Rightarrow E\{y(n)x^*(n-l)\} &= \sum_k h(n-k)E\{x(k)x^*(n-l)\} \\
\Rightarrow r_{yx}(l) &= \sum_k h(n-k)r_x(k-n+l) \quad [\text{let } m = n-k] \\
&= \sum_m h(m)r_x(l-m) \\
\Rightarrow r_{yx}(l) &= h(l) * r_x(l)
\end{aligned}
$$

Known results:

$$
\begin{aligned}
r_{yx}(l) &= h(l) * r_x(l) \qquad \text{[new result]}\\
r_x(l) &= r_x^*(-l) \qquad \text{[prior result]}\\
r_{yx}(l) &= r_{xy}^*(-l) \qquad \text{[prior result]}
\end{aligned}
$$

Using these results:

$$
\begin{aligned}
r_{xy}^*(-l) &= r_{yx}(l)\\
&= h(l) * r_x(l)\\
\Rightarrow r_{xy}^*(l) &= h(-l) * r_x(-l)\\
&= h(-l) * r_x^*(l)\\
\Rightarrow r_{xy}(l) &= h^*(-l) * r_x(l)
\end{aligned}
$$

Note: $r_{yx}(l) = h(l) * r_x(l)$ while $r_{xy}(l) = h^*(-l) * r_x(l)$

Lastly, examine $r_y(l)$

$$\begin{aligned}
y(n) &= \sum_k h(n-k)x(k) \\
\Rightarrow E\{y(n)y^*(n-l)\} &= \sum_k h(n-k)E\{x(k)y^*(n-l)\} \\
\Rightarrow r_y(l) &= \sum_k h(n-k)r_{xy}(k-n+l) \quad [\text{let } m = n-k] \\
&= \sum_m h(m)r_{xy}(l-m) \\
&= h(l) * r_{xy}(l)
\end{aligned}$$

or, substituting back using $r_{xy}(l) = h^*(-l) * r_x(l)$

$$\Rightarrow r_y(l) = h(l) * h^*(-l) * r_x(l)$$

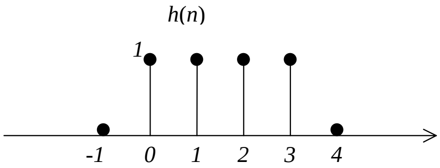
Summary of Results (Correlations for a LTI System)

$$
\begin{aligned}
r_{yx}(l) &= h(l) * r_x(l) \\
r_{xy}(l) &= h^*(-l) * r_x(l) \\
r_y(l) &= h(l) * r_{xy}(l) \\
&= h(l) * h^*(-l) * r_x(l)
\end{aligned}
$$

Note: Similar results hold for the covariance.

Example

Suppose the impulse response of a LTI is



Question: If the system is driven by white zero mean noise, what are m_y, $r_{yx}(l)$, $r_{xy}(l)$, and $r_y(l)$?

Consider m_y first:

$$m_y = E\{y(n)\} = m_x \sum_k h(k) = 0$$

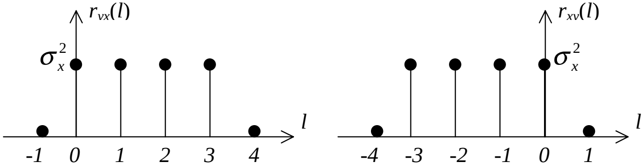
Also note $r_x(l) = \sigma_x^2 \delta(l)$. Thus,

$$\begin{aligned} r_{yx}(l) &= h(l) * r_x(l) \\ &= h(l) * \sigma_x^2 \delta(l) \\ &= \sigma_x^2 h(l) \end{aligned}$$

By the symmetry property

$$r_{xy}(l) = r_{yx}^*(-l) = \sigma_x^2 h^*(-l)$$

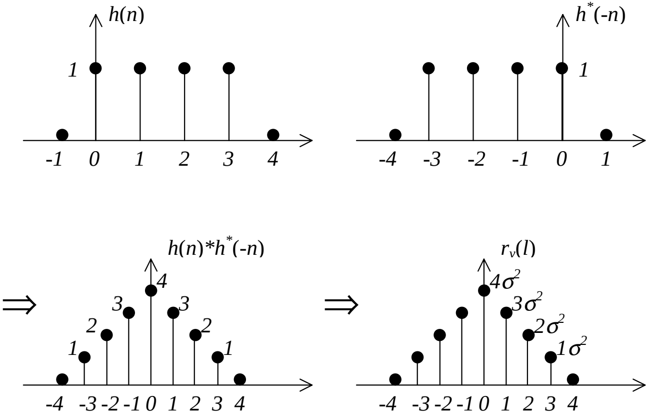
Plotting the results:



Lastly

$$
\begin{aligned}
r_y(l) &= h(l) * h^*(-l) * r_x(l) \\
&= \sigma_x^2 (h(l) * h^*(-l))
\end{aligned}
$$

Generating the result graphically:



Power Spectral Density

Definition (Power Spectral Density)

The autocorrelation function of a wide-sense stationary process and the power spectral density form a Fourier transform pair

$$\begin{aligned} S(\omega) &= \sum_{l=-\infty}^{\infty} r(l)e^{-j\omega l} \qquad -\pi < \omega \leq \pi \\ r(l) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)e^{j\omega l}d\omega \quad l = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

PSD Properties:

- The PSD is periodic, with period 2π

$$S(\omega + 2k\pi) = S(\omega) \qquad k = 0, \pm 1, \pm 2, \ldots$$

$\Rightarrow$ All information contained in the Nyquist interval $[-\pi, \pi]$

- The PSD of a stationary process is real

  To prove this, note

$$
\begin{aligned}
S(\omega) &= \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k} \\
&= r(0) + \sum_{k=1}^{\infty} r(k)e^{-j\omega k} + \sum_{k=-\infty}^{-1} r(k)e^{-j\omega k} \\
&= r(0) + \sum_{k=1}^{\infty} r(k)e^{-j\omega k} + \sum_{k=1}^{\infty} r(-k)e^{j\omega k} \\
&= r(0) + \sum_{k=1}^{\infty} r(k)e^{-j\omega k} + r^{*}(k)e^{j\omega k} \\
&= r(0) + 2\sum_{k=1}^{\infty} \text{Re}[r(k)e^{-j\omega k}]
\end{aligned}
$$

  QED

- For real-valued sequences, $S(\omega) \geq 0$ and has even symmetry (prove yourself)
- The PSD is (within a constant) a density describing the concentration of power

Proof: Relate area under the PSD to signal power. Thus, let $l = 0$ in the inverse transform,

$$\begin{aligned} r(l) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)e^{j\omega l}d\omega \\ \Rightarrow r(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)d\omega \\ \Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)d\omega &= E\{x^2(n)\} \quad \text{[signal power]} \end{aligned}$$

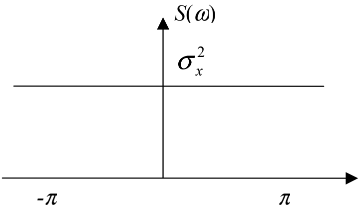
$\Rightarrow$ Area under the PSD is (scaled by the $1/2\pi$ constant) equal to the signal power. Moreover, it characterizes the signal power by frequency

Example

If $x(n)$ is zero mean i.i.d. with variance σ_x^2, what is the PSD?

In the case,

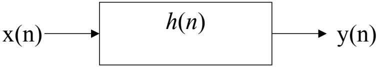
$$\begin{aligned} r(l) &= \sigma_x^2 \delta(l) \\ \Rightarrow S(\omega) &= \sum_l r(l) e^{-j\omega l} \\ &= \sum_l \sigma_x^2 \delta(l) e^{-j\omega l} = \sigma_x^2 \qquad -\pi < \omega \leq \pi \end{aligned}$$



Note: Flat spectrums are said to be white because they have a uniform distribution of power

Transmission Through a Linear System

Let $x(n)$ be a stationary discrete time process that is passed through a LTI system characterized by $h(n)$.



Then by known properties and time/frequency relations

$$
\begin{aligned}
r_{yx}(l) &= h(l) * r_x(l) \\
\Rightarrow S_{yx}(\omega) &= H(\omega)S_x(\omega)
\end{aligned}
$$

where

$$H(\omega) = \sum_l h(l)e^{-j\omega l}$$

Also note that

$$\begin{aligned} FT\{h^*(-l)\} &= \sum_l h^*(-l)e^{-j\omega l} \\ &= \sum_l h^*(l)e^{j\omega l} \\ &= \left[\sum_l h(l)e^{-j\omega l}\right]^* = H^*(\omega) \end{aligned}$$

Thus

$$\begin{aligned} r_y(l) &= h(l) * h^*(-l) * r_x(l) \\ \Rightarrow S_y(\omega) &= H(\omega)H^*(\omega)S_x(\omega) \\ &= |H(\omega)|^2 S_x(\omega) \end{aligned}$$

Note: Only the PSD magnitude is affected by the system

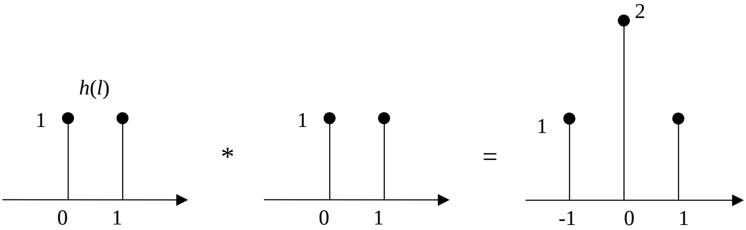
Example

Let $x(n)$ have correlation $r_x(l) = (\frac{1}{2})^{|l|}$ and be the input to a LTI system with impulse response

$$h(n) = \delta(n) + \delta(n-1).$$

Find $r_y(l)$ and $S_y(\omega)$

Note $r_y(l) = h(l) * h(-l) * r_x(l)$



Thus

$$\begin{aligned} h(l) * h(-l) &= \delta(l+1) + 2\delta(l) + \delta(l-1) \\ \Rightarrow r_y(l) &= h(l) * h(-l) * r_x(l) \\ &= [\delta(l+1) + 2\delta(l) + \delta(l-1)] * \left(\frac{1}{2}\right)^{|l|} \\ &= 2\left(\frac{1}{2}\right)^{|l|} + \left(\frac{1}{2}\right)^{|l+1|} + \left(\frac{1}{2}\right)^{|l-1|} \end{aligned}$$

The result is a consequence of convolving with shifted delta functions

Next, compute $S_x(\omega)$ and $|H(\omega)|^2$

Aside: Recall for $|p| < 1$,

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \quad \text{and} \quad \sum_{k=1}^{\infty} p^k = \frac{p}{1-p}$$

Given $r_x(k) = (\frac{1}{2})^{|k|}$

$$\begin{aligned} S_x(\omega) &= \sum_k \left(\frac{1}{2}\right)^{|k|} e^{-j\omega k} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{-j\omega k} + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k} e^{j\omega k} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}} = \frac{\frac{3}{4}}{\frac{5}{4}-\cos(\omega)} \end{aligned}$$

Also

$$\begin{aligned} H(\omega) &= \sum_k h(k)e^{-j\omega k} \\ &= \sum_k [\delta(k) + \delta(k-1)]e^{-j\omega k} \\ &= 1 + e^{-j\omega} \end{aligned}$$

Taking the magnitude,

$$\begin{aligned} |H(\omega)|^2 &= (1+e^{-j\omega})(1+e^{-j\omega})^* \\ &= 1+e^{-j\omega}+e^{j\omega}+1 \\ &= 2+2\cos(\omega) \\ &= 2(1+\cos(\omega)) \end{aligned}$$

Thus the output PSD is

$$\begin{aligned} S_y(\omega) &= |H(\omega)|^2 S_x(\omega) \\ &= 2(1+\cos(\omega))\left(\frac{\frac{3}{4}}{\frac{5}{4}-\cos(\omega)}\right) \\ &= \frac{\frac{3}{2}(1+\cos(\omega))}{\frac{5}{4}-\cos(\omega)} \end{aligned}$$

Note: The system only affects the magnitude and $S_y(\omega)$ has even symmetry. Question: Where is the power concentrated?

Spectral Factorization

Suppose a system is described by the difference equation

$$\sum_{k=0}^{M} a_k x(n-k) = \sum_{l=0}^{K} b_l v(n-l)$$

Then taking the z-transform

$$\sum_{k=0}^{M} a_k z^{-k} X(z) = \sum_{l=0}^{K} b_l z^{-l} V(z)$$

or

$$\frac{X(z)}{V(z)} \triangleq H(z) = \frac{\sum_{l=0}^{K} b_l z^{-l}}{\sum_{k=0}^{M} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

Note: $H(z)$ is defined as the transfer function

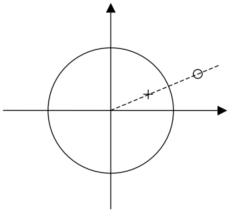
All pass systems

Suppose a stable first-order system has a transfer function

$$H(z) = \frac{1 - a^*z}{z - a}$$

Note: System has a zero at $z = 1/a^*$ and a pole at $z = a$.

Let $a = re^{j\theta}$. Then $\frac{1}{a^*} = \frac{1}{re^{-j\theta}} = \frac{1}{r}e^{j\theta}$. For a stable ($r < 1$) system:



Note: Pole and zero are at the same angle

Evaluating $H(\omega) = H(z)|_{z=e^{j\omega}}$

$$H(\omega) = \frac{1 - a^* e^{j\omega}}{e^{j\omega} - a} = \frac{e^{j\omega}(e^{-j\omega} - a^*)}{e^{j\omega} - a}$$

$$\Rightarrow |H(\omega)| = |e^{j\omega}| \frac{|e^{-j\omega} - a^*|}{|e^{j\omega} - a|} = 1 \quad \text{[All pass filter]}$$

General Result – All Pass Systems

Systems of the form

$$H_{\text{ap}}(z) = \prod_{i=1}^{N} \frac{(1 - a_i^* z)}{(z - a_i)}$$

are all pass. For stability, $|a_i| < 1$.

Minimum Phase Systems

A system is minimum phase if it is causal and has all its poles and zeros inside the unit circle

- Minimum phase systems are stable
- If $H(z)$ is minimum phase, then $H^{-1}(z)$ is minimum phase and stable

Maximum Phase Systems

A system is maximum phase if it has all its zeros outside the unit circle

- The inverse of a maximum phase systems is not stable

Mixed Phase Systems

A system is mixed phase if it has zeros inside and outside the unit circle

- The inverse of a mixed phase systems is not stable

Factorization Result

Any rational $H(z)$ can be factored as

$$H(z) = \underbrace{H_{\min}(z)}_{\text{Min. phase}} \; \underbrace{H_{\text{ap}}(z)}_{\text{All pass}}$$

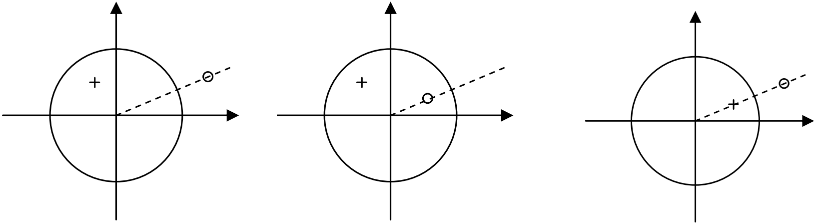
Consider a non–minimum phase $H(z)$ with a (single) zero outside the unit circle, at $z = 1/a^*$ where $|a| < 1$

$$\begin{aligned} H(z) &= H_1(z)(1 - a^* z) \qquad \text{[factor out non–min. phase comp.]} \\ &= H_1(z)(1 - a^* z)\left(\frac{z-a}{z-a}\right) \\ &= \underbrace{H_1(z)(z-a)}_{\text{Minimum phase}} \underbrace{\left(\frac{1 - a^* z}{z-a}\right)}_{\text{All pass}} \end{aligned}$$

Note: Magnitude response is determined by the minimum phase component, while both terms contribute to phase response

Example

Consider a stable system with a single zero outside the unit circle (maximum phase system):



$H(z)$	$H_{min}(z)$	$H_{ap}(z)$
System with 1 zero outside UC	System with reflected zero; min phase	System with original zero & reflected/canceling pole; all pass

Note: The zero in $H_{\min}(z)$ and the pole in $H_{\text{ap}}(z)$ cancel each other out