FSAN-815/ELEG-815: Foundations of Statistical Learning

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Chapter 6: Wiener Filtering

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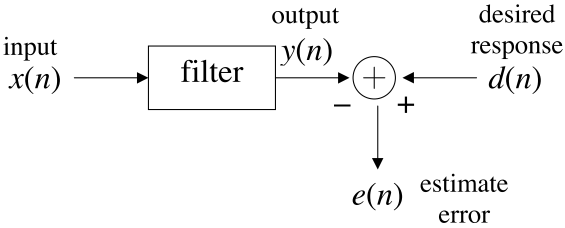
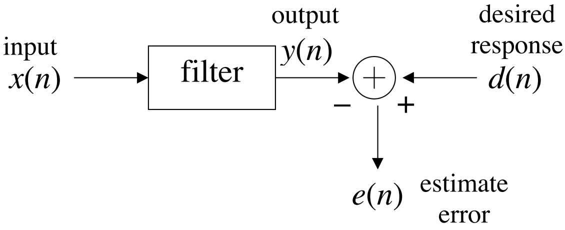
Problem Statement

Produce an estimate of a desired process statistically related to a set of observations



Historical Notes: The linear filtering problem was solved by

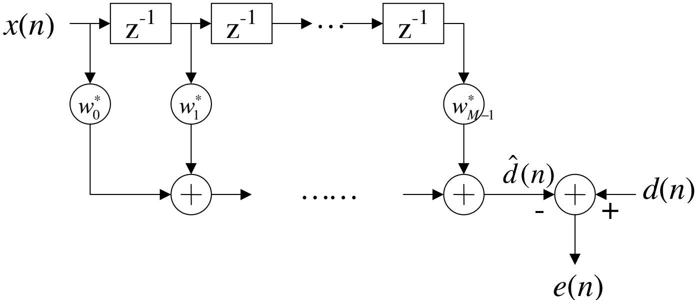
- Andrey Kolmogorov for discrete time – his 1938 paper "established the basic theorems for smoothing and predicting stationary stochastic processes"
- Norbert Wiener in 1941 for continuous time – not published until the 1949 paper *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*



System restrictions and considerations:

- Filter is linear
- Filter is discrete time
- Filter is finite impulse response (FIR)
- The process is WSS
- Statistical optimization is employed

For the discrete time case



- The filter impulse response is finite and given by

$$h_k = \begin{cases} w_k^* & \text{for } k = 0, 1, \cdots, M-1 \\ 0 & \text{otherwise} \end{cases}$$

- The output $\hat{d}(n)$ is an estimate of the desired signal $d(n)$
    - $x(n)$ and $d(n)$ are statistically related $\Rightarrow$ $\hat{d}(n)$ and $d(n)$ are statistically related

In convolution and vector form

$$\hat{d}(n) = \sum_{k=0}^{M-1} w_k^* x(n-k) = \mathbf{w}^H \mathbf{x}(n)$$

where

$$\begin{aligned} \mathbf{w} &= [w_0, w_1, \cdots, w_{M-1}]^T \qquad \text{[filter coefficient vector]} \\ \mathbf{x} &= [x(n), x(n-1), \cdots, x(n-M+1)]^T \qquad \text{[observation vector]} \end{aligned}$$

The error can now be written as

$$e(n) = d(n) - \hat{d}(n) = d(n) - \mathbf{w}^H \mathbf{x}(n)$$

Question: Under what criteria should the error be minimized?

Selected Criteria: Mean squared-error (MSE)

$$J(\mathbf{w}) = E\{e(n)e^*(n)\} \qquad (*)$$

Result: The **w** that minimizes $J(\mathbf{w})$ is the optimal (Wiener) filter

Utilizing $e(n) = d(n) - \mathbf{w}^H \mathbf{x}(n)$ in $(*)$ and expanding,

$$\begin{aligned} J(\mathbf{w}) &= E\{e(n)e^*(n)\} \\ &= E\{(d(n) - \mathbf{w}^H\mathbf{x}(n))(d^*(n) - \mathbf{x}^H(n)\mathbf{w})\} \\ &= E\{|d(n)|^2 - d(n)\mathbf{x}^H(n)\mathbf{w} - \mathbf{w}^H\mathbf{x}(n)d^*(n) \\ &\quad +\mathbf{w}^H\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{w}\} \\ &= E\{|d(n)|^2\} - E\{d(n)\mathbf{x}^H(n)\}\mathbf{w} - \mathbf{w}^H E\{\mathbf{x}(n)d^*(n)\} \\ &\quad +\mathbf{w}^H E\{\mathbf{x}(n)\mathbf{x}^H(n)\}\mathbf{w} \qquad (**) \end{aligned}$$

Let

$$\begin{aligned} \mathbf{R} &= E\{\mathbf{x}(n)\mathbf{x}^H(n)\} && [\text{autocorrelation of } \mathbf{x}(n)] \\ \mathbf{p} &= E\{\mathbf{x}(n)d^*(n)\} && [\text{cross correlation between } \mathbf{x}(n) \text{ and } d(n)] \end{aligned}$$

Then $(**)$ can be compactly expressed as

$$J(\mathbf{w}) = \sigma_d^2 - \mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w}$$

where we have assumed $x(n)$ & $d(n)$ are zero mean, WSS

The MSE criteria as a function of the filter weight vector **w**

$$J(\mathbf{w}) = \sigma_d^2 - \mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w}$$

Observation: The error is a quadratic function of **w**

Consequences: The error is an M–dimensional bowl–shaped function of **w** with a unique minimum

Result: The optimal weight vector, $\mathbf{w}_0$, is determined by differentiating $J(\mathbf{w})$ and setting the result to zero

$$\nabla_{\mathbf{w}} J(\mathbf{w})|_{\mathbf{w}=\mathbf{w}_0} = 0$$

- A closed form solution exists

Example

Consider a two dimensional case, i.e., a $M = 2$ tap filter. Plot the error surface and error contours.

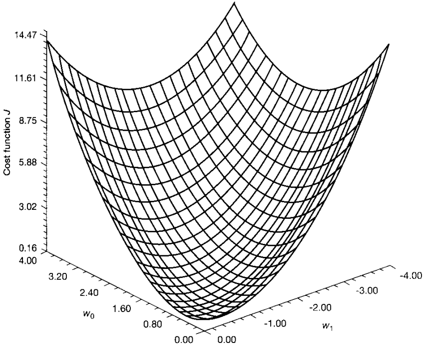


Figure 5.6 Error-performance surface of the two-tap transversal filter described in the numerical example.

Error Surface

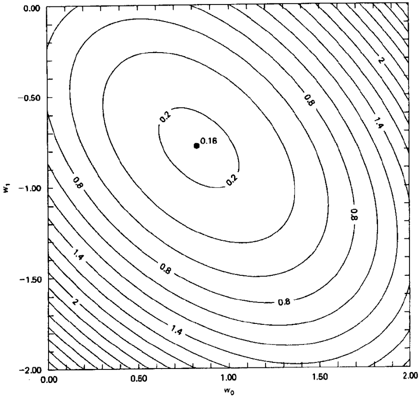


Figure 5.7 Contour plots of the error-performance surface depicted in Fig. 5.6.

Error Contours

Aside (Matrix Differentiation): For complex data,

$$w_k = a_k + jb_k, \qquad k = 0, 1, \cdots, M-1$$

the gradient, with respect to w_k, is

$$\nabla_k(J) = \frac{\partial J}{\partial a_k} + j\frac{\partial J}{\partial b_k}, \qquad k = 0, 1, \cdots, M-1$$

The complete gradient is thus given by

$$\nabla_{\mathbf{w}}(J) = \begin{bmatrix} \nabla_0(J) \\ \nabla_1(J) \\ \vdots \\ \nabla_{M-1}(J) \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial a_0} + j\frac{\partial J}{\partial b_0} \\ \frac{\partial J}{\partial a_1} + j\frac{\partial J}{\partial b_1} \\ \vdots \\ \frac{\partial J}{\partial a_{M-1}} + j\frac{\partial J}{\partial b_{M-1}} \end{bmatrix}$$

Example

Let $\mathbf{c}$ and $\mathbf{w}$ be $M \times 1$ complex vectors.
For $g = \mathbf{c}^H\mathbf{w}$, find $\nabla_{\mathbf{w}}(g)$

Note

$$g = \mathbf{c}^H\mathbf{w} = \sum_{k=0}^{M-1} c_k^* w_k = \sum_{k=0}^{M-1} c_k^*(a_k + jb_k)$$

Thus

$$\begin{aligned}\nabla_k(g) &= \frac{\partial g}{\partial a_k} + j\frac{\partial g}{\partial b_k} \\ &= c_k^* + j(jc_k^*) = 0, \qquad k = 0, 1, \cdots, M-1\end{aligned}$$

Result: For $g = \mathbf{c}^H\mathbf{w}$

$$\nabla_{\mathbf{w}}(g) = \begin{bmatrix} \nabla_0(g) \\ \nabla_1(g) \\ \vdots \\ \nabla_{M\text{-}1}(g) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

Example

Now suppose $g = \mathbf{w}^H \mathbf{c}$.
Find $\nabla_{\mathbf{w}}(g)$

In this case,

$$g = \mathbf{w}^H \mathbf{c} = \sum_{k=0}^{M-1} w_k^* c_k = \sum_{k=0}^{M-1} c_k (a_k - jb_k)$$

and

$$\begin{aligned} \nabla_k(g) &= \frac{\partial g}{\partial a_k} + j \frac{\partial g}{\partial b_k} \\ &= c_k + j(-jc_k) = 2c_k, \qquad k = 0, 1, \cdots, M-1 \end{aligned}$$

Result: For $g = \mathbf{w}^H \mathbf{c}$

$$\nabla_{\mathbf{w}}(g) = \begin{bmatrix} \nabla_0(g) \\ \nabla_1(g) \\ \vdots \\ \nabla_{M-1}(g) \end{bmatrix} = \begin{bmatrix} 2c_0 \\ 2c_1 \\ \vdots \\ 2c_{M-1} \end{bmatrix} = 2\mathbf{c}$$

Example

Lastly, suppose $g = \mathbf{w}^H \mathbf{Q}\mathbf{w}$.
Find $\nabla_{\mathbf{w}}(g)$

In this case,

$$\begin{aligned} g &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} w_i^* w_j q_{i,j} \\ &= \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} (a_i - jb_i)(a_j + jb_j) q_{i,j} \\ \Rightarrow \nabla_k(g) &= \frac{\partial g}{\partial a_k} + j\frac{\partial g}{\partial b_k} \\ &= 2\sum_{j=0}^{M-1} (a_j + jb_j) q_{k,j} + 0 \\ &= 2\sum_{j=0}^{M-1} w_j q_{k,j} \end{aligned}$$

Result: For $g = \mathbf{w}^H \mathbf{Q} \mathbf{w}$

$$\nabla_{\mathbf{w}}(g) = \begin{bmatrix} \nabla_0(g) \\ \nabla_1(g) \\ \vdots \\ \nabla_{M-1}(g) \end{bmatrix} = 2 \begin{bmatrix} \sum_{i=0}^{M-1} q_{0,i} w_i \\ \sum_{i=0}^{M-1} q_{1,i} w_i \\ \vdots \\ \sum_{i=0}^{M-1} q_{M-1,i} w_i \end{bmatrix} = 2\mathbf{Q}\mathbf{w}$$

- Observation: Differentiation result depends on matrix ordering

Returning to the MSE performance criteria

$$J(\mathbf{w}) = \sigma_d^2 - \mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w}$$

Approach: Minimize error by differentiating with respect to $\mathbf{w}$ and set result to 0

$$\begin{aligned} \nabla_{\mathbf{w}}(J) &= \mathbf{0} - \mathbf{0} - 2\mathbf{p} + 2\mathbf{R}\mathbf{w} \\ &= \mathbf{0} \\ \Rightarrow \mathbf{R}\mathbf{w}_0 &= \mathbf{p} \qquad \text{[normal equation]} \end{aligned}$$

Result: The Wiener filter coefficients are defined by

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$$

Question: Does $\mathbf{R}^{-1}$ always exist? Recall $\mathbf{R}$ is positive semi-definite, and usually positive definite

Orthogonality Principle

Consider again the normal equation that defines the optimal solution

$$\begin{aligned} \mathbf{R}\mathbf{w}_0 &= \mathbf{p} \\ \Rightarrow E\{\mathbf{x}(n)\mathbf{x}^H(n)\}\mathbf{w}_0 &= E\{\mathbf{x}(n)d^*(n)\} \end{aligned}$$

Rearranging

$$\begin{aligned} E\{\mathbf{x}(n)d^*(n)\} - E\{\mathbf{x}(n)\mathbf{x}^H(n)\}\mathbf{w}_0 &= \mathbf{0} \\ E\{\mathbf{x}(n)[d^*(n) - \mathbf{x}^H(n)\mathbf{w}_0]\} &= \mathbf{0} \\ E\{\mathbf{x}(n)e_0^*(n)\} &= \mathbf{0} \end{aligned}$$

Note: $e_0^*(n)$ is the error when the optimal weights are used, i.e.,

$$e_0^*(n) = d^*(n) - \mathbf{x}^H(n)\mathbf{w}_0$$

Thus

$$E\{\mathbf{x}(n)e_0^*(n)\} = E\begin{bmatrix} x(n)e_0^*(n) \\ x(n-1)e_0^*(n) \\ \vdots \\ x(n-M+1)e_0^*(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Orthogonality Principle

A necessary and sufficient condition for a filter to be optimal is that the estimate error, $e^*(n)$, be orthogonal to each input sample in $\mathbf{x}(n)$

Interpretation: The observations samples and error are orthogonal and contain no mutual "information"

Objective: Determine the minimum MSE

Approach: Use the optimal weights $\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$ in the MSE expression

$$\begin{aligned} J(\mathbf{w}) &= \sigma_d^2 - \mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w} \\ \Rightarrow J_{\min} &= \sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 - \mathbf{w}_0^H\mathbf{p} + \mathbf{w}_0^H\mathbf{R}(\mathbf{R}^{-1}\mathbf{p}) \\ &= \sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 - \mathbf{w}_0^H\mathbf{p} + \mathbf{w}_0^H\mathbf{p} \\ &= \sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 \end{aligned}$$

Result:

$$J_{\min} = \sigma_d^2 - \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$

where the substitution $\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$ has been employed

Objective: Consider the excess MSE introduced by using a weighted vector that is not optimal.

$$J(\mathbf{w}) - J_{\min} = (\sigma_d^2 - \mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w}) - (\sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 - \mathbf{w}_0^H\mathbf{p} + \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0)$$

Using the fact that

$$\mathbf{p} = \mathbf{R}\mathbf{w}_0 \quad \text{and} \quad \mathbf{p}^H = \mathbf{w}_0^H\mathbf{R}$$

yields

$$\begin{aligned}
J(\mathbf{w}) - J_{\min} &= -\mathbf{p}^H\mathbf{w} - \mathbf{w}^H\mathbf{p} + \mathbf{w}^H\mathbf{R}\mathbf{w} + \mathbf{p}^H\mathbf{w}_0 + \mathbf{w}_0^H\mathbf{p} - \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0 \\
&= -\mathbf{w}_0^H\mathbf{R}\mathbf{w} - \mathbf{w}^H\mathbf{R}\mathbf{w}_0 + \mathbf{w}^H\mathbf{R}\mathbf{w} + \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0 \\
&\quad + \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0 - \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0 \\
&= -\mathbf{w}_0^H\mathbf{R}\mathbf{w} - \mathbf{w}^H\mathbf{R}\mathbf{w}_0 + \mathbf{w}^H\mathbf{R}\mathbf{w} + \mathbf{w}_0^H\mathbf{R}\mathbf{w}_0 \\
&= (\mathbf{w} - \mathbf{w}_0)^H\mathbf{R}(\mathbf{w} - \mathbf{w}_0) \\
\Rightarrow J(\mathbf{w}) &= J_{\min} + (\mathbf{w} - \mathbf{w}_0)^H\mathbf{R}(\mathbf{w} - \mathbf{w}_0)
\end{aligned}$$

Finally, using the eigenvalue and vector representation $\mathbf{R} = \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^H$

$$J(\mathbf{w}) = J_{\min} + (\mathbf{w} - \mathbf{w}_0)^H \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^H (\mathbf{w} - \mathbf{w}_0)$$

or defining the eigenvector transformed difference

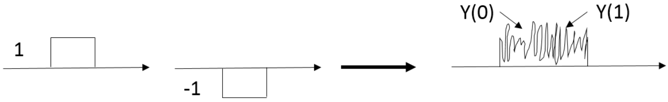
$$\begin{aligned} \mathbf{v} &= \mathbf{Q}^H(\mathbf{w} - \mathbf{w}_0) \qquad (*) \\ \Rightarrow J(\mathbf{w}) &= J_{\min} + \mathbf{v}^H \boldsymbol{\Omega} \mathbf{v} \\ &= J_{\min} + \sum_{k=1}^{M} \lambda_k v_k v_k^* \end{aligned}$$

Result:

$$J(\mathbf{w}) = J_{\min} + \sum_{k=1}^{M} \lambda_k |v_k|^2$$

Note: $(*)$ shows that v_k is the difference $(\mathbf{w} - \mathbf{w}_0)$ projected onto eigenvector $\mathbf{q}_k$

Example: Binary Phase-Shift Keying Symbol Estimate



Let x be a signal that is either -1 or 1 with probability $1/2$.
Collect two noisy measurements of the same value of x:

$$y(0) = x + v(0);$$

$$y(1) = x + v(1);$$

where $v(0)$ and $v(1)$ are independent zero-mean Gaussian with $\sigma_v^2 = 1$.
The optimal linear estimator of x given $\mathbf{y} = [y(0), y(1)]^T$ is

$$\hat{x} = \mathbf{w^H y}.$$

The autocorrelation matrix of **y** is

$$\mathbf{R}_y = \begin{bmatrix} E[y(0)^2] & E[y(0)y^*(1)] \\ E[y(1)y^*(0)] & E[y(1)^2] \end{bmatrix}.$$

Notice that x, $v(0)$ and $v(1)$ are independent, we get

$$E[y(0)^2] = E[x^2] + E[v(0)^2] = 1 + 1 = 2;$$

$$E[y(1)^2] = E[x^2] + E[v(1)^2] = 1 + 1 = 2;$$

$$E[y(0)y^*(1)] = E[(x + v(0))(x + v(1))^*] = E[x^2] = 1;$$

$$E[y(1)y^*(0)] = E[(x + v(1))(x + v(0))^*] = E[x^2] = 1.$$

So we have

$$\mathbf{R}_y = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The cross-correlation vector of the desired value x and the measurements y is

$$\mathbf{P} = \begin{bmatrix} E[xy^*(0)] & E[xy^*(1)] \end{bmatrix}^H,$$

where

$$E[xy^*(0)] = E[x(x+v(0))] = E[x^2] = 1;$$
$$E[xy^*(1)] = E[x(x+v(1))] = E[x^2] = 1.$$

So we have

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \end{bmatrix}^H,$$

The weights of the estimator are:

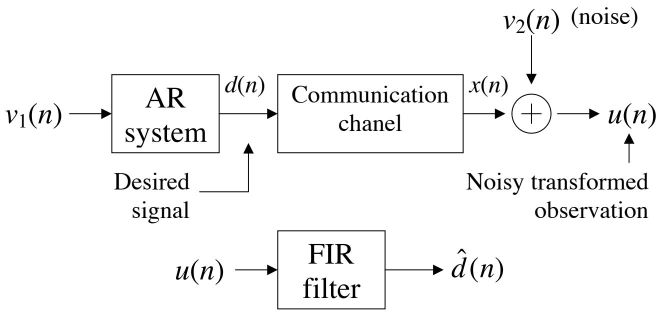
$$\mathbf{w} = \mathbf{R}_y^{-1}\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

That is

$$\hat{x} = \frac{1}{3}(y(0) + y(1)).$$

Example

Consider the following system



Objective: Determine the optimal filter for a given system and channel

Specific Objective

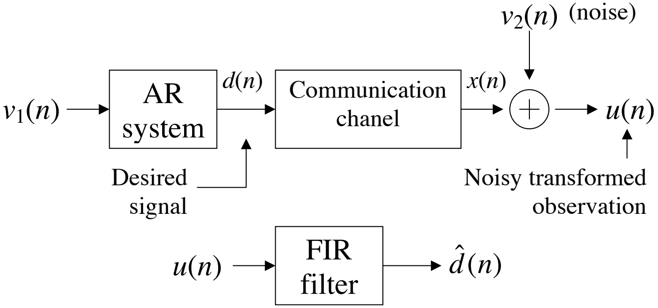
Determine the optimal order two filter weights, $\mathbf{w}_0$, for

$$\begin{aligned} H_1(z) &= \frac{1}{1+0.8458z^{-1}} \qquad \text{[AR process]} \\ H_2(z) &= \frac{1}{1-0.9458z^{-1}} \qquad \text{[communication channel]} \end{aligned}$$

where $v_1(n)$ and $v_2(n)$ zero mean white noise processes with $\sigma_1^2 = 0.27$ and $\sigma_2^2 = 0.1$

Note: To determine $\mathbf{w}_0$, we need:

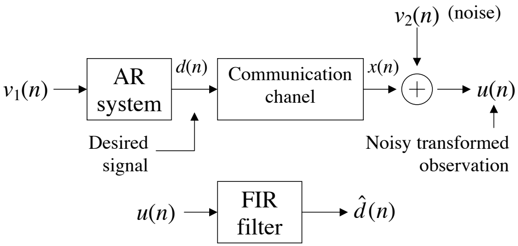
- $\mathbf{R}_\mathbf{u}$ — auto–correlation of the received signal
- $\mathbf{p}$ — the cross correlation between received signal $\mathbf{u}(n)$ and the desired signal $d(n)$



Procedure: Consider $\mathbf{R_u}$ first

Since $u(n) = x(n) + v_2(n)$, where $v_2(n)$ is white with $\sigma_2^2 = 0.1$ and is uncorrelated with $x(n)$

$$\mathbf{R_u} = \mathbf{R_x} + \mathbf{R_{v_2}} = \mathbf{R_x} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$



Next, consider the relation between $x(n)$ and $v_1(n)$

$$X(z) = H_1(z)H_2(z)V_1(z)$$

where for the give systems

$$H_1(z)H_2(z) = \frac{1}{(1+0.8458z^{-1})(1-0.9458z^{-1})}$$

Converting to the time domain, we see $x(n)$ is an order 2 AR process

$$\begin{aligned} x(n) - 0.1x(n-1) - 0.8x(n-2) &= v_1(n) \\ x(n) + a_1 x(n-1) + a_2 x(n-2) &= v_1(n) \end{aligned}$$

Since $x(n)$ is a real valued order two AR process, the Yule-Walker equations are given by

$$\begin{bmatrix} r(0) & r(1) \\ r^*(1) & r(0) \end{bmatrix} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} r^*(1) \\ r^*(2) \end{bmatrix}$$

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}$$

Solving for the coefficients:

$$\begin{aligned} -a_1 &= \frac{r(1)[r(0) - r(2)]}{r^2(0) - r^2(1)} \\ -a_2 &= \frac{r(0)r(2) - r^2(1)}{r^2(0) - r^2(1)} \end{aligned}$$

Question: What are the known and unknown terms in this system?

Note: We must solve the system to obtain the unknown $r(\cdot)$ values

Noting $r(0) = \sigma_x^2$ and rearranging to solve for $r(1)$ and $r(2)$

$$\begin{aligned} r(1) &= \frac{-a_1}{1+a_2}\sigma_x^2 && (*) \\ r(2) &= \left(-a_2 + \frac{a_1^2}{1+a_2}\right)\sigma_x^2 && (**) \end{aligned}$$

The Yule-Walker equations also stipulate

$$\sigma_{v_1}^2 = r(0) + a_1 r(1) + a_2 r(2) \qquad (***)$$

Next, utilize the determined $r(1), r(2)$ and given $a_1, a_2, \sigma_{v_1}^2$ values to determine $r(0) = \sigma_x^2$

Substituting $(*)$ and $(**)$ into $(***)$, utilize $r(0) = \sigma_x^2$, and rearranging

$$
\begin{aligned}
\sigma_{v_1}^2 &= r(0) + a_1 r(1) + a_2 r(2) \\
&= \sigma_x^2 + a_1 \left(\frac{-a_1}{1+a_2} \right) \sigma_x^2 + a_2 \left(-a_2 + \frac{a_1^2}{1+a_2} \right) \sigma_x^2 \\
&= \left(1 + \frac{-a_1^2}{1+a_2} - a_2^2 + \frac{a_1^2 a_2}{1+a_2} \right) \sigma_x^2 \\
\Rightarrow \sigma_x^2 &= \left(\frac{1+a_2}{1-a_2} \right) \frac{\sigma_{v_1}^2}{(1+a_2)^2 - a_1^2}
\end{aligned}
$$

Note: All correlation terms have been determined since $r(0) = \sigma_x^2$

Using $a_1 = -0.1$, $a_2 = -0.8$, and $\sigma_{v_1}^2 = 0.27$,

$$\begin{aligned} \sigma_x^2 &= \left(\frac{1+a_2}{1-a_2}\right)\frac{\sigma_{v_1}^2}{(1+a_2)^2 - a_1^2} \\ &= 1 = \end{aligned}$$

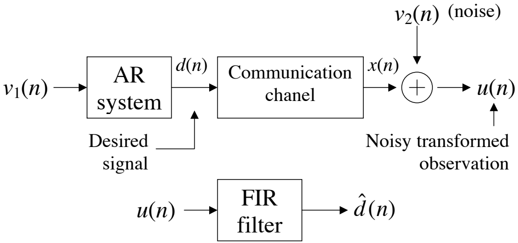
Thus $r(0) = 1$.
Similarly

$$r(1) = \frac{-a_1}{1+a_2}\sigma_x^2 = 0.5$$

and finally, $\mathbf{R_x}$ is

$$\mathbf{R_x} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Recall the overall system



Putting the pieces of $\mathbf{R_u}$ together

$$\begin{aligned}\mathbf{R_u} &= \mathbf{R_x} + \mathbf{R_{v_2}} \\ &= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \\ &= \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}\end{aligned}$$

Recall: The Wiener solution is given by $\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$

Note: $\mathbf{R}$ is known, but $\mathbf{p}$ is still to be determined

$$\mathbf{p} = E\left\{ \begin{bmatrix} d(n)u(n) \\ d(n)u(n-1) \end{bmatrix} \right\}$$

Recall

$$X(z) = H_2(z)D(z) = \frac{D(z)}{1-0.9458z^{-1}}$$

or in the time domain

$$x(n) - 0.9458x(n-1) = d(n)$$

Lastly, the observation is corrupted by additive noise

$$u(n) = x(n) + v_2(n)$$

Thus

$$
\begin{aligned}
E\{u(n)d(n)\} &= E\{[x(n)+v_2(n)][x(n)-0.9458x(n-1)]\} \\
&= E\{x^2(n)\}+E\{x(n)v_2(n)\}-0.9458E\{x(n)x(n-1)\} \\
&\quad -0.9458E\{v_2(n)x(n-1)\} \\
&= \sigma_x^2+0-0.9458r(1)-0 \\
&= 1-0.9458\left(\frac{1}{2}\right) \\
&= 0.5272
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\{u(n-1)d(n)\} &= E\{[x(n-1)+v_2(n-1)][x(n)-0.9458x(n-1)]\} \\
&= r(1)-0.9458r(0) \\
&= -0.4458
\end{aligned}
$$

Thus

$$\mathbf{p} = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$$

Final Solution:

$$\begin{aligned} \mathbf{w}_0 &= \mathbf{R}^{-1}\mathbf{p} \\ &= \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix} \\ &= \begin{bmatrix} 0.8360 \\ -0.7853 \end{bmatrix} \end{aligned}$$

- The optimal filter weights are a function of the source signal statistics and the communications channel
- Question: How do we optimized a filter of when the statistics are not known in closed form or *a priori*?