ELEG-636: Statistical Signal Processing

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Course Objectives & Structure

Objective: Given a discrete time sequence $\{x(n)\}$, develop

- Statistical and spectral signal representation
- Filtering, prediction, and system identification algorithms
- Optimization methods that are
    - Statistical
    - Adaptive

Course Structure:

- Weekly lectures [notes: www.ece.udel.edu/∼arce]
- Periodic homework (theory & Matlab implementations) [15%]
- Midterm & Final examinations [85%]

Textbook:

- Haykin, Adaptive Filter Theory.

Course Objectives & Structure

- Broad Applications in Communications, Imaging, Sensors.
- Emerging application in
  - Brain-imaging techniques
  - Brain-machine interfaces,
  - Implantable devices.
- Neurofeedback presents real-time physiological signals from MRIs in a visual or auditory form to provide information about brain activity. These signals are used to train the patient to alter neural activity in a desired direction.
- Traditionally, feedback using EEGs or other mechanisms has not focused on the brain because the resolution is not good enough.

Method of Least Squares (LS)

Definition (Method of Least Squares (LS))

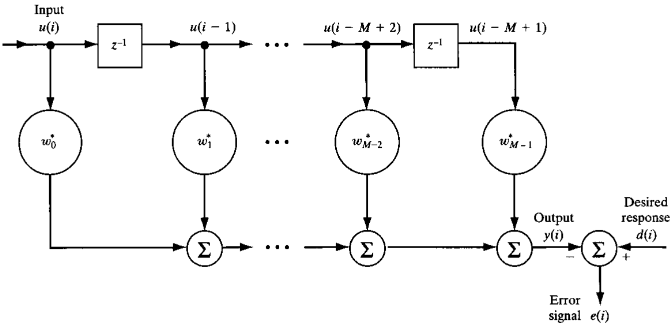
Motivation: Develop a general method for optimally adjusting parameters to model observed data

Solution: Set the sum of squared residuals (errors) as the performance criteria and restrict the model to be linear

- The LS filtering method is a deterministic method
- Can be applied to linear and nonlinear systems
- LS corresponds to the ML criterion if the errors have a normal distribution
- The method is related to linear regression
- Optimization procedure results in a LS best fit for a filter over the observed (training) samples

Historical Note: Gauss developed LS in 1795 at the age of 18

Consider the linear transversal filter



and a fixed number of observed samples: $i = 1, 2, \cdots, N$.

- M – the number of taps in the filter
- $\{x(i)\}$ – input sequence
- $\{d(i)\}$ – desired output sequence

Objective: Set the tap weights to minimize the sum of squared errors

$$\epsilon(\mathbf{w}) = \sum_{i=M}^{N} |e(i)|^2$$

Let

$$\begin{aligned} \mathbf{w} &= [w_0, w_1, \cdots, w_{M-1}]^T \qquad \text{[weight vector]} \\ \mathbf{x}(i) &= [x(i), x(i-1), \cdots, x(i-M+1)]^T, M \leq i \leq N \quad \text{[obs. vect.]} \end{aligned}$$

The error at time i is

$$e(i) = d(i) - \mathbf{w}^H \mathbf{x}(i)$$

The full set of error values can be compiled into a vector

Define the $(N-M+1)\times 1$ vectors:

$$\begin{aligned}\boldsymbol{\epsilon}^H &= [e(M), e(M+1), \cdots, e(N)] \qquad \text{[error vector]}\\ \mathbf{d}^H &= [d(M), d(M+1), \cdots, d(N)] \qquad \text{[desired vector]}\end{aligned}$$

Denoting the filter output as $\hat{d}(i)$ and using vector form:

$$\begin{aligned}\hat{\mathbf{d}}^H &= [\hat{d}(M), \hat{d}(M+1), \cdots, \hat{d}(N)]\\ &= [\mathbf{w}^H\mathbf{x}(M), \mathbf{w}^H\mathbf{x}(M+1), \cdots, \mathbf{w}^H\mathbf{x}(N)]\\ &= \mathbf{w}^H[\mathbf{x}(M), \mathbf{x}(M+1), \cdots, \mathbf{x}(N)]\\ &= \mathbf{w}^H\mathbf{A}^H\end{aligned}$$

where

$$\mathbf{A}^H = [\mathbf{x}(M), \mathbf{x}(M+1), \cdots, \mathbf{x}(N)]$$

is the observation data matrix

Expanding the data matrix

$$\begin{aligned}\mathbf{A}^H &= [\mathbf{x}(M), \mathbf{x}(M+1), \cdots, \mathbf{x}(N)] \\ &= \begin{bmatrix} x(M) & x(M+1) & \cdots & x(N) \\ x(M-1) & x(M) & \cdots & x(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ x(1) & x(2) & \cdots & x(N-M+1) \end{bmatrix}\end{aligned}$$

$\Rightarrow$ $\mathbf{A}^H$ is a $M \times (N-M+1)$ rectangular toplitz matrix.

Combining all the above:

$$\begin{aligned} \text{Filter output vector:} \quad & \hat{\mathbf{d}}^H = \mathbf{w}^H \mathbf{A}^H \\ \text{Desired output vector:} \quad & \mathbf{d}^H \\ \text{Error vector:} \quad & \epsilon^H = \mathbf{d}^H - \hat{\mathbf{d}}^H = \mathbf{d}^H - \mathbf{w}^H \mathbf{A}^H \end{aligned}$$

Note: All incorporate samples for $M \leq i \leq N$

The sum of the squared estimate errors can now be written as

$$\begin{aligned}\epsilon(\mathbf{w}) &= \sum_{i=M}^{N} |e(i)|^2 \\ &= \boldsymbol{\epsilon}^H \boldsymbol{\epsilon} \\ &= (\mathbf{d}^H - \mathbf{w}^H \mathbf{A}^H)(\mathbf{d} - \mathbf{A}\mathbf{w}) \\ &= \mathbf{d}^H\mathbf{d} - \mathbf{d}^H\mathbf{A}\mathbf{w} - \mathbf{w}^H\mathbf{A}^H\mathbf{d} + \mathbf{w}^H\mathbf{A}^H\mathbf{A}\mathbf{w}\end{aligned}$$

Minimizing with respect to **w**,

$$\frac{\partial \epsilon(\mathbf{w})}{\partial \mathbf{w}} = -2\mathbf{A}^H\mathbf{d} + 2\mathbf{A}^H\mathbf{A}\mathbf{w} \qquad (*)$$

Setting $(*)$ equal to zero gives the optimal LS weight $\hat{\mathbf{w}}$

$$\Rightarrow \mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} = \mathbf{A}^H\mathbf{d} \qquad [\text{Deterministic normal equation}]$$

Note: $\mathbf{A}$ is not generally square, and thus not invertible, but $\mathbf{A}^H\mathbf{A}$ is square and generally invertible

$$\begin{aligned} \mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} &= \mathbf{A}^H\mathbf{d} \\ \Rightarrow \hat{\mathbf{w}} &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d} \end{aligned}$$

The deterministic normal equation can be rearranged as

$$\begin{aligned} \mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} - \mathbf{A}^H\mathbf{d} &= \mathbf{0} \\ \mathbf{A}^H(\mathbf{A}\hat{\mathbf{w}} - \mathbf{d}) &= \mathbf{0} \qquad \text{[or using } \epsilon_{\min} = \mathbf{d} - \mathbf{A}\hat{\mathbf{w}}\text{]} \\ \mathbf{A}^H\epsilon_{\min} &= \mathbf{0} \end{aligned}$$

Observation: The LS orthogonality principle states that the estimate error $\epsilon_{\min}$ is orthogonal to the row vectors of the data matrix $\mathbf{A}^H$

Objective: Determine the minimum sum of squared errors ($e_{\min}$)

$$\begin{aligned} e_{\min} &= \epsilon_{\min}^H \epsilon_{\min} \\ &= (\mathbf{d}^H - \hat{\mathbf{w}}^H \mathbf{A}^H)(\mathbf{d} - \mathbf{A}\hat{\mathbf{w}}) \\ &= \mathbf{d}^H\mathbf{d} - \hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{d} - \mathbf{d}^H \mathbf{A}\hat{\mathbf{w}} + \hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{A}\hat{\mathbf{w}} \end{aligned}$$

Utilizing the normal equations $\hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{d} = \hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{A}\hat{\mathbf{w}}$

$$\begin{aligned} e_{\min} &= \mathbf{d}^H\mathbf{d} - \underbrace{\hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{d}}_{\hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{A}\hat{\mathbf{w}}} - \mathbf{d}^H \mathbf{A}\hat{\mathbf{w}} + \hat{\mathbf{w}}^H \mathbf{A}^H \mathbf{A}\hat{\mathbf{w}} \\ &= \mathbf{d}^H\mathbf{d} - \mathbf{d}^H \mathbf{A}\hat{\mathbf{w}} \end{aligned}$$

or using $\hat{\mathbf{w}} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d}$

$$e_{\min} = \mathbf{d}^H\mathbf{d} - \mathbf{d}^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d} \qquad (*)$$

Note that

$$\mathbf{d}^H\mathbf{d} = \sum_{i=M}^{N} |d(i)|^2 \qquad \text{[energy of desired response]}$$

Consider again the deterministic normal equation

$$\mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} = \mathbf{A}^H\mathbf{d}$$

Note that

$$\begin{aligned}
\mathbf{A}^H\mathbf{A} &= [\mathbf{x}(M), \mathbf{x}(M+1), \cdots, \mathbf{x}(N)] \begin{bmatrix} \mathbf{x}^H(M) \\ \mathbf{x}^H(M+1) \\ \vdots \\ \mathbf{x}^H(N) \end{bmatrix} \\
&= \sum_{i=M}^{N} \mathbf{x}(i)\mathbf{x}^H(i) \\
&= \mathbf{\Phi} \qquad [\text{time averaged correlation matrix, size } M \times M\ ]
\end{aligned}$$

From $\boldsymbol{\Phi} = \sum_{i=M}^{N} \mathbf{x}(i)\mathbf{x}^H(i)$ it can be shown that:

1. $\boldsymbol{\Phi}$ is Hermitian
2. $\boldsymbol{\Phi}$ is nonnegative definite

   To prove this, note that for any $\mathbf{a}$

$$\begin{aligned}
\mathbf{a}^H\boldsymbol{\Phi}\mathbf{a} &= \sum_{i=M}^{N} \mathbf{a}^H\mathbf{x}(i)\mathbf{x}^H(i)\mathbf{a} \\
&= \sum_{i=M}^{N} [\mathbf{a}^H\mathbf{x}(i)][\mathbf{a}^H\mathbf{x}(i)]^H \\
&= \sum_{i=M}^{N} |\mathbf{a}^H\mathbf{x}(i)|^2 \geq 0
\end{aligned}$$

3. From (1) and (2) we can prove that the eigenvalues of $\boldsymbol{\Phi}$ are real and nonnegative

The deterministic normal equation,

$$\mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} = \mathbf{A}^H\mathbf{d}$$

also employs

$$\begin{aligned}\mathbf{A}^H\mathbf{d} &= [\mathbf{x}(M), \mathbf{x}(M+1), \cdots, \mathbf{x}(N)] \begin{bmatrix} d^*(M) \\ d^*(M+1) \\ \vdots \\ d^*(N) \end{bmatrix} \\ &= \sum_{i=M}^{N} \mathbf{x}(i)d^*(i) \\ &= \boldsymbol{\theta} \qquad [\text{Time averaged cross-correlation vector}, \text{ size } M \times 1]\end{aligned}$$

Thus the deterministic normal equation, $\mathbf{A}^H\mathbf{A}\hat{\mathbf{w}} = \mathbf{A}^H\mathbf{d}$, reduces to

$$\boldsymbol{\Phi}\hat{\mathbf{w}} = \boldsymbol{\theta}$$

$\boldsymbol{\Phi}$ is usually positive definite (always positive semi-definite) $\Rightarrow$ the solution is well defined

$$\hat{\mathbf{w}} = \boldsymbol{\Phi}^{-1}\boldsymbol{\theta} \qquad \text{[LS optimal weight vector]}$$

Also, recall from $(*)$ that $e_{\min}$ can be expressed as

$$\begin{aligned} e_{\min} &= \mathbf{d}^H\mathbf{d} - \underbrace{\mathbf{d}^H\mathbf{A}}_{\boldsymbol{\theta}^H}\underbrace{(\mathbf{A}^H\mathbf{A})^{-1}}_{\boldsymbol{\Phi}^{-1}}\underbrace{\mathbf{A}^H\mathbf{d}}_{\boldsymbol{\theta}} \\ &= e_d - \boldsymbol{\theta}^H\boldsymbol{\Phi}^{-1}\boldsymbol{\theta} \end{aligned}$$

where e_d is the energy of desired signal

Consider again the orthogonality principle

$$\mathbf{A}^H \epsilon_{\min} = \mathbf{0}$$

Recall that $\hat{\mathbf{d}} = \mathbf{A}\hat{\mathbf{w}}$. Thus

$$\begin{aligned} \mathbf{A}^H \epsilon_{\min} &= \mathbf{0} \\ \Rightarrow \hat{\mathbf{w}}^H \mathbf{A}^H \epsilon_{\min} &= \hat{\mathbf{w}}^H \mathbf{0} \\ \Rightarrow \hat{\mathbf{d}}^H \epsilon_{\min} &= \mathbf{0} \end{aligned}$$

Result: The minimum estimation error vector, $\epsilon_{\min}$, is orthogonal to the data matrix $\mathbf{A}^H$ and the LS estimate $\hat{\mathbf{d}}$

Objective: Analyze the Least Squares solution in terms of

- Bias – it is the LS solution unbiased?
- BLUE – is the LS solution the Best Linear Unbiased Estimate?

Assumption: Take the true underlying system to be a linear

$$\begin{aligned} d(i) &= \sum_{k=0}^{M-1} w_{0k}^{*} x(i-k) + e_0(i) \\ &= \mathbf{w}_0^H \mathbf{x}(i) + e_0(i) \end{aligned}$$

$e_0(i)$ is the unobservable measurement error

$\Rightarrow$ $e_0(i)$ is white (uncorrelated) with zero mean and variance σ^2

Express the desired signal in vector form

$$\mathbf{d} = \mathbf{A}\mathbf{w}_0 + \epsilon_0$$

where $\epsilon_0^H = [e_0(M), e_0(M+1), \cdots, e_0(N)]$

Objective: Evaluate the bias of $\hat{\mathbf{w}}$

Recall that

$$\hat{\mathbf{w}} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d}$$

Using $\mathbf{d} = \mathbf{A}\mathbf{w}_0 + \epsilon_0$ in the above

$$\begin{aligned}\hat{\mathbf{w}} &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d} \\ &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H(\mathbf{A}\mathbf{w}_0 + \epsilon_0) \\ &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{A}\mathbf{w}_0 + (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\epsilon_0 \\ &= \mathbf{w}_0 + (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\epsilon_0 \qquad (*)\end{aligned}$$

Note $\mathbf{A}$ is fixed. Thus taking the expectation of $(*)$ yields

$$\begin{aligned}E\{\hat{\mathbf{w}}\} &= \mathbf{w}_0 + (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H E\{\epsilon_0\} \\ &= \mathbf{w}_0\end{aligned}$$

Result: The LS estimate, $\hat{\mathbf{w}}$, is unbiased

Objective: Evaluate the covariance of $\hat{\mathbf{w}}$

Note that from $(*)$

$$\begin{aligned} \hat{\mathbf{w}} &= \mathbf{w}_0 + (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\epsilon_0 \\ \Rightarrow \hat{\mathbf{w}} - \mathbf{w}_0 &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\epsilon_0 \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{cov}[\hat{\mathbf{w}}] &= E\{(\hat{\mathbf{w}} - \mathbf{w}_0)(\hat{\mathbf{w}} - \mathbf{w}_0)^H\} \\ &= E\{(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\epsilon_0\epsilon_0^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\} \\ &= \mathbf{\Phi}^{-1}\mathbf{A}^H\underbrace{E\{\epsilon_0\epsilon_0^H\}}_{\sigma^2\mathbf{I}}\mathbf{A}\mathbf{\Phi}^{-1} \\ &= \sigma^2\mathbf{\Phi}^{-1}\mathbf{\Phi}\mathbf{\Phi}^{-1} = \sigma^2\mathbf{\Phi}^{-1} \qquad (\maltese_1) \end{aligned}$$

Result: The covariance of $\hat{\mathbf{w}}$ is proportional to: (1) the variance of the measurement noise and (2) the inverse of the time average correlation matrix

Objective: Show that the LS estimate $\hat{\mathbf{w}}$ is the Best Linear Unbiased Estimate (BLUE)

- Consider any linear unbiased estimate $\tilde{\mathbf{w}}$
- Note that $\tilde{\mathbf{w}}$ is a linear function of the observed date and can thus be written as

$$\tilde{\mathbf{w}} = \mathbf{B}\mathbf{d}$$

  where $\mathbf{B}$ is a $M \times (N - M + 1)$ matrix

Substituting $\mathbf{d} = \mathbf{A}\mathbf{w}_0 + \epsilon_0$ into the above,

$$\begin{aligned} \tilde{\mathbf{w}} &= \mathbf{B}\mathbf{A}\mathbf{w}_0 + \mathbf{B}\epsilon_0 \qquad (*) \\ \Rightarrow E\{\tilde{\mathbf{w}}\} &= \mathbf{B}\mathbf{A}\mathbf{w}_0 \\ \Rightarrow \mathbf{B}\mathbf{A} &= \mathbf{I} \qquad [\text{since } \tilde{\mathbf{w}} \text{ unbiased}] \end{aligned}$$

Thus $\mathbf{B}\mathbf{A} = \mathbf{I}$ and $(*) \Rightarrow$

$$\tilde{\mathbf{w}} = \mathbf{w}_0 + \mathbf{B}\epsilon_0$$

Rearranging $\tilde{\mathbf{w}} = \mathbf{w}_0 + \mathbf{B}\epsilon_0$,

$$\begin{aligned}
\tilde{\mathbf{w}} - \mathbf{w}_0 &= \mathbf{B}\epsilon_0 \\
\Rightarrow \text{cov}[\tilde{\mathbf{w}}] &= E\{(\tilde{\mathbf{w}} - \mathbf{w}_0)(\tilde{\mathbf{w}} - \mathbf{w}_0)^H\} \\
&= E\{\mathbf{B}\epsilon_0\epsilon_0^H\mathbf{B}^H\} \\
&= \sigma^2\mathbf{B}\mathbf{B}^H \qquad (\maltese_2)
\end{aligned}$$

Now define

$$\begin{aligned}
\Psi &= \mathbf{B} - (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H \\
\Rightarrow \Psi\Psi^H &= [\mathbf{B} - \Phi^{-1}\mathbf{A}^H][\mathbf{B}^H - \mathbf{A}\Phi^{-1}] \\
&= \mathbf{B}\mathbf{B}^H - \underbrace{\mathbf{B}\mathbf{A}}_{\mathbf{I}}\Phi^{-1} - \Phi^{-1}\underbrace{\mathbf{A}^H\mathbf{B}^H}_{\mathbf{I}} + \underbrace{\Phi^{-1}\mathbf{A}^H\mathbf{A}\Phi^{-1}}_{\Phi^{-1}\Phi\Phi^{-1}} \\
&= \mathbf{B}\mathbf{B}^H - \Phi^{-1} - \Phi^{-1} + \Phi^{-1} \\
&= \mathbf{B}\mathbf{B}^H - \Phi^{-1} \\
&= \mathbf{B}\mathbf{B}^H - (\mathbf{A}^H\mathbf{A})^{-1}
\end{aligned}$$

Observation: The diagonal elements at $\Psi\Psi^H$ must be ≥ 0

Thus $\Psi\Psi^H = \mathbf{B}\mathbf{B}^H - (\mathbf{A}^H\mathbf{A})^{-1} \Rightarrow$

$$\begin{aligned} \text{diag}[\mathbf{B}\mathbf{B}^H] &\geq \text{diag}[(\mathbf{A}^H\mathbf{A})^{-1}] \\ \Rightarrow \text{diag}[\sigma^2\mathbf{B}\mathbf{B}^H] &\geq \text{diag}[\sigma^2(\mathbf{A}^H\mathbf{A})^{-1}] \qquad (*) \end{aligned}$$

But recall from $(\maltese_1)$ and $(\maltese_2)$ that

$$\text{cov}[\hat{\mathbf{w}}] = \sigma^2(\mathbf{A}^H\mathbf{A})^{-1} \qquad \text{and} \qquad \text{cov}[\tilde{\mathbf{w}}] = \sigma^2\mathbf{B}\mathbf{B}^H$$

Utilizing these results in $(*) \Rightarrow$

$$\text{variance}[\tilde{w}_i] \geq \text{variance}[\hat{w}_i] \quad i = 1, 2, \cdots, M$$

Thus the weights in $\hat{\mathbf{w}}$ have lower variance than any other linear estimates

Result: The LS estimate $\hat{\mathbf{w}}$ is unbiased and has the smallest weight variance $\Rightarrow$ it is the Best Linear Unbiased Estimate (BLUE)

Definition (Recursive Least Squares (RLS))

Motivation: LS requires solving

$$\begin{aligned}\hat{\mathbf{w}} &= (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{d} \\ &= \boldsymbol{\Phi}^{-1}\boldsymbol{\theta}\end{aligned}$$

where

$$\boldsymbol{\Phi} = \sum_{i=M}^{N} \mathbf{x}(i)\mathbf{x}^H(i) \quad \text{and} \quad \boldsymbol{\theta} = \sum_{i=M}^{N} \mathbf{x}(i)d^*(i)$$

- $(\mathbf{A}^H\mathbf{A})$ is $M \times M$ and inversion requires $O(M^3)$ multiplications and additions

Approach: Suppose the LS optimal weights are known at time n, $\hat{\mathbf{w}}(n)$. As time evolves, find the new estimate, $\hat{\mathbf{w}}(n+1)$, in terms of $\hat{\mathbf{w}}(n)$.

- Employ the matrix inversion lemma to reduce the number of computations

Let the observation sequence be $x(1), x(2), \cdots, x(n)$

- ⇒ Assume $x(l) = 0$ for $l \leq 0$

Define the error as

$$\epsilon(n) = \sum_{i=1}^{n} \beta(n,i)|e(i)|^2$$

where

$$\begin{aligned} e(i) &= d(i) - \mathbf{w}^H(n)\mathbf{x}(i) \\ \mathbf{x}(i) &= [x(i), x(i-1), \cdots, x(i-M+1)]^T \\ \mathbf{w}(n) &= [w_0(n), w_1(n), \cdots, w_{M-1}(n)]^T \end{aligned}$$

- ⇒ $\beta(n,i) \in (0,1]$ is a forgetting factor used in non–stationary statistics cases

A commonly used forgetting factor is the exponential forgetting factor

$$\beta(n,i) = \lambda^{n-i} \quad i = 1, 2, \cdots, n, \quad \lambda \in (0, 1]$$

Thus,

$$\epsilon(n) = \sum_{i=1}^{n} \lambda^{n-i} |e(i)|^2$$

The LS solution is given by the deterministic normal equation

$$\boldsymbol{\Phi}(n)\hat{\mathbf{w}}(n) = \boldsymbol{\theta}(n)$$

where now

$$\boldsymbol{\Phi}(n) = \sum_{i=1}^{n} \lambda^{n-i} \mathbf{x}(i)\mathbf{x}^H(i)$$

$$\boldsymbol{\theta}(n) = \sum_{i=1}^{n} \lambda^{n-i} \mathbf{x}(i) d^*(i)$$

The normal equation terms can be updated recursively,

$$\begin{aligned}
\mathbf{\Phi}(n) &= \sum_{i=1}^{n} \lambda^{n-i}\mathbf{x}(i)\mathbf{x}^H(i) \\
&= \underbrace{\lambda\left[\sum_{i=1}^{n-1} \lambda^{(n-1)-i}\mathbf{x}(i)\mathbf{x}^H(i)\right]}_{\mathbf{\Phi}(n-1)} + \mathbf{x}(n)\mathbf{x}^H(n) \\
&= \lambda\mathbf{\Phi}(n-1) + \mathbf{x}(n)\mathbf{x}^H(n)
\end{aligned}$$

Similarly

$$\begin{aligned}
\boldsymbol{\theta}(n) &= \sum_{i=1}^{n} \lambda^{n-i}\mathbf{x}(i)d^*(i) \\
&= \lambda\left[\sum_{i=1}^{n-1} \lambda^{(n-1)-i}\mathbf{x}(i)d^*(i)\right] + \mathbf{x}(n)d^*(n) \\
&= \lambda\boldsymbol{\theta}(n-1) + \mathbf{x}(n)d^*(n)
\end{aligned}$$

Aside: *Matrix inversion lemma*: If

$$\underbrace{\mathbf{A}}_{M\times M} = \underbrace{\mathbf{B}^{-1}}_{M\times M} + \underbrace{\mathbf{C}}_{M\times L}\underbrace{\mathbf{D}^{-1}}_{L\times L}\underbrace{\mathbf{C}^{H}}_{L\times M}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{D}$ are positive definite (non-singular), then

$$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{B}\mathbf{C}[\mathbf{D} + \mathbf{C}^{H}\mathbf{B}\mathbf{C}]^{-1}\mathbf{C}^{H}\mathbf{B}$$

Apply the lemma to

$$\Phi(n) = \lambda\Phi(n-1) + \mathbf{x}(n)\mathbf{x}^{H}(n)$$

Accordingly, set

$$\begin{array}{llll} \mathbf{A} = \Phi(n) & [M \times M] & \mathbf{B}^{-1} = \lambda\Phi(n-1) & [M \times M] \\ \mathbf{C} = \mathbf{x}(n) & [M \times 1] & \mathbf{D} = 1 & [1 \times 1] \end{array}$$

Utilizing

$$\mathbf{A} = \mathbf{\Phi}(n) \qquad \mathbf{B}^{-1} = \lambda\mathbf{\Phi}(n-1)$$
$$\mathbf{C} = \mathbf{x}(n) \qquad \mathbf{D} = 1$$

and

$$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{BC}[\mathbf{D} + \mathbf{C}^H\mathbf{BC}]^{-1}\mathbf{C}^H\mathbf{B} \qquad (*)$$

we get

$$[\mathbf{D} + \mathbf{C}^H\mathbf{BC}]^{-1} = [1 + \lambda^{-1}\mathbf{x}^H(n)\mathbf{\Phi}^{-1}(n-1)\mathbf{x}(n)]^{-1}$$

which is a scalar. Thus evaluating $(*)$ yields

$$\mathbf{\Phi}^{-1}(n) = \lambda^{-1}\mathbf{\Phi}^{-1}(n-1) - \frac{\lambda^{-2}\mathbf{\Phi}^{-1}(n-1)\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{\Phi}^{-1}(n-1)}{1 + \lambda^{-1}\mathbf{x}^H(n)\mathbf{\Phi}^{-1}(n-1)\mathbf{x}(n)}$$

To simplify the result, let $\mathbf{P}(n) = \mathbf{\Phi}^{-1}(n)$ and

$$\underbrace{\mathbf{k}(n)}_{\text{Gain vector}} = \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1 + \lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n)}$$

Utilizing $\mathbf{P}(n) = \Phi^{-1}(n)$ and $\mathbf{k}(n) = \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1+\lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n)}$

$$\begin{aligned}\Phi^{-1}(n) &= \lambda^{-1}\Phi^{-1}(n-1) - \frac{\lambda^{-2}\Phi^{-1}(n-1)\mathbf{x}(n)\mathbf{x}^H(n)\Phi^{-1}(n-1)}{1+\lambda^{-1}\mathbf{x}^H(n)\Phi^{-1}(n-1)\mathbf{x}(n)} \\ \Rightarrow \mathbf{P}(n) &= \lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1) \qquad (*)\end{aligned}$$

Also, the gain vector can be simplified as

$$\begin{aligned}\mathbf{k}(n) &= \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1+\lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n)} \qquad \text{[multiply by denom.]} \\ \Rightarrow \mathbf{k}(n) &= \lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n) \\ &= \underbrace{[\lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)]}_{=\mathbf{P}(n)\text{ from }(*)}\mathbf{x}(n) \\ &= \mathbf{P}(n)\mathbf{x}(n) = \Phi^{-1}(n)\mathbf{x}(n) \qquad (**)\end{aligned}$$

We must now derive an update for the tap weight vector. Recall,

$$\hat{\mathbf{w}}(n) = \mathbf{\Phi}^{-1}(n)\boldsymbol{\theta}(n) = \mathbf{P}(n)\boldsymbol{\theta}(n)$$

Using the recursion $\boldsymbol{\theta}(n) = \lambda\boldsymbol{\theta}(n-1) + \mathbf{x}(n)d^*(n)$ in the above

$$\hat{\mathbf{w}}(n) = \lambda\mathbf{P}(n)\boldsymbol{\theta}(n-1) + \mathbf{P}(n)\mathbf{x}(n)d^*(n) \qquad (***)$$

Using the update $(*)$

$$\mathbf{P}(n) = \lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)$$

in the first $\mathbf{P}(n)$ term of $(***)$

$$\begin{aligned}\hat{\mathbf{w}}(n) &= \lambda\mathbf{P}(n)\boldsymbol{\theta}(n-1) + \mathbf{P}(n)\mathbf{x}(n)d^*(n) \\ &= \lambda[\lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)]\boldsymbol{\theta}(n-1) \\ &\quad + \mathbf{P}(n)\mathbf{x}(n)d^*(n)\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{w}}(n) &= \lambda[\lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)]\boldsymbol{\theta}(n-1) \\ &\quad +\mathbf{P}(n)\mathbf{x}(n)d^*(n) \\ &= \underbrace{\mathbf{P}(n-1)\boldsymbol{\theta}(n-1)}_{\hat{\mathbf{w}}(n-1)} - \mathbf{k}(n)\mathbf{x}^H(n)\underbrace{\mathbf{P}(n-1)\boldsymbol{\theta}(n-1)}_{\hat{\mathbf{w}}(n-1)} \\ &\quad +\mathbf{P}(n)\mathbf{x}(n)d^*(n) \\ &= \hat{\mathbf{w}}(n-1) - \mathbf{k}(n)\mathbf{x}^H(n)\hat{\mathbf{w}}(n-1) + \underbrace{\mathbf{P}(n)\mathbf{x}(n)}_{=\mathbf{k}(n)\text{ from }(**)}\ d^*(n) \\ &= \hat{\mathbf{w}}(n-1) - \mathbf{k}(n)[\mathbf{x}^H(n)\hat{\mathbf{w}}(n-1) - d^*(n)] \\ &= \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)\alpha^*(n)\end{aligned}$$

where $\alpha(n) = d(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)$

Observation: Difference between $e(n)$ and $\alpha(n)$:

$e(n) = d(n) - \hat{\mathbf{w}}^H(n)\mathbf{x}(n) \Rightarrow$ a posteriori error

$\alpha(n) = d(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n) \Rightarrow$ a priori error

RLS Algorithm Summary

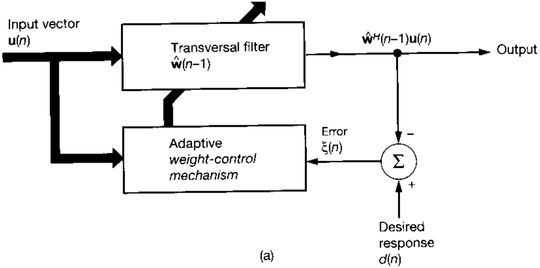
1. Given a new sample $x(n)$, update the gain vector

$$\mathbf{k}(n) = \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1+\lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n)}$$

2. Update the innovation: $\alpha(n) = d(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)$
3. Update the tap weight vector: $\hat{\mathbf{w}}(n) = \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)\alpha^*(n)$
4. Update inverse correlation matrix

$$\mathbf{P}(n) = \lambda^{-1}\mathbf{P}(n-1) - \lambda^{-1}\mathbf{k}(n)\mathbf{x}^H(n)\mathbf{P}(n-1)$$

Initial Conditions: $\hat{\mathbf{w}}(0) = \mathbf{0}$ and $\Phi(0) = \delta\mathbf{I}$, where δ is a small positive constant, $\delta \approx 0.01\sigma_x^2$.



Algorithm Comparison: RLS and LMS algorithm terms:

Entity	RLS	LMS
Error	$\alpha(n) = d(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)$ (a priori error)	$e(n) = d(n) - \hat{\mathbf{w}}^H(n)\mathbf{x}(n)$ (a posteriari error)
Weight Update	$\hat{\mathbf{w}}(n) = \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)\alpha^*(n)$	$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu\mathbf{x}(n)e^*(n)$
Gain of error update	$\left(\frac{\lambda^{-1}\mathbf{P}(n-1)}{1+\lambda^{-1}\mathbf{x}^H(n)\mathbf{P}(n-1)\mathbf{x}(n)}\right)\mathbf{x}(n)$	$(\mu)\mathbf{x}(n)$

Objective: Compare the complexities (number of additions and multiplies) for the LMS, LS, and RLS algorithms.

- Assume the data is real and the filter is of size M

Case 1 – The LMS algorithm: Algorithm stages:

1. $\hat{d}(n) = \mathbf{w}^T(n)\mathbf{x}(n)$
2. $e(n) = d(n) - \hat{d}(n)$
3. $\mathbf{w}(n+1) = \mathbf{w}(n) + \mu\mathbf{x}(n)e(n)$

Complexity		
Stage	$O_\times$	O_+
(1)	M	$M-1$
(2)	0	1
(3)	$M+1$	M
Total complexity per iteration	$O_\times(2M+1)$	$O_+(2M)$

Case 2 – The LS algorithm: Algorithm solves

$$\hat{\mathbf{w}}(n) = \mathbf{\Phi}^{-1}(n)\boldsymbol{\theta}(n)$$

and has stages:

1. $\mathbf{\Phi}(n+1) = \mathbf{\Phi}(n) + \mathbf{x}(n+1)\mathbf{x}^H(n+1)$
2. $\boldsymbol{\theta}(n+1) = \boldsymbol{\theta}(n) + \mathbf{x}(n+1)d(n+1)$
3. $\hat{\mathbf{w}}(n+1) = \mathbf{\Phi}^{-1}(n+1)\boldsymbol{\theta}(n+1)$

Complexity		
Stage	$O_\times$	O_+
(1)	M^2	M^2
(2)	M	M
(3)	$M^3 + M^2$	$M^3 + M(M-1)$
Total complexity per iteration	$O_\times(M^3 + 2M^2 + M)$	$O_+(M^3 + 2M^2)$

Case 3 – The RLS algorithm: Algorithm has stages (assuming $\lambda = 1$):

1. $\mathbf{k}(n) = \frac{\lambda^{-1}\mathbf{P}(n-1)\mathbf{x}(n)}{1+\mathbf{x}^T(n)\mathbf{P}(n-1)\mathbf{x}(n)}$
2. $\alpha(n) = d(n) - \hat{\mathbf{w}}^T(n-1)\mathbf{x}(n)$
3. $\hat{\mathbf{w}}(n) = \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)\alpha(n)$
4. $\mathbf{P}(n) = \mathbf{P}(n-1) - \mathbf{k}(n)\mathbf{x}^T(n)\mathbf{P}(n-1)$

Note: The operation $\mathbf{x}^T(n)\mathbf{P}(n-1)$ is repeated (but only performed once). Corresponding steps are underlined in the chart.

Complexity		
Stage	$O_\times$	O_+
(1) numerator (1) denominator (1) division	M^2 $\underline{M^2} + M$ M	$M(M-1)$ $\underline{M(M-1)} + M$
(2)	M	M
(3)	M	M
(4)	$\underline{M^2} + M^2$	$\underline{M(M-1)} + M^2$
Total complexity per iteration	$O_\times(3M^2 + 4M)$	$O_+(3M^2 + M)$

Objective: Analyze the RLS algorithm in terms of

- Bias
- Convergence in the mean; Convergence in the mean square
- Learning curve decay rate

Assumptions:

1. The desired signal is formed by the regression model
$$d(n) = \mathbf{w}_0^H \mathbf{x}(n) + e_0(n)$$
where $e_0(n)$ is white with variance σ^2.
2. $\lambda = 1$ and $n \geq M$.

Then
$$\hat{\mathbf{w}}(n) = \mathbf{\Phi}^{-1}(n)\boldsymbol{\theta}(n)$$

where
$$\mathbf{\Phi}(n) = \sum_{i=1}^{n} \mathbf{x}(i)\mathbf{x}^H(i) \quad \text{and} \quad \boldsymbol{\theta}(n) = \sum_{i=1}^{n} \mathbf{x}(i)d^*(i)$$

Substituting $d^*(n) = \mathbf{x}^H(n)\mathbf{w}_0 + e_0^*(n)$ into $\theta(n)$

$$\begin{aligned}\theta(n) &= \sum_{i=1}^{n} \mathbf{x}(i)[\mathbf{x}^H(i)\mathbf{w}_0 + e_0^*(i)] \\ &= \sum_{i=1}^{n} \mathbf{x}(i)\mathbf{x}^H(i)\mathbf{w}_0 + \sum_{i=1}^{n} \mathbf{x}(i)e_0^*(i) \\ &= \mathbf{\Phi}(n)\mathbf{w}_0 + \sum_{i=1}^{n} \mathbf{x}(i)e_0^*(i)\end{aligned}$$

Thus

$$\begin{aligned}\hat{\mathbf{w}}(n) &= \mathbf{\Phi}^{-1}(n)\theta(n) \\ &= \mathbf{\Phi}^{-1}(n)[\mathbf{\Phi}(n)\mathbf{w}_0 + \sum_{i=1}^{n} \mathbf{x}(i)e_0^*(i)] \\ &= \mathbf{w}_0 + \mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n} \mathbf{x}(i)e_0^*(i) \qquad (*)\end{aligned}$$

Note that $E\{A\} = E\{E\{A|B\}\}$. Thus

$$\begin{aligned}
\hat{\mathbf{w}}(n) &= \mathbf{w}_0 + \mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n}\mathbf{x}(i)e_0^*(i) \\
\Rightarrow E\{\hat{\mathbf{w}}(n)\} &= \mathbf{w}_0 + E\{E\{\mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n}\mathbf{x}(i)e_0^*(i)|x(i), i = 1, 2, \cdots, n\}\} \\
&= \mathbf{w}_0 + E\{\mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n}\mathbf{x}(i)E\{e_0^*(i)\}\} = \mathbf{w}_0
\end{aligned}$$

The above follows from the fact that $\Phi(n)$ and $e_0^*(i)$ are independent.

Why?$e_0(i)$ is independent of all observations and the $x(i)$ terms are given, uniquely defining $\Phi(n)$. $\Rightarrow$ independence of $\Phi(n)$ and $e_0^*(i)$.

Result: The RLS algorithm is unbiased and convergent in the mean for $n \geq M$.

Question: How does this compare to the LMS algorithm?

Next, consider the convergence in the mean square. Recall $(*)$

$$\hat{\mathbf{w}}(n) = \mathbf{w}_0 + \mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n}\mathbf{x}(i)e_0^*(i)$$

which gives

$$\boldsymbol{\epsilon}(n) = \hat{\mathbf{w}}(n) - \mathbf{w}_0 = \mathbf{\Phi}^{-1}(n)\sum_{i=1}^{n}\mathbf{x}(i)e_0^*(i)$$

Thus the weight error correlation matrix is

$$\begin{aligned}\mathbf{K}(n) &= E\{\boldsymbol{\epsilon}(n)\boldsymbol{\epsilon}^H(n)\}\\ &= E\{\mathbf{\Phi}^{-1}(n)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{x}(i)e_0^*(i)e_0(j)\mathbf{x}^H(j)\right)\mathbf{\Phi}^{-1}(n)\}\end{aligned}$$

Again using $E\{A\} = E\{E\{A|B\}\}$ yields

$$\begin{aligned}
\mathbf{K}(n) &= E\left\{\mathbf{\Phi}^{-1}(n)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{x}(i)\underbrace{E\{e_0^*(i)e_0(j)\}}_{\sigma^2\delta(i-j)}\mathbf{x}^H(j)\right)\mathbf{\Phi}^{-1}(n)\right\} \\
&= \sigma^2 E\left\{\mathbf{\Phi}^{-1}(n)\left(\sum_{i=1}^{n}\mathbf{x}(i)\mathbf{x}^H(i)\right)\mathbf{\Phi}^{-1}(n)\right\} \\
&= \sigma^2 E\{\mathbf{\Phi}^{-1}(n)\mathbf{\Phi}(n)\mathbf{\Phi}^{-1}(n)\} \\
&= \sigma^2 E\{\mathbf{\Phi}^{-1}(n)\}
\end{aligned}$$

Note: $\mathbf{\Phi}^{-1}(n)$ has a Wishart distribution, the expectation of which is

$$E\{\mathbf{\Phi}^{-1}(n)\} = \frac{1}{n-M-1}\mathbf{R}^{-1} \quad n > M+1$$

Using $\mathbf{K}(n) = \frac{\sigma^2}{n-M-1}\mathbf{R}^{-1}$ and the trace

$$\begin{aligned} E\{||\epsilon(n)||^2\} &= E\{\epsilon^H(n)\epsilon(n)\} \\ &= E\{\mathrm{trace}[\epsilon^H(n)\epsilon(n)]\} \\ &= E\{\mathrm{trace}[\epsilon(n)\epsilon^H(n)]\} \\ &= \mathrm{trace}E\{\epsilon(n)\epsilon^H(n)\} \\ &= \mathrm{trace}[\mathbf{K}(n)] \\ &= \frac{\sigma^2}{n-M-1}\mathrm{trace}[\mathbf{R}^{-1}] \\ &= \frac{\sigma^2}{n-M-1}\sum_{i=1}^{M}\frac{1}{\lambda_i} \quad n > M+1 \end{aligned}$$

Results:

- The weight vector MSE is initially proportional to $\sum_{i=1}^{M}\frac{1}{\lambda_i}$
- The weight vector converges linearly in the mean squared sense

Objective: Evaluate the RLS (error) learning curve

Recall the *a priori* estimation error

$$\begin{aligned}\alpha(n) &= d(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)\\ &= d(n) - \hat{d}_0(n) + \hat{d}_0(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)\\ &= e_0(n) + \mathbf{w}_0^H\mathbf{x}(n) - \hat{\mathbf{w}}^H(n-1)\mathbf{x}(n)\\ &= e_0(n) - \boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\end{aligned}$$

Now consider the MSE of $\alpha(n)$

$$\begin{aligned}J_\alpha(n) &= E\{|\alpha(n)|^2\}\\ &= E\{[e_0^*(n) - \mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)][e_0(n) - \boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)]\}\\ &= E\{|e_0(n)|^2\} - E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)e_0(n)\}\\ &\quad - E\{\epsilon^H(n-1)\mathbf{x}(n)e_0^*(n)\} + E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\}\end{aligned}$$

To analyze $J_\alpha(n)$, consider each term individually

$$\begin{aligned} J_\alpha(n) &= E\{|e_0(n)|^2\} - E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)e_0(n)\} \\ &\quad -E\{\epsilon^H(n-1)\mathbf{x}(n)e_0^*(n)\} + E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\} \end{aligned}$$

Term: $E\{|e_0(n)|^2\}$. Clearly,

$$E\{|e_0(n)|^2\} = \sigma^2$$

Term: $E\{\epsilon^H(n-1)\mathbf{x}(n)e_0^*(n)\}$. By the independence theorem, $\boldsymbol{\epsilon}(n-1)$ is independent of $\mathbf{x}(n)$ and $e_0(n)$. Thus,

$$\begin{aligned} E\{\epsilon^H(n-1)\mathbf{x}(n)e_0^*(n)\} &= E\{\epsilon^H(n-1)\}E\{\mathbf{x}(n)e_0^*(n)\} \\ &= 0 \end{aligned}$$

where the final result is due to the orthogonality principle.

Term: $E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)e_0(n)\} \rightarrow 0$ by similar arguments

$$\begin{aligned} J_\alpha(n) &= E\{|e_0(n)|^2\} - E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)e_0(n)\} \\ &\quad - E\{\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)e_0^*(n)\} + E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\} \end{aligned}$$

Term: $E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\}$

$$\begin{aligned} E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\} &= E\{\mathrm{trace}[\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)]\} \\ &= E\{\mathrm{trace}[\mathbf{x}(n)\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)]\} \end{aligned}$$

Invoking the independence theorem

$$\begin{aligned} & E\{\mathbf{x}^H(n)\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\mathbf{x}(n)\} \\ &\qquad = \mathrm{trace}[E\{\mathbf{x}(n)\mathbf{x}^H(n)\}E\{\boldsymbol{\epsilon}(n-1)\boldsymbol{\epsilon}^H(n-1)\}] \\ &\qquad = \mathrm{trace}[\mathbf{RK}(n-1)] \end{aligned}$$

Utilizing $\mathbf{K}(n-1) = \frac{\sigma^2}{n-M-2}\mathbf{R}^{-1}$ and substituting back each of the components

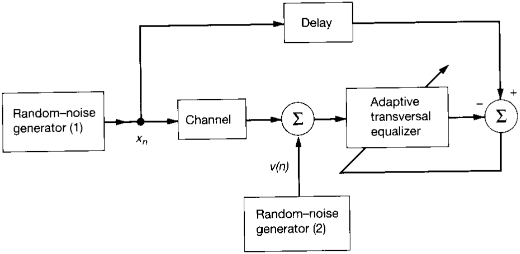
$$\begin{aligned} J_\alpha(n) &= \sigma^2 + \mathrm{trace}[\mathbf{RK}(n-1)] \\ &= \sigma^2 + \frac{M\sigma^2}{n-M-2} \quad n > M+1 \end{aligned}$$

Results:

- The ensemble average learning curve of the RLS converges in about $2M$ iterations, which is typically an order of magnitude faster than the LMS
- $\lim_{n\to\infty} J_\alpha(n) = \sigma^2$ thus there is no excess MSE
- Convergence of the RLS algorithm is independent of the eigenvalues of $\Phi(n)$

Example

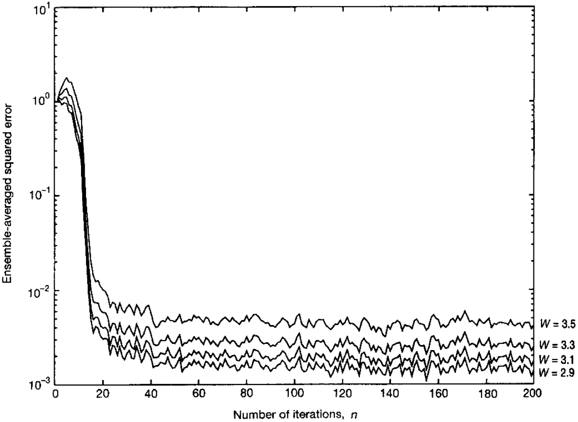
Consider again the channel equalization problem



where

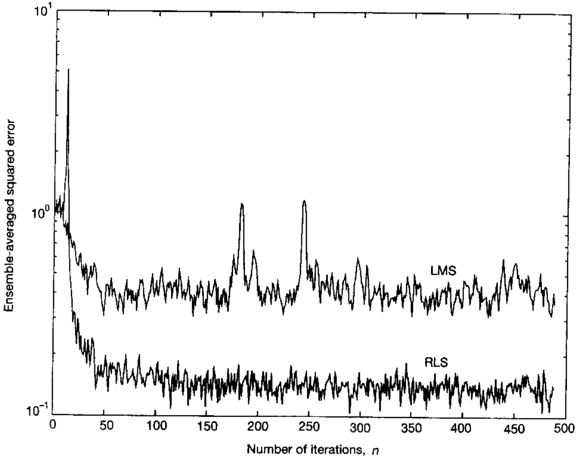
$$h_n = \begin{cases} \frac{1}{2}[1 + \cos(\frac{2\pi}{W}(n-1))] & n = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

- As before an 11-tap filter is used
- The SNR is 30dB and W is varied to control the eigenvalue spread



Observations:

- The RLS algorithm converges in about 20 iterations (twice the number of filter taps)
- The convergence (rate) is insensitive to the eigenvalue spread



Observations:

- The RLS algorithm converges faster than the LMS algorithm
- The RLS algorithm has lower steady state error than the LMS algorithm