Nonlinear Signal Processing
ELEG 833

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Outline

Order Statistics

If the random variables $X_1, X_2, \ldots, X_N$ are arranged in ascending order of magnitude such that

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)},$$

we denote $X_{(i)}$ as the ith *order statistic* for $i = 1, \ldots, N$.
The extremes $X_{(N)}$ and $X_{(1)}$, are useful tools in the detection of outliers.
The range $X_{(N)} - X_{(1)}$ is a quick estimator of the dispersion.

Outline

1. Order Statistics
   - Distributions Of Order Statistics
   - Moments Of Order Statistics
     - Order Statistics From Uniform Distributions
   - Order Statistics Containing Outliers

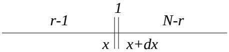
Distributions Of Order Statistics

For continuous, independent and identically distributed (i.i.d.) samples, the density of the rth order statistic is formed as follows. First, decompose the event that $x < X_{(r)} \leq x + dx$ into three exclusive parts:

A) that $r - 1$ of the samples X_i are less than or equal to x

B) that one is between x and $x + dx$

C) that $N - r$ are greater than $x + dx$.



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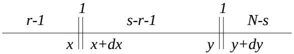


(b)

A) The probability that $N-r$ are greater than or equal to $x + dx$ is $[1-F(x+dx)]^{N-r}$

B) The probability that one is between x and $x + dx$ is $f_x(x)\ dx$

C) The probability that $r-1$ are less than or equal to x is $F(x)^{r-1}$

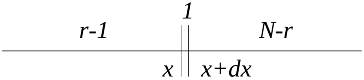
The probability of having more than one sample in $(x, x+dx]$ is on the order of $(dx)^2$ and is negligible as dx approaches zero.
Counting all enumerations of N samples in the three respective groups:

$$\begin{aligned} f_{(r)}(x)\ dx &= Pr\left[\ x < X_{(r)} \leq x + dx\ \right] \\ &= \frac{N!}{(r-1)!\ (N-r)!} F(x)^{r-1}\ [1-F(x)]^{N-r}\ f_x(x)\ dx. \end{aligned}$$

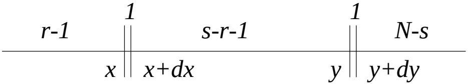
The trinomial coefficient above follows from the multinomial coefficient. Given a set of N objects, k_1 labels of type 1, k_2 labels of type 2, $\cdots$, and k_m labels of type m where $k_1 + k_2 + \cdots + k_m = N$, the number of ways in which we may assign the labels to the N objects is given by

$$\frac{N!}{k_1!\ k_2! \cdots k_m!}\ . \tag{1}$$

The trinomial coefficient in (1) is a special case of (1) with $k_1 = r - 1$, $k_2 = 1$, and $k_3 = N - r$.



(a)



(b)

Figure: The event $x < X_{(r)} \leq x + dx$ and $y < X_{(s)} \leq y + dy$ can be seen as $r-1$ of the samples X_i are less than x, that one of the samples is between x and $x + dx$, that $s-r-1$ of the samples X_i are less than y but greater than x, that one of the samples is between y and $y + dy$, and finally that $N-s$ of the samples are greater than y.

For $1 \leq r < s \leq N$ and $x \leq y$, $f_{(r,s)}(x, y)$ is obtained by decomposing the event

$$x < X_{(r)} \leq x + dx < y < X_{(s)} \leq y + dy \tag{2}$$

into five mutually exclusive parts:

A) That $r - 1$ of the samples X_i are less than x

B) That one of the samples is between x and $x + dx$

C) That $s - r - 1$ of the samples X_i are less than y but greater than $x + dx$

D) That one of the samples is between y and $y + dy$

E) That $N - s$ of the samples are greater than $y + dy$.

The probability of occurrence for each of the five listed parts is

A) $F(x)^{r-1}$

B) $f_x(x)\ dx$

C) $[F(y)\ -\ F(x+dx)]^{s-r-1}$

D) $f_x(y)\ dy,$

E) $[1\ -\ F(y+dy)]^{N-s}$

Using the multinomial counting principle to enumerate all possible occurrences in each part, and the fact that $F(x + dx) \sim F(x)$ and $F(y + dy) \sim F(y)$ as $dx \to 0$ we obtain the joint density function

$$
\begin{aligned}
f_{(r,s)}(x, y) \quad = \quad & \frac{N!}{(r-1)!\;(s-r-1)!\;(N-s)!} F(x)^{r-1}\; f_x(x) \\
& [F(y) \;-\; F(x)]^{s-r-1}\; f_x(y)\; [1 \;-\; F(y)]^{N-s}.
\end{aligned}
$$

These density functions are only valid for continuous random variables

For continuous and discontinuous distributions: let the i.i.d. variables $X_1, X_2, \ldots, X_N$ have a parent distribution $F(x)$, the pdf of $X_{(N)}$ is

$$\begin{aligned} F_{(N)}(x) &= Pr\{X_{(N)} \leq x\} \\ &= Pr\{\text{all } X_{(i)} \leq x\} \\ &= Pr\{\text{all } X_i \leq x\} = [F(x)]^N. \end{aligned}$$

due to independence. Similarly, the pdf of the minimum sample $X_{(1)}$ is

$$\begin{aligned} F_{(1)}(x) &= Pr\{X_{(1)} \leq x\} = 1 - Pr\{X_{(1)} > x\} \\ &= 1 - Pr\{\text{all } X_i > x\} = 1 - [1 - F(x)]^N, \end{aligned}$$

since $X_{(1)}$ is less than, or equal to, all the samples in the set.

The distribution function for the general case is

$$
\begin{aligned}
F_{(r)}(x) &= Pr\{X_{(r)} \leq x\} \\
&= Pr\{\text{at least } r \text{ of the } X_i \text{ are less than or equal to } x\} \\
&= \sum_{i=r}^{N} Pr\{\text{exactly } i \text{ of the } X_i \text{ are less than or equal to } x\} \\
&= \sum_{i=r}^{N} \binom{N}{i} [F(x)]^i [1 - F(x)]^{N-i}.
\end{aligned}
$$

The joint distribution function $F_{(r,s)}(x, y)$ of $X_{(r)}$ and $X_{(s)}$, for $1 \leq r < s \leq N$, is (for $x < y$)

$$
\begin{aligned}
F_{(r,s)}(x, y) &= Pr\{\text{at least } r \text{ of the } X_i \leq x, \text{ at least } s \text{ of the } X_i \leq y\ \} \\
&= \sum_{j=s}^{N} \sum_{i=r}^{j} Pr\{\text{exactly } i \text{ of } X_1, X_2 \ldots, X_n \text{ are at most } x \\
&\qquad \text{and exactly } j \text{ of } X_1, X_2 \ldots, X_n \text{ are at most } y\} \\
&= \sum_{j=s}^{N} \sum_{i=r}^{j} \frac{N!}{i!(j-i)!(N-j)!} [F(x)]^i [F(y) - F(x)]^{j-i} \\
&\qquad \times [1 - F(y)]^{N-j}. \qquad (3)
\end{aligned}
$$

Notice that for $x \geq y$, the ordering $X_{(r)} < x$ with $X_{(s)} \leq y$, implies that $F_{(r,s)}(x, y) = F_{(s)}(y)$.

Outline

Moments Of Order Statistics

Moments of order statistics are defined in the same fashion as moments of arbitrary random variables. Here we always assume that the sample size is N. The expected value of the rth order statistic is denoted as $\mu_{(r)}$ and is found as

$$\begin{aligned}
\mu_{(r)} &= \int_{-\infty}^{\infty} x \; f_{(r)}(x) \; dx \\
&= \frac{N!}{(r-1)! \; (N-r)!} \int_{-\infty}^{\infty} x \; F(x)^{r-1}[1-F(x)]^{N-r} \; f_x(x) \; dx.
\end{aligned}$$

The statistical characteristics of the order-statistics $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$ are not homogeneous since

$$EX_{(r)} \neq EX_{(s)}$$

for $r \neq s$, as expected since $E\{X_{(r)}\}$ should be less than $E\{X_{(r+1)}\}$.

In general, the expectation of products of order statistics are not symmetric

$$E(X_{(r)}X_{(r+s)}) \neq E(X_{(r)}X_{(r-s)}). \tag{4}$$

This symmetry only holds in very special cases. One such case is when the parent distribution is symmetric and where $r = (N+1)/2$ such that $X_{(r)}$ is the median. The covariance of $X_{(r)}$ and $X_{(s)}$ is written as

$$\mathrm{cov}\left[X_{(r)}X_{(s)}\right] = E\left\{\left(X_{(r)} - \mu_{(r)}\right)\left(X_{(s)} - \mu_{(s)}\right)\right\}. \tag{5}$$

The covariance of order statistics satisfies $cov[X_{(r)}X_{(s)}] \geq 0$.

Order Statistics From Uniform Distributions

Consider N samples of a standard uniform distribution with density function $f_u(u) = 1$, for $0 \leq u \leq 1$. The density function of the rth order-statistic $U_{(r)}$ is

$$f_{(r)}(u) = \frac{N!}{(r-1)!\,(N-r)!} u^{r-1}(1-u)^{N-r} \tag{6}$$

in the interval $0 \leq u \leq 1$. The mode of the density function can be found at $(r-1)/(N-1)$.

The kth moment of $U_{(r)}$ is found from the above as

$$\begin{aligned} \mu_{(r)}^{(k)} &= \int_0^1 u^k f_{(r)}(u)\, du \\ &= \frac{N!}{(r-1)!(N-r)!} \int_0^1 u^k u^{r-1}(1-u)^{N-r}\, du \qquad (7) \\ &= B(r+k, N-r+1)/B(r, N-r+1), \qquad (8) \end{aligned}$$

where we make use of the complete beta function

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \qquad (9)$$

for $p, q > 0$. Simplifying leads to

$$\mu_{(r)}^{(k)} = \frac{N!\ \ (r+k-1)!}{(N+k)!\ (r-1)!}. \qquad (10)$$

In particular, the first moment of the rth order statistic is

$$\mu_{(r)}^{(1)} = r/(N+1).$$

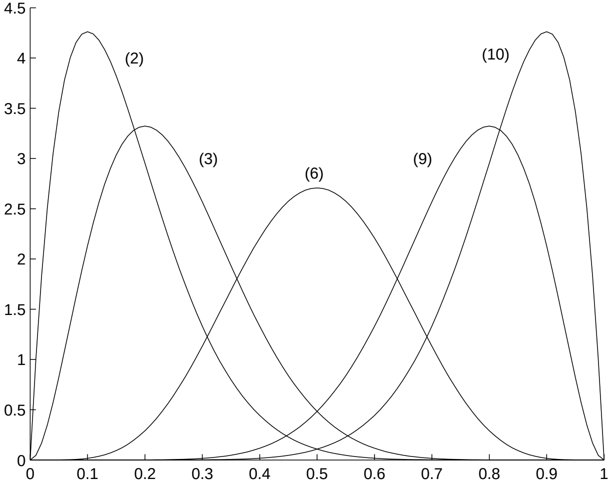


Figure: Density functions of $X_{(2)}, X_{(3)}, X_{(6)}$ (median), $X_{(9)}$, and $X_{(10)}$ for a set of eleven uniformly distributed samples.

Outline

Order Statistics Containing Outliers

The N sample set consist of $N-1$ i.i.d. variates X_i, $i = 1, \ldots, N-1$, and the contaminant variable Y which is also independent.
Let $F(x)$ and $G(x)$ be the continuous parent distributions of X_i and Y.
Furthermore, let

$$Z_{(1):N} \leq Z_{(2):N} \leq \cdots \leq \quad Z_{(N):N}$$

be the order statistics obtained by arranging the N samples in increasing order of magnitude.

The distribution of the maxima denoted as $H_{(N):N}(x)$ is

$$\begin{aligned} H_{(N):N}(x) &= Pr\{\text{all of } X_1, \ldots, X_{N-1}, \text{ and } Y \leq x\} \\ &= F(x)^{N-1} \;\; G(x). \end{aligned}$$

The distribution of the *ith* order statistic, for $1 < i \leq N-1$ is

$$\begin{aligned} H_{(i):N}(x) &= Pr\{\text{ at least } i \text{ of } X_1, X_2, \ldots, X_{N-1}, Y \leq x\} \\ &= Pr\{\text{exactly } i-1 \text{ of } X_1, X_2, \ldots, X_{N-1} \leq x \text{ and } Y \leq x\} \\ &\quad + Pr\{\text{at least } i \text{ of } X_1, X_2, \ldots, X_{N-1} \leq x\} \\ &= \S \begin{pmatrix} N-1 \\ i-1 \end{pmatrix} (F(x))^{i-1}(1-F(x))^{N-i} G(x) + F_{(i):N-1}(x) \end{aligned}$$

where $F_{(i):N-1}(x)$ is the distribution of the *ith* order statistic in a sample of size $N-1$ drawn from a parent distribution $F(x)$.

The density function of $Z_{(i):N}$ can be obtained by differentiating the above or by direct derivation which is left as an exercise:

$$
\begin{aligned}
h_{(i):N}(x) &= \tfrac{(N-1)!}{(i-2)!(N-i)!}(F(x))^{i-2}(1-F(x))^{N-i}G(x)f(x) \\
&+ \tfrac{(N-1)!}{(i-1)!(N-i)!}(F(x))^{i-1}(1-F(x))^{N-i}g(x) \\
&+ \tfrac{(N-1)!}{(i-1)!(N-i-1)!}(F(x))^{i-1}(1-F(x))^{N-i-1}(1-G(x))f(x)
\end{aligned}
$$

where the first term drops out if $i = 1$, and the last term if $N = i$.

Densities of $Z_{(2)}, Z_{(6)}$ (median), and $Z_{(10)}$ for a sample set of 11, laplacian random variables. In the contaminated case, one the mean of one sample is shifted to 20.

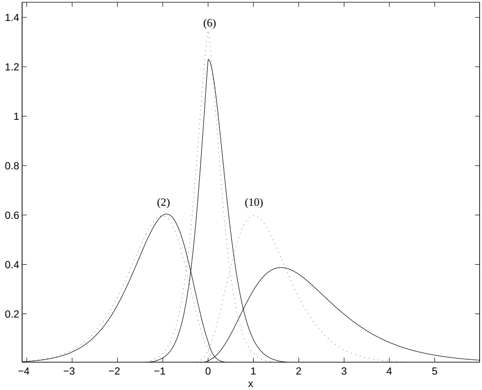


Figure: Density functions of $Z_{(2)}, Z_{(6)}$ (median), and $Z_{(10)}$ with (solid) and without contamination (dotted).