Nonlinear Signal Processing
ELEG 833

Gonzalo R. Arce

Department of Electrical and Computer Engineering
University of Delaware
arce@ee.udel.edu

Fall 2008

Outline

1. Non-Gaussian Models
   - Generalized Gaussian Distributions
   - Stable Distributions
   - Symmetric Stable Distributions
   - Generalized Central Limit Theorem
   - Lower Order Moments
   - Zero Order Statistics

Non-Gaussian Models

A number of distributions with heavier-than-Gaussian tails have been proposed. A popular example is the *contaminated Gaussian* model

$$f(x) = (1-\epsilon)f_n(x) + \epsilon f_c(x) \tag{1}$$

- $f_n(x)$ is the *nominal* Gaussian density with variance σ_n^2
- ϵ is a small positive constant determining the percentage of contamination,
- $f_c(x)$ is the contaminating Gaussian density with $\sigma_c^2 \gg \sigma_n^2$.

Intuitively, one out of $1/\epsilon$ samples is contaminated.
Drawback: *Over-parameterized.*

Outline

Generalized Gaussian Distributions

Definition (Generalized Gaussian Distribution)

The p.d.f. for the generalized Gaussian distribution is

$$f(x) = \frac{k}{2\sigma\Gamma(1/k)} \exp^{-(|x-\beta|/\sigma)^k}, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

The scale is determined by $\sigma > 0$; impulsiveness related to $k > 0$.

- The standard Gaussian distribution is a special case for $k = 2$.
- For $k = 1$, the Laplacian, distribution is

$$f(x) = \frac{1}{2\sigma} e^{-|x-\beta|/\sigma}. \tag{3}$$

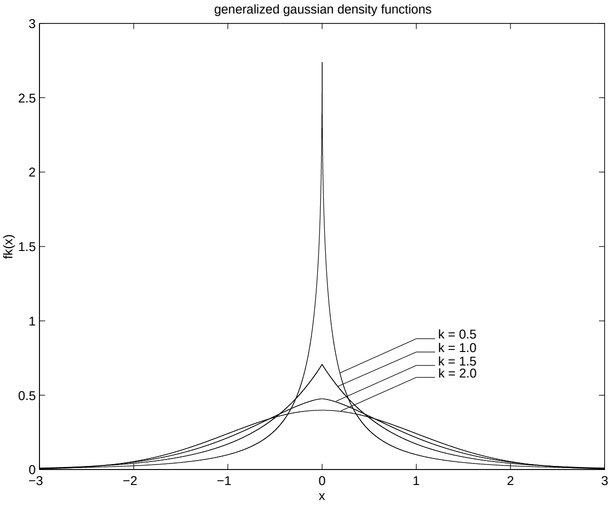


Figure: Generalized Gaussian density functions for different values of k.

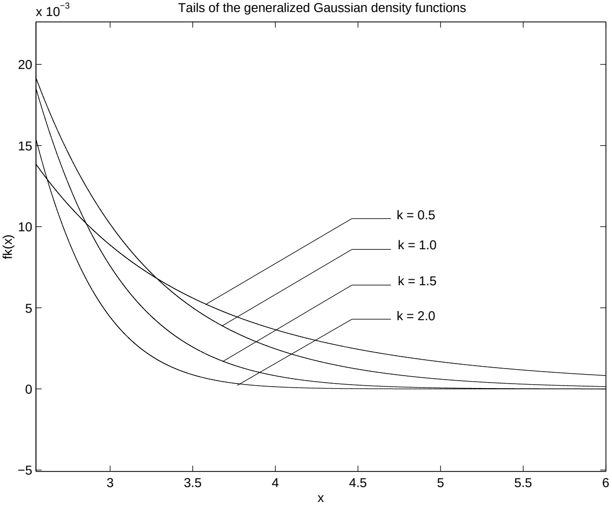


Figure: Tails of the Generalized Gaussian density functions for different k.

Outline

1. Non-Gaussian Models
   - Generalized Gaussian Distributions
   - Stable Distributions
   - Symmetric Stable Distributions
   - Generalized Central Limit Theorem
   - Lower Order Moments
   - Zero Order Statistics

Stable Distributions

Stable distributions are described by four parameters (Lévy 1925):

- an *index of stability* $\alpha \in (0, 2]$ (tail thickness)
- a scale parameter $\gamma > 0$
- a skewness parameter $\delta \in [-1, 1]$
- a location parameter $\beta \in \mathcal{R}$.

For $\delta = 0$, the stable distribution is symmetric about β.

Definition (Stable Random Variables)

A random variable X is *stable* if for X_1 and X_2 independent copies of X and for arbitrary positive constants a and b, there are constants c and d such that

$$aX_1 \; + \; bX_2 \stackrel{d}{=} cX + d. \tag{4}$$

Shape of X is preserved under addition up to scale and shift.

For Gaussian random variables, $c^2 = a^2 + b^2$ and $d = (a + b - c)\mu$ where μ is the mean of the parent Gaussian distribution.

Other stable distributions are the Cauchy and Lévy distributions. The density function, for $X \sim \text{Cauchy}(\gamma, \beta)$ has the form

$$f(x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x - \beta)^2}, \quad -\infty < x < \infty. \tag{5}$$

The Lévy density function is totally skewed concentrating on $(0, \infty)$. The density function for $X \sim \text{Lévy}(\gamma, \delta)$ has the form

$$f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x - \delta)^{3/2}} \exp\left(-\frac{\gamma}{2(x - \delta)}\right), \quad -\delta < x < \infty. \tag{6}$$

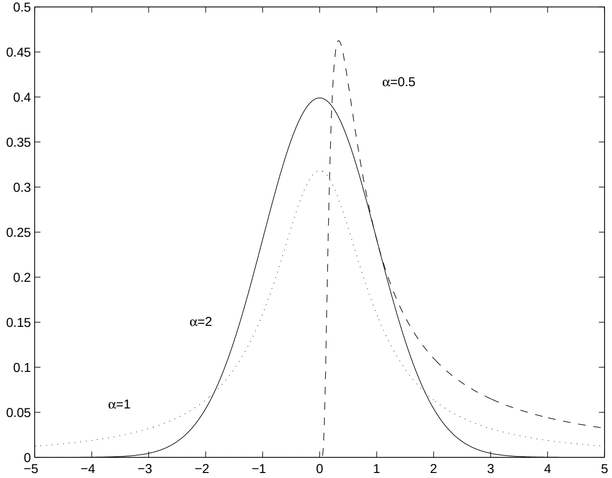


Figure: Density functions of standarized Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1$), and Lévy ($\alpha = 0.5,\ \delta = 1$).

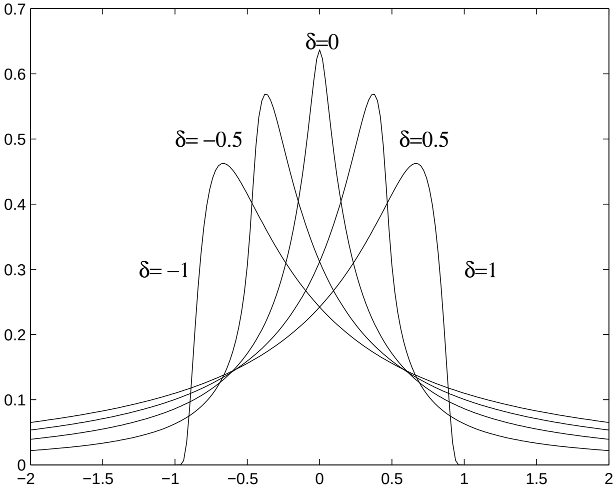


Figure: Density functions of skewed stable variables ($\alpha = 0.5$).

Outline

Symmetric Stable Distributions

Symmetric α-stable or $S\alpha S$ distributions are defined when the skewness parameter δ is set to zero. These can be characterized by the characteristic function

$$\phi(\omega) = E \exp(j\omega X) = \int_{-\infty}^{\infty} \exp(j\omega x) f(x) dx \tag{7}$$

Definition (Characteristic Function of SαS Distributions)

A *symmetrically stable* random variable is characterized by

$$\phi(\omega) = e^{-\gamma|\omega|^{\alpha}}. \tag{8}$$

Definition (Symmetric Stable Density Functions)

A general, "zero-centered", symmetric stable random variable with unitary dispersion can be characterized by:

$$f_\alpha(x) = \begin{cases} SS\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(k\alpha+1)\sin(\frac{\pi k\alpha}{2})|x|^{-k\alpha-1} & SS\text{for } 0<\alpha<1, \quad x \neq \\ \frac{1}{\pi(x^2+1)} & \text{for } \alpha = 1 \\ \frac{1}{\pi\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\Gamma(\frac{2k+1}{\alpha})x^{2k} & \text{for } 1<\alpha<2 \\ \frac{1}{2\sqrt{\pi}}\exp[-\frac{x^2}{4}] & \text{for } \alpha = 2. \end{cases} \tag{9}$$

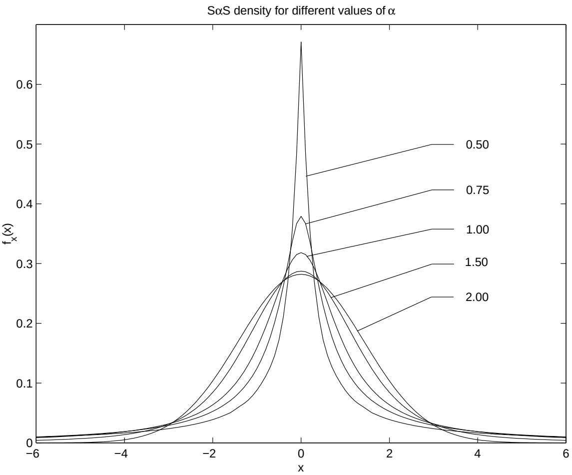


Figure: Density functions of Symmetric stable distributions for different values of the tail constant α.

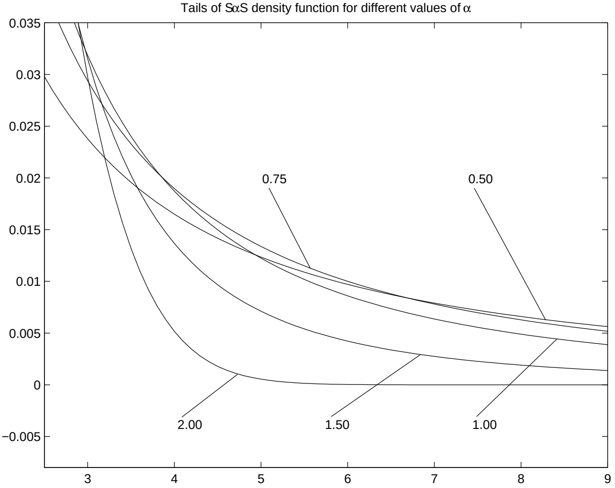


FIGURE: Tails of symmetric stable distributions for different values of the tail constant α.

Outline

1. Non-Gaussian Models
   - Generalized Gaussian Distributions
   - Stable Distributions
   - Symmetric Stable Distributions
   - Generalized Central Limit Theorem
   - Lower Order Moments
   - Zero Order Statistics

Generalized Central Limit Theorem

Theorem (Generalized Central Limit Theorem)

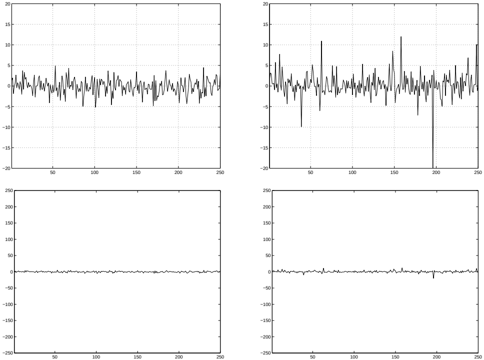
Let X_1,
$X_2, \cdots$ be an independent, identically distributed sequence of (possibly shift corrected) random variables. There exist constants a_n such that as $n \to \infty$ the sum

$$a_n(X_1 + X_2 + \cdots) \xrightarrow{d} Z \tag{10}$$

if and only if Z is a stable random variable with some $0 < \alpha \leq 2$.

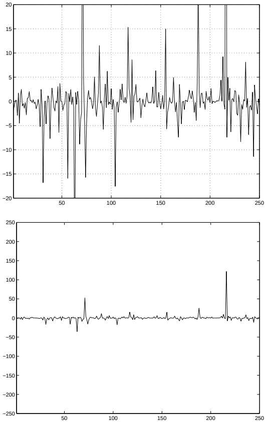
The generalized CLT constitutes a strong argument compelling the use of stable models in practice.

Figures 7 and 8 illustrate the impulsive behavior of symmetric stable processes as the characteristic exponent α is varied.

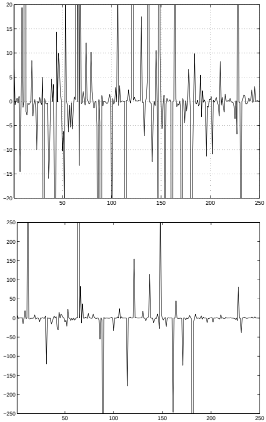


$\alpha = 2$ $\qquad\qquad$ $\alpha = 1.7$

FIGURE: Impulsive behavior of i.i.d. α-stable signals as the tail constant α is varied. Signals are plotted twice under two different scales.



$\alpha = 1$



$\alpha = 0.6$

Figure: Impulsive behavior of i.i.d. α-stable signals as the tail constant α is varied. Signals are plotted twice under two different scales.

Outline

1. Non-Gaussian Models
   - Generalized Gaussian Distributions
   - Stable Distributions
   - Symmetric Stable Distributions
   - Generalized Central Limit Theorem
   - Lower Order Moments
   - Zero Order Statistics

Lower Order Moments

The existance of second-order moments depends on the tail of the distribution: $P(|X| > x)$ as $x \to \infty$.
The tails of the Laplacian distribution are heavier than that of the Gaussian distribution but remain of exponential order with

$$P(|X| > x) \sim \exp^{-x/\sigma} . \tag{11}$$

Infinite variance processes are modeled by pdf's with algebraic tails for which

$$P(X > x) \sim cx^{-\alpha} \tag{12}$$

for some fixed c and $\alpha > 0$.

Theorem (Stable Distribution Tails)

Let $X \sim S(\alpha)$ be a symmetric stable random variable with $0 < \alpha < 2$, then as $x \to \infty$,

$$P(X > x) \sim \Gamma(\alpha)\frac{sin(\pi\alpha/2)}{\pi}x^{-\alpha}. \tag{13}$$

Theorem

Algebraic-tailed random variables exhibit finite absolute moments for orders less than α

$$E(|X|^p) < \infty, \quad \text{if } p < \alpha. \tag{14}$$

Conversely, if $p \geq \alpha$*, the absolute moments become infinite.*

For algebraic-tailed processes, it is better to rely on *fractional lower-order moments* (FLOMs): $E|X|^p = \int_{-\infty}^{\infty} |x|^p f(x) dx$, which exist for $0 < p < \alpha$.

Proof: The variable Y is replaced by $|X|^p$ in the first moment relationship

$$EY = \int_{-\infty}^{\infty} P(Y > y)dy \tag{15}$$

yielding

$$E(|X|^p) = \int_0^{\infty} P(|X|^p > t)dt \tag{16}$$

$$= \int_0^{\infty} pu^{p-1} P(|X| > u)du \tag{17}$$

Since

$$P(X > x) \sim cx^{-\alpha} \tag{18}$$

$E(|X|^p)$ diverges for any distribution having algebraic tails.

Proposition

The FLOMs for a $S\alpha S$ random variable with zero location parameter and dispersion γ is given by

$$E(|X|^p) = C(p,\alpha)\gamma^{p/\alpha} \qquad 0 < p < \infty, \tag{19}$$

where

$$C(p,\alpha) = \frac{2^{p+1}\Gamma\left(\frac{p+1}{2}\right)\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)}. \tag{20}$$

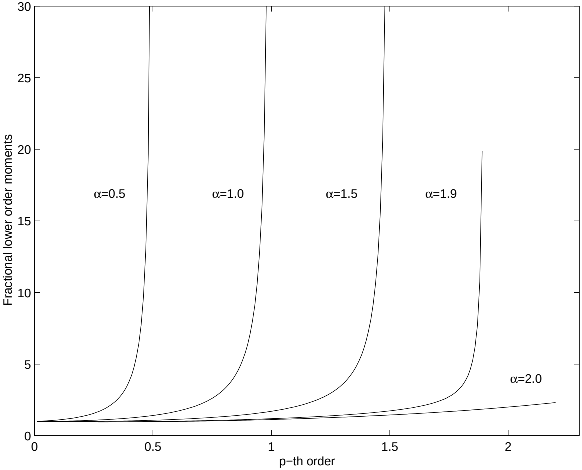


Figure: Fractional lower-order moments of the standardized $S\alpha S$ random variable.

Outline

1. Non-Gaussian Models
   - Generalized Gaussian Distributions
   - Stable Distributions
   - Symmetric Stable Distributions
   - Generalized Central Limit Theorem
   - Lower Order Moments
   - Zero Order Statistics

Zero Order Statistics

For a given $p > 0$, there will always be a "remaining" class of processes (those with $\alpha \leq p$) for which the associated FLOS do not exist.

- Restricting the values of p to the valid interval $(0; \alpha)$ requires previous knowledge of α.

Zero-order statistics (ZOS) provide a common ground for the analysis of basically *any* distribution of practical use.

Zero-order statistics are based on logarithmic "moments" of the form $E\log|X|$.

Theorem

Let X be a random variable with algebraic or lighter tails. Then, $E\log|X| < \infty$.

Proof: If X has algebraic or lighter tails, there exists a $p > 0$ such that $E|X|^p < \infty$. Jensen's inequality guarantees that for a concave function ϕ, and a random variable Z, $E\phi(Z) \leq \phi(EZ)$. Letting $\phi(x) = \log|x|/p$ and $Z = |X|^p$ leads to

$$E\log|X| = E\left(\frac{\log|X|^p}{p}\right) \leq \frac{\log(E|X|^p)}{p} < \infty, \qquad (21)$$

which is the desired result.

The *power* EX^2 is a widely accepted measure of signal strength. EX^2, is infinite when the processes exhibit algebraic tails. Zero-order statistics can be used to define an alternative strength measure referred to as the *geometric power*.

Definition (Geometric power)

Let X be a logarithmic-order random variable. The *geometric power* of X is defined as

$$S_0 = S_0(X) = e^{E \log |X|}. \tag{22}$$

The geometric power is a scale parameter satisfying $S_0(X) \geq 0$ and $S_0(cX) = |c| S_0(X)$. It takes on the value $S_0(X) = 0$ if and only if $\Pr(X = 0) > 0$ (zero power is only attained when there is a discrete probability mass located in zero).

Proposition (Geometric Power of Stable Processes)

The geometric power of a symmetric stable variable is given by

$$S_0 = \frac{(C_g \gamma)^{1/\alpha}}{C_g}, \tag{23}$$

where $C_g = e^{C_e} \approx 1.78$*, is the exponential of the Euler constant.*

Proof: From [Zolotarev 86], p. 215, the logarithmic moment of a zero-centered symmetric α-stable random variable with unitary dispersion is given by

$$E \log |X| = \left(\frac{1}{\alpha} - 1 \right) C_e, \tag{24}$$

where $C_e = 0.5772...$ is the Euler constant.

This gives

$$S_0(X)\Big|_{\gamma=1} = e^{E\log|X|} = \left(e^{C_e}\right)^{\frac{1}{\alpha}-1} = \frac{C_g^{1/\alpha}}{C_g}, \tag{25}$$

where $C_g = e^{C_e} \approx 1.78$. If X has a non-unitary dispersion γ, it is easy to see that

$$S_0(X) = \gamma^{1/\alpha}\left[S_0(X)\mid_{\gamma=1}\right] = \frac{(C_g\gamma)^{1/\alpha}}{C_g}. \tag{26}$$

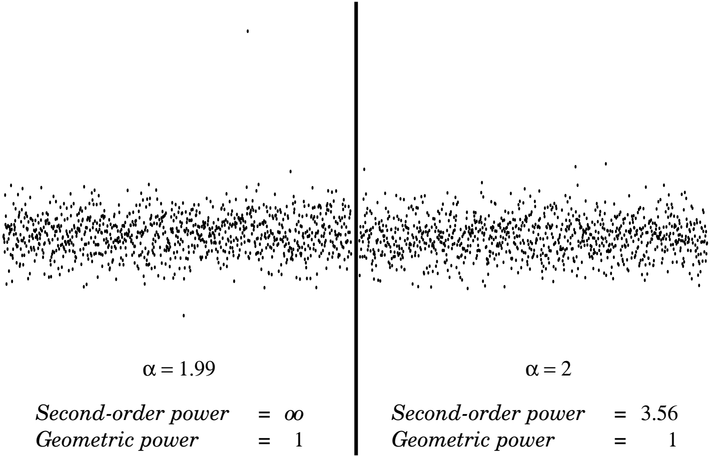


Figure: Comparison of second -order power vs. geometric power for i.i.d. α-stable processes. Left: $\alpha = 1.99$. Right: $\alpha = 2$.