ELEG-636: Statistical Signal Processing

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Spring 2010

Course Objectives & Structure

Objective: Given a discrete time sequence $\{x(n)\}$, develop

- Statistical and spectral signal representation
- Filtering, prediction, and system identification algorithms
- Optimization methods that are
  - Statistical
  - Adaptive

Course Structure:

- Weekly lectures [notes: www.ece.udel.edu/∼arce]
- Periodic homework (theory & Matlab implementations) [15%]
- Midterm & Final examinations [85%]

Textbook:

- Haykin, Adaptive Filter Theory.

Course Objectives & Structure

- Broad Applications in Communications, Imaging, Sensors.
- Emerging application in
  - Brain-imaging techniques
  - Brain-machine interfaces,
  - Implantable devices.
- Neurofeedback presents real-time physiological signals from MRIs in a visual or auditory form to provide information about brain activity. These signals are used to train the patient to alter neural activity in a desired direction.
- Traditionally, feedback using EEGs or other mechanisms has not focused on the brain because the resolution is not good enough.

Outline

Outline I

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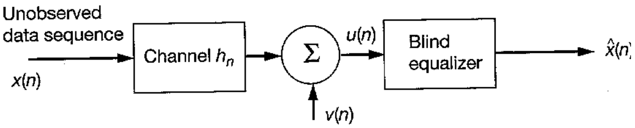
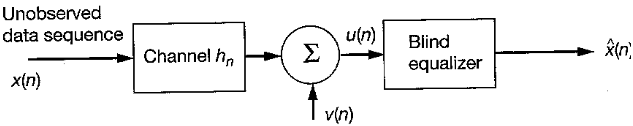
- This known signal training is not always appropriate, e.g., in mobile communications
  - Cost is too high (time/bandwidth)
  - Multipathing or other time varying interference
- In such cases, we must use blind equalization


Unobserved
data sequence
x(n)
Channel h_n
Σ
u(n)
v(n)
Blind
equalizer
x̂(n)



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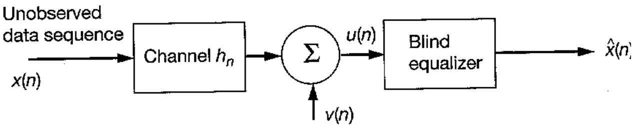
- Channel introduces distortion (dominant)



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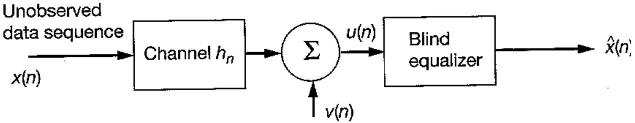

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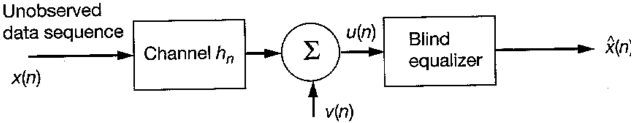
    - Multilevel pulse amplitude modulation (M-ary PAM)



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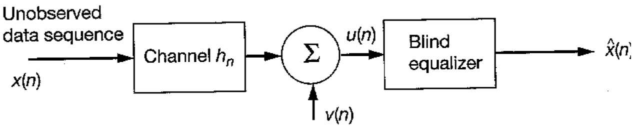

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  - Dominating interference is due to intersymbol interference (ISI) from channel distortion
    $\Rightarrow$ The noise is ignored

Also assume that:

$$h \neq 0 \text{ for } n < 0 \quad \text{(noncausal)}$$

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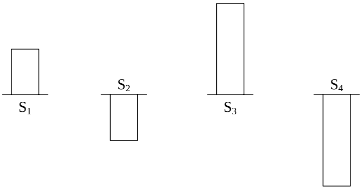
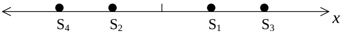
Example (4-ary PAM modulation)

A 4–ary PAM modulation scheme uses 4 signals



Or



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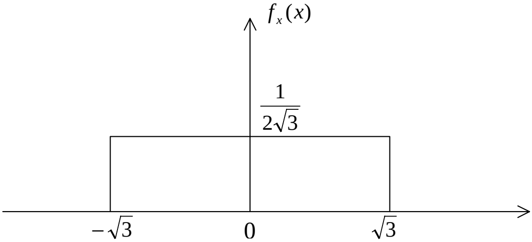
2. The pdf of $x(n)$ is symmetric and uniform



Deconvolution Objective: If $\{w_i\}$ are the coefficients of the ideal inverse filter, then

$$\sum_i w_i h_{l-i} = \delta_l = \begin{cases} 1 & l = 0 \\ 0 & \text{else} \end{cases}$$

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Problem: h_n is not known $\Rightarrow$ the exact inverse can not be used

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Note $v(n) = \sum_i [\hat{w}_i(n) - w_i] u(n-i)$ is the residual ISI

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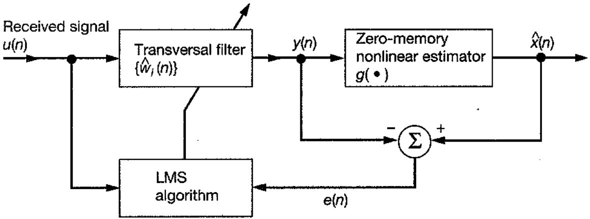

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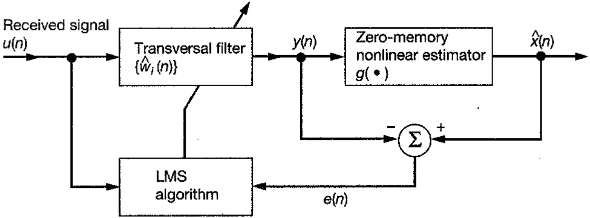

- The nonlinear estimate $g[y(n)]$ can be used to update the equalizer to produce a better estimate at time $n+1$

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- The cost function can have numerous local minima

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where recall

$$\begin{aligned} w_i &\equiv \text{perfect equalizer weights} \\ \hat{w}_i(n) &\equiv \text{finite approximate equalizer with } \hat{w}_i(n) = 0 \text{ for } |i| > L \end{aligned}$$

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- $\nabla(n)$ is the residual impulse response of the channel due to imperfect equalization
- $\nabla(n)$ is small in value, but long and oscillatory

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Observation: $v(n) = \sum_{l} x(l)\nabla(n-l)$ is a weighted sum of i.i.d. RVs

Since $\nabla(n)$ is long and oscillatory

$$E\{v(n)v(n-j)\} = \sum_l \nabla(n-l)\nabla(n-l-j)$$

tends to average to 0 for $j \neq 0$. Thus

$$E\{v(n)v(n-j)\} = \begin{cases} \sigma^2 & j = 0 \\ 0 & j \neq 0 \end{cases}$$

where

$$\sigma^2 = \sigma^2(n) = \sum_l \nabla^2(n-l)$$

Observation: $v(n) = \sum_l x(l)\nabla(n-l)$ is a weighted sum of i.i.d. RVs $\Rightarrow$ $v(n)$ is Gaussian (central limit theorem)

Lastly, consider the cross correlation of $x(n)$ and $v(n)$,

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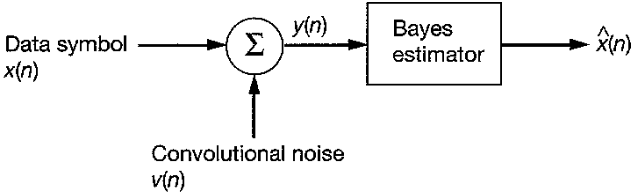
Observation: Thus we can say $x(n)$ and $v(n)$ are essentially independent

Finally, consider the nonlinear estimation of $x(n)$



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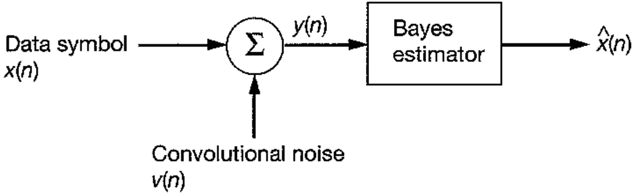

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- $x(n)$ is uniformly distributed with zero mean and unit variance

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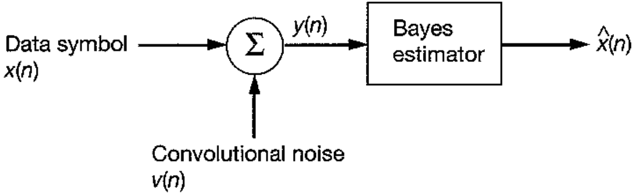

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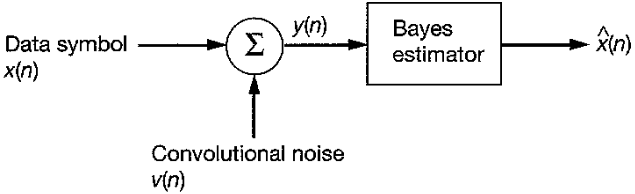

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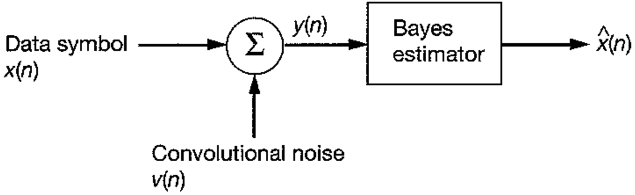

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Estimation Approach: Utilize Bayes estimation, which exploits knowledge of the distributions

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where we can use the fact that

$$f_x(x|y) = \frac{f_y(y|x) f_x(x)}{f_y(y)}$$

Employ the current model,

$$y = c_0 x + v$$

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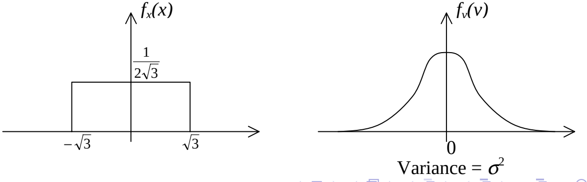

Evaluating this yields (Berllini, 1988)

$$\hat{x} = \frac{1}{c_0 y} - \frac{\sigma}{c_0} \frac{Z(y_1) - Z(y_2)}{Q(y_1) - Q(y_2)}$$

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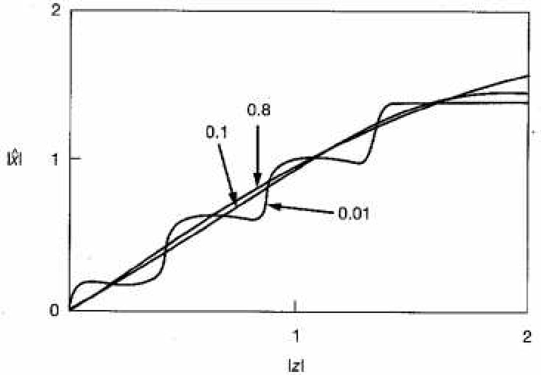
and

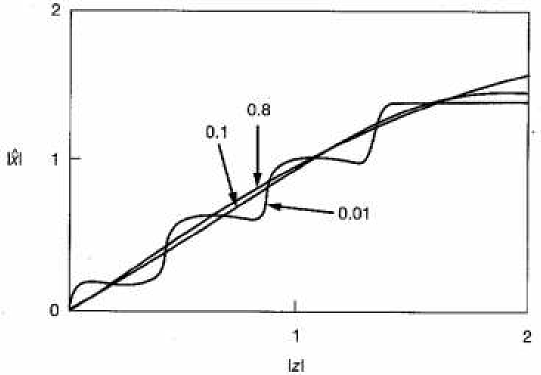
$$Z(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad \text{[normalized Gaussian pdf]}$$

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-\frac{u^2}{2}} du \qquad \text{[normalized Gaussian cdf]}$$

Results for an 8 level PAM system:



Results for an 8 level PAM system:



- Plotted for three different noise to (signal + noise) ratios
- Note that this tends to a step function as the noise goes to zero

Implementation Point: When the blind equalizer has converged, the algorithm is switched to decision-directed mode.

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In the case of binary symbols,

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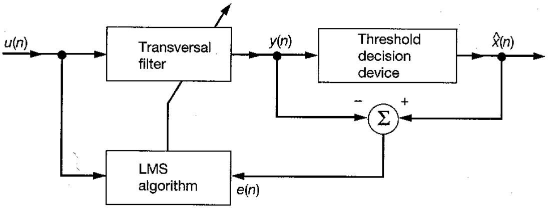
which results in the sample estimate

$$\Rightarrow \hat{x}(n) = \mathrm{sgn}(y(n)) = \begin{cases} +1 & \text{if } y(n) \geq 0 \\ -1 & \text{else} \end{cases}$$

Final System:



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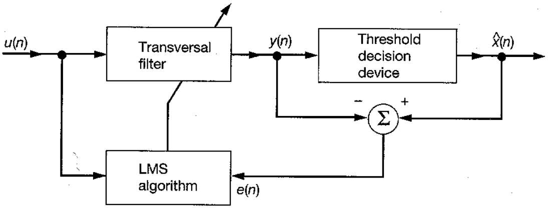

Final Observation: This system works well as long as what?