Numeric-Symbolic Exact Rational Linear System Solver

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Motivation

$Ax = b$

Given $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, compute $x \in \mathbb{Q}^n$.

Core problem specifics:

- Square $m = n$ matrices
- Matrix entries of length d bits or fewer
- Dense matrices

Background

Symbolic Approaches

- Dixon
  - Solve system modulo p
  - Use Hensel lifting to obtain a p-adic expansion of solution
  - Rational reconstruction from p-adic approximants

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- Wan
  - Numeric iterative refinement to obtain dyadic number solution of high accuracy
  - Specifically, $2h$, the Hadamard bound of the input system.
  - Rational reconstruction from dyadic approximants

Fact

If two rational numbers $r_1 = a/b, r_2 = c/d$ are given with gcd$(a, b) = 1$ and gcd$(c, d) = 1$, and $r_1 \neq r_2$, then $|r_1 - r_2| \geq 1/bd$.

That is, though dense in the real number line, rational numbers with bound denominators are discrete.

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A given real number r can be represented by a continued fraction: $r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$, where $a_i \in \mathbb{Z}$.

Confirmed Continuation (Overlap)

At each iteration, we scale our partial solution $\hat{x}$, then split into two parts:

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Before updating dyadic approximants, we *confirm continuation* of the iteration by verifying **overlap** between the current $\hat{x}'$ and the previous $\hat{x}_{frac}$.

- e.g. $\max |\hat{x}' - \hat{x}_{frac}| \leq \frac{1}{2^b}$ ensures b bits of overlap

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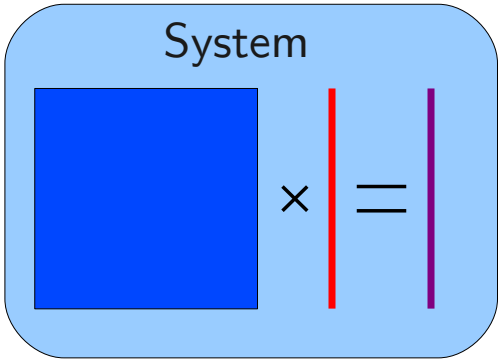
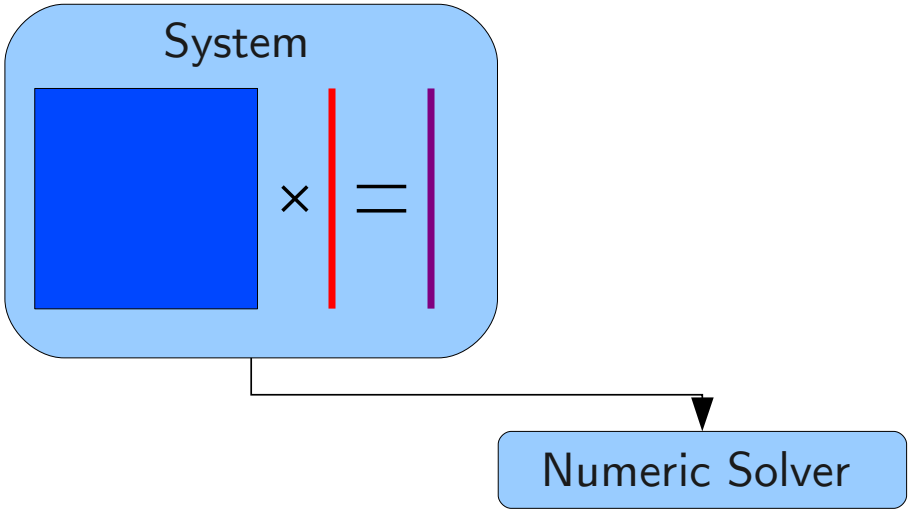
- One bit is (typically) sufficient.

Iterative Refinement



Iterative Refinement



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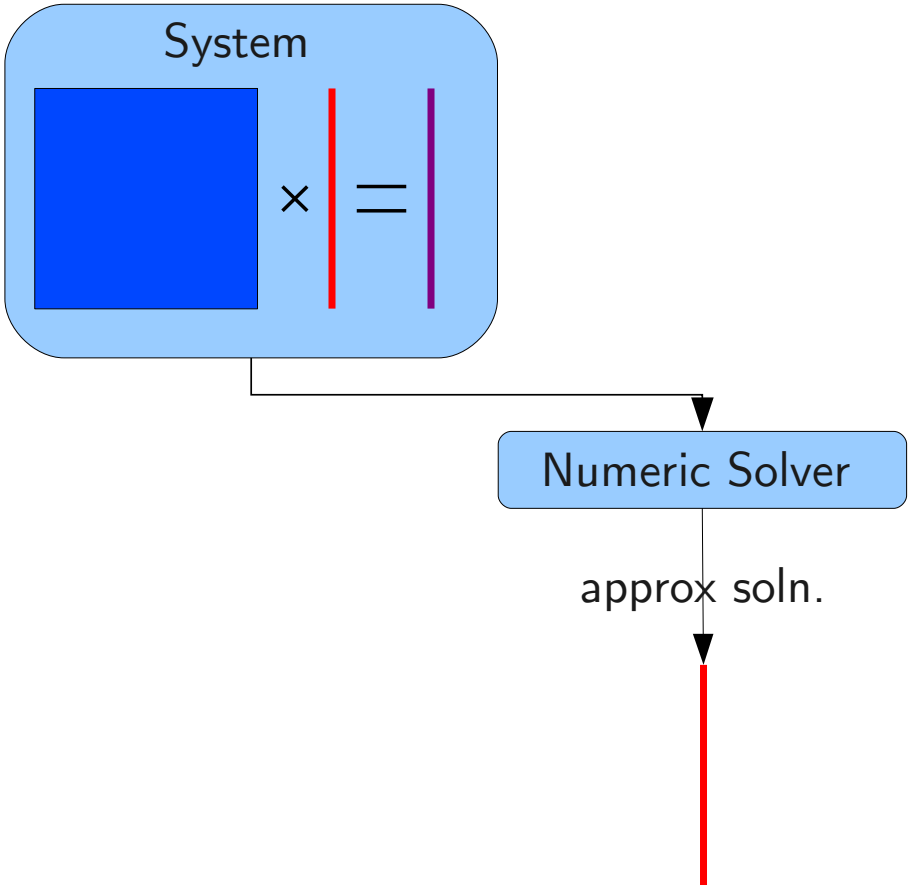

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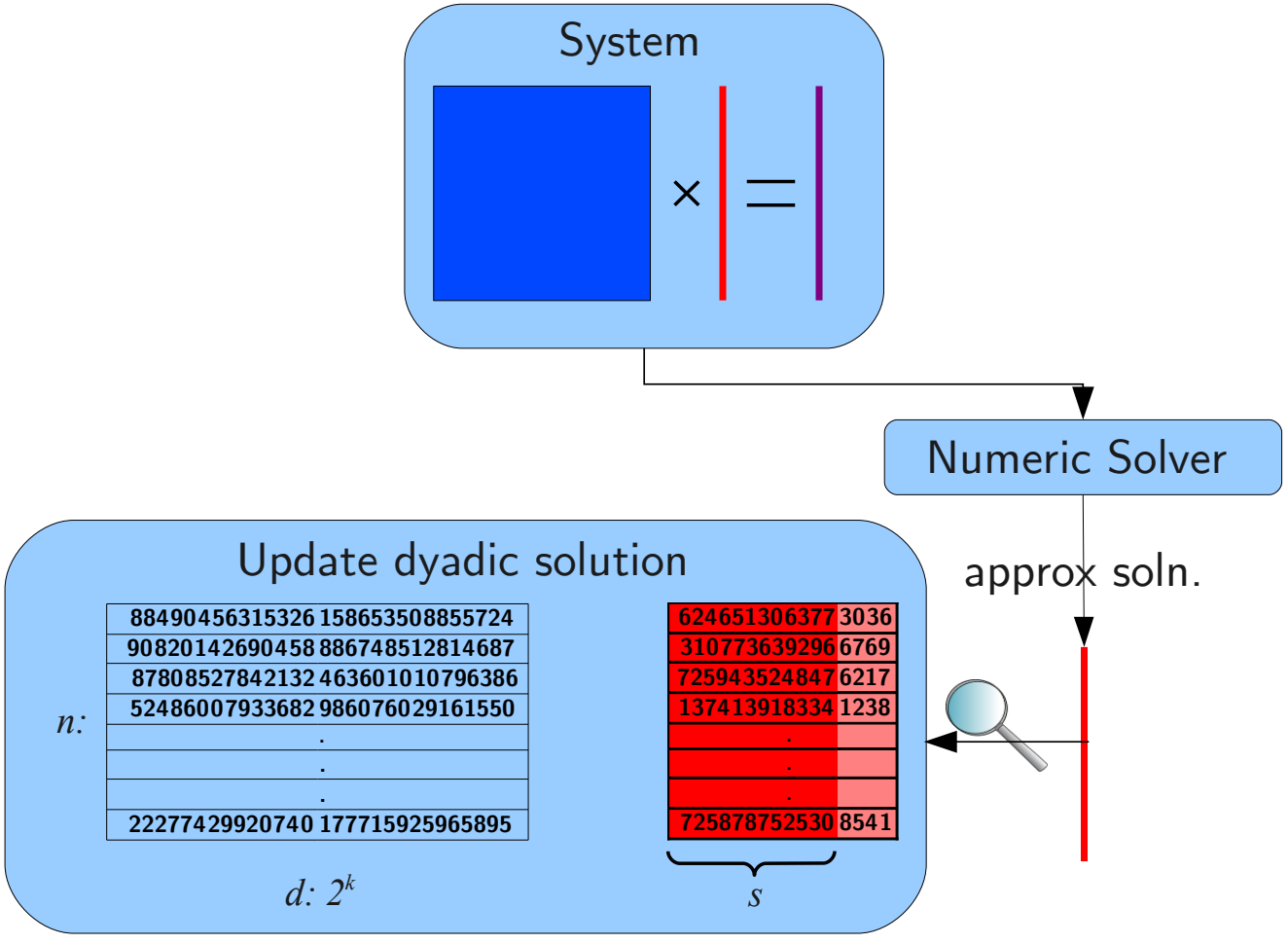

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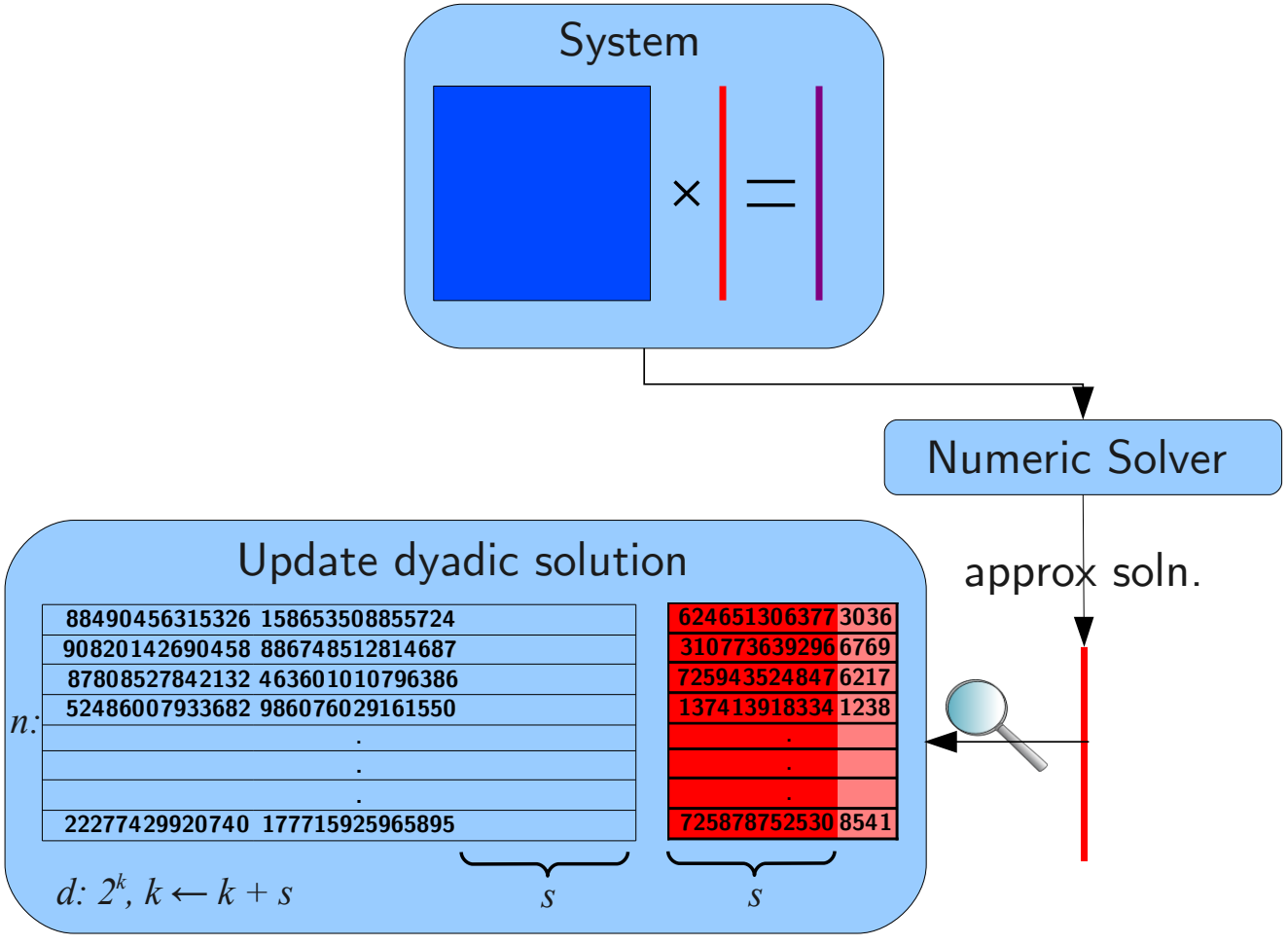

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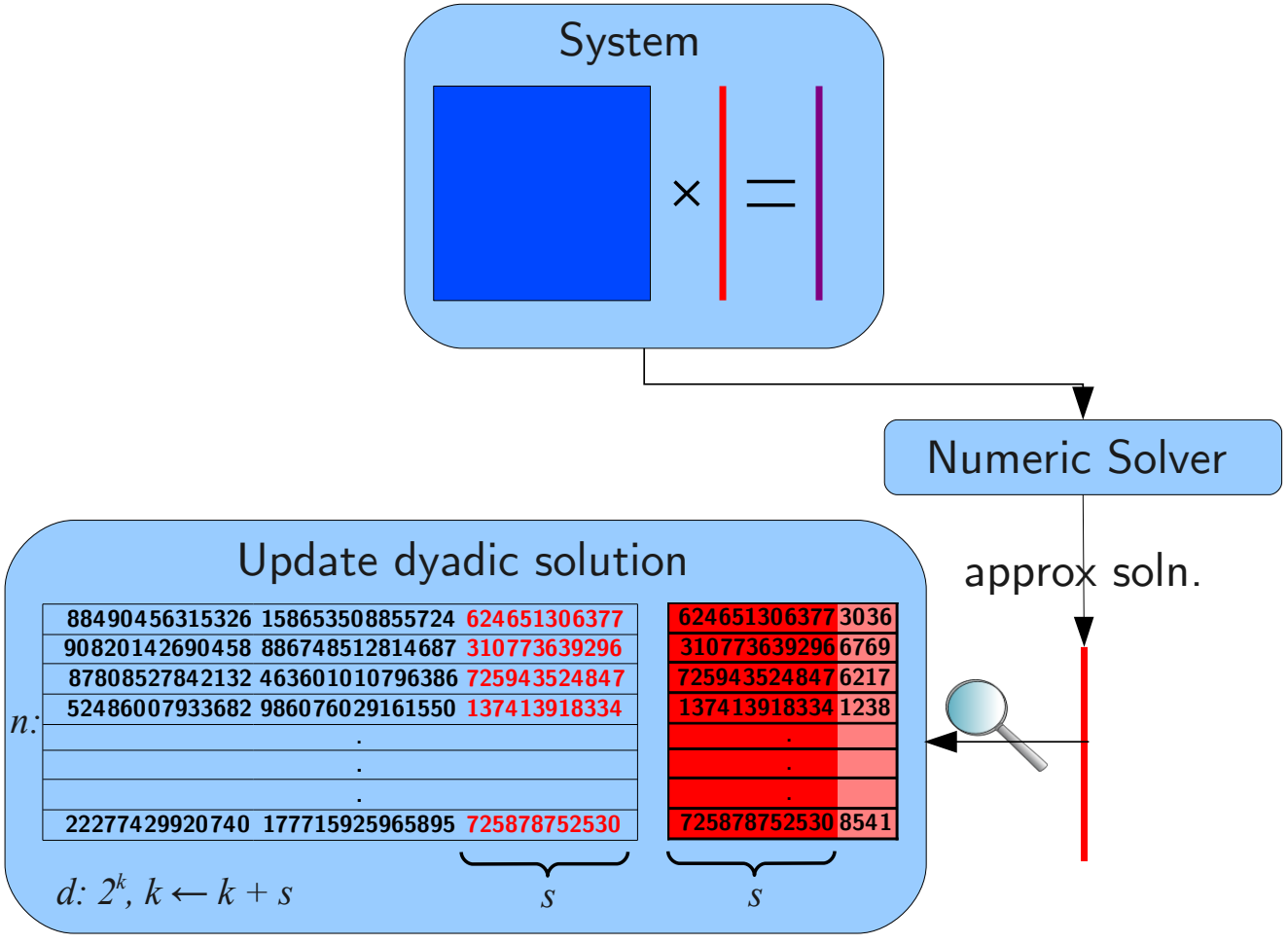

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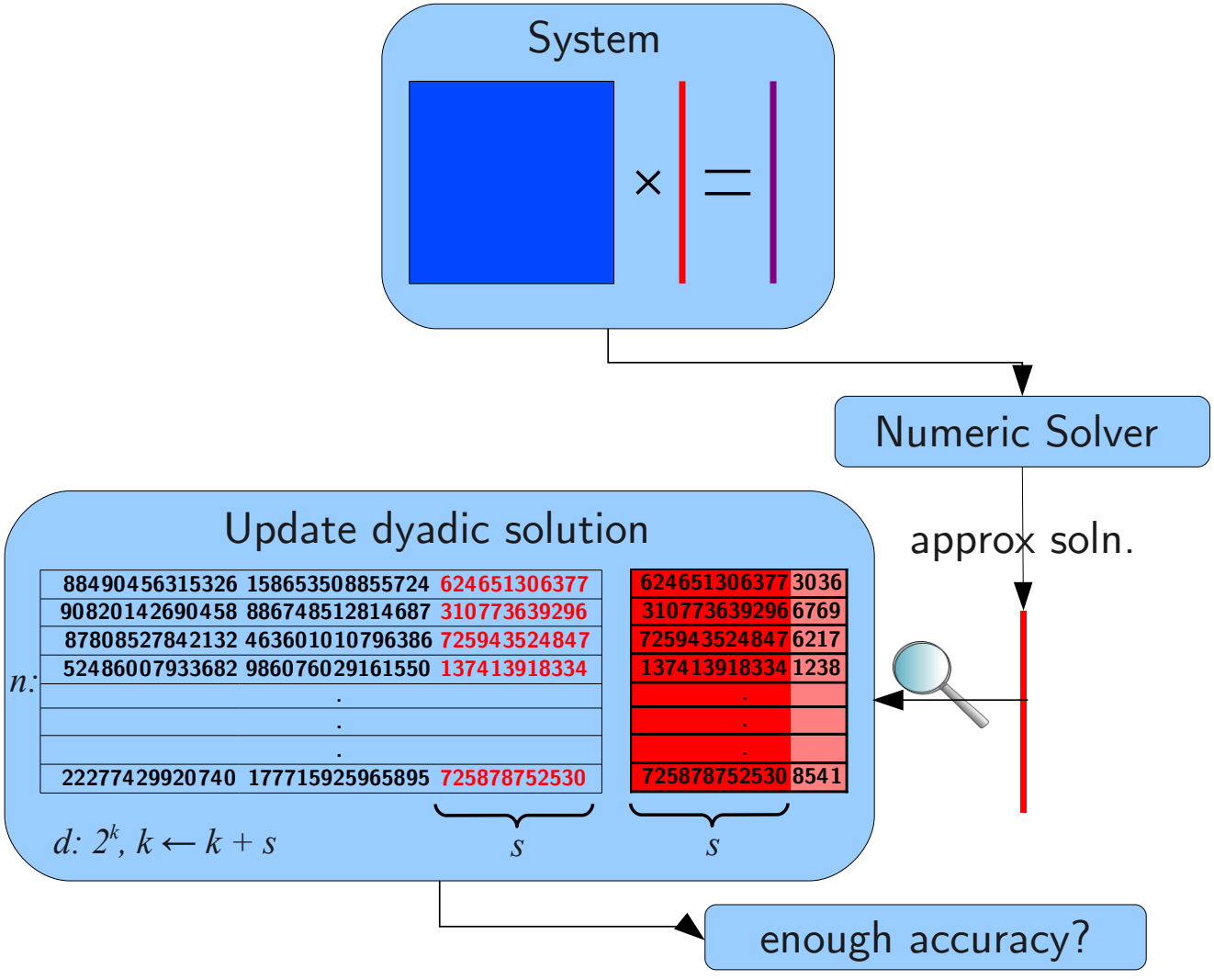

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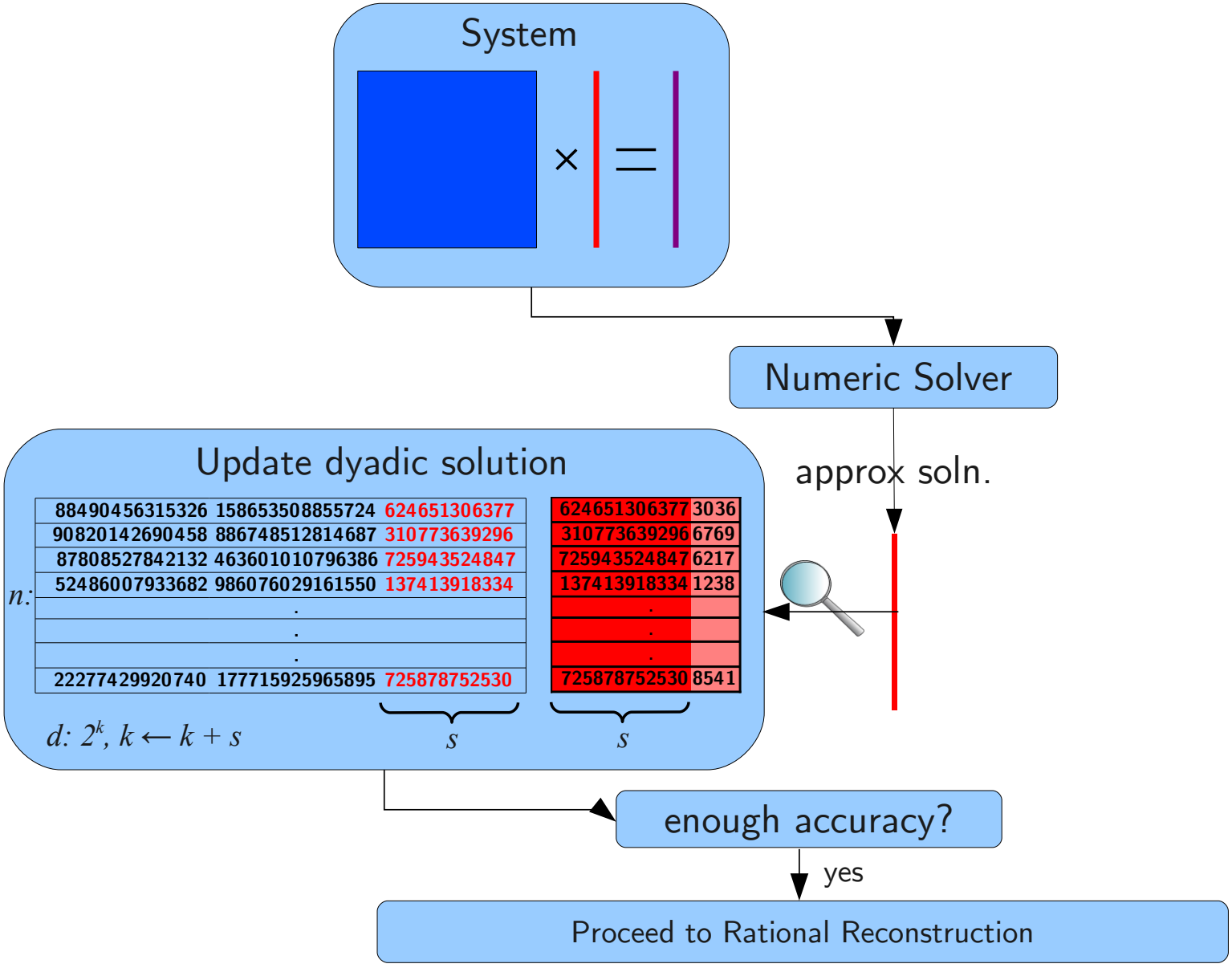

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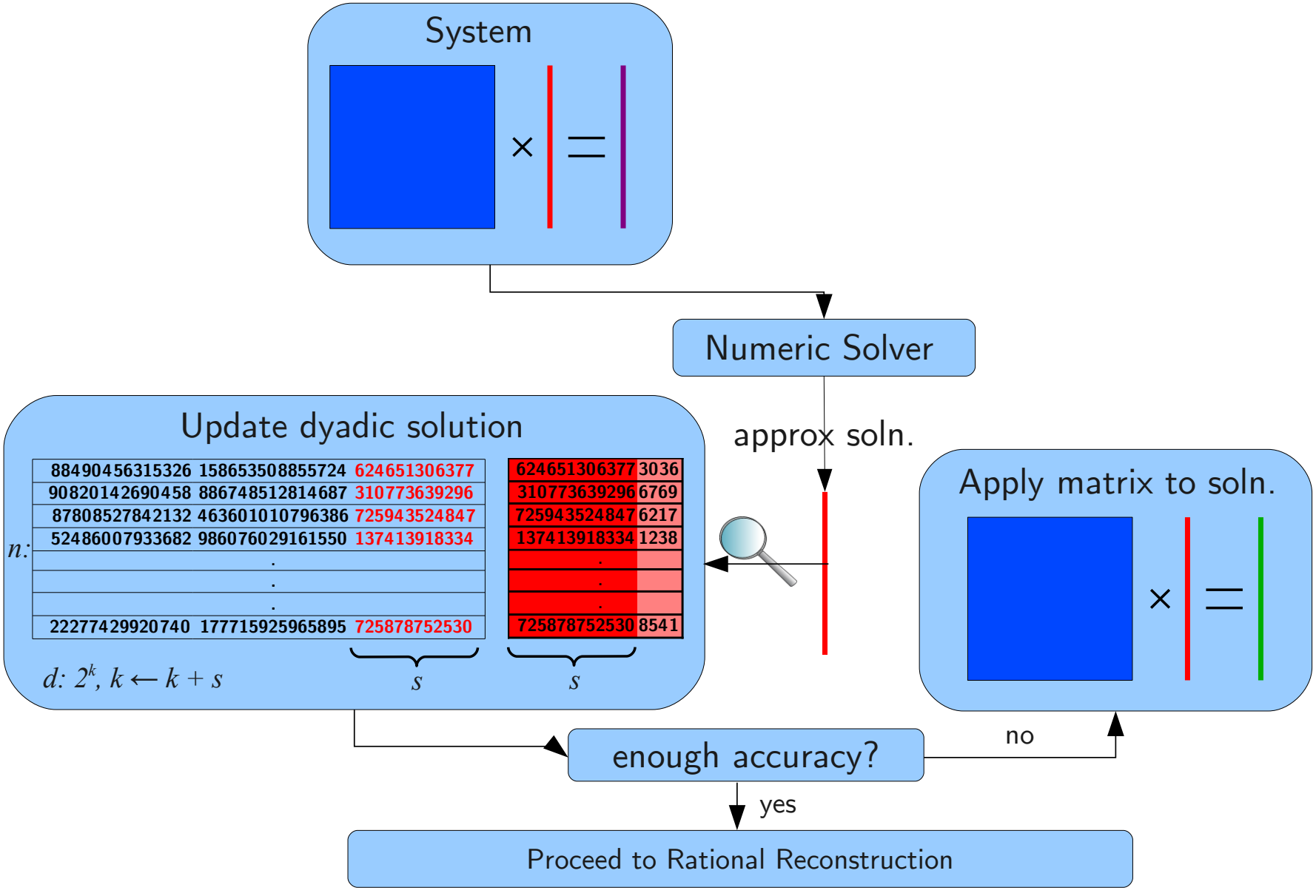

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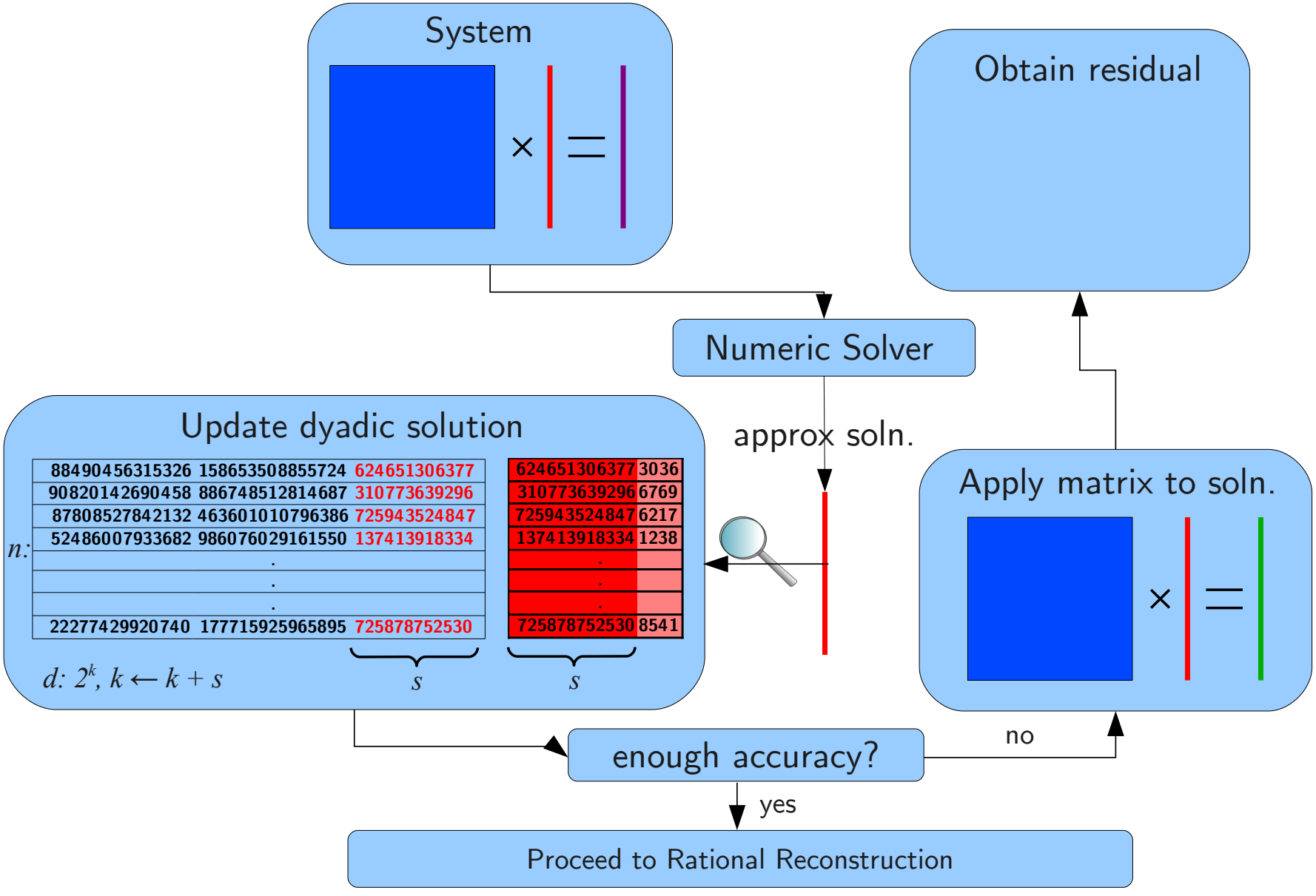

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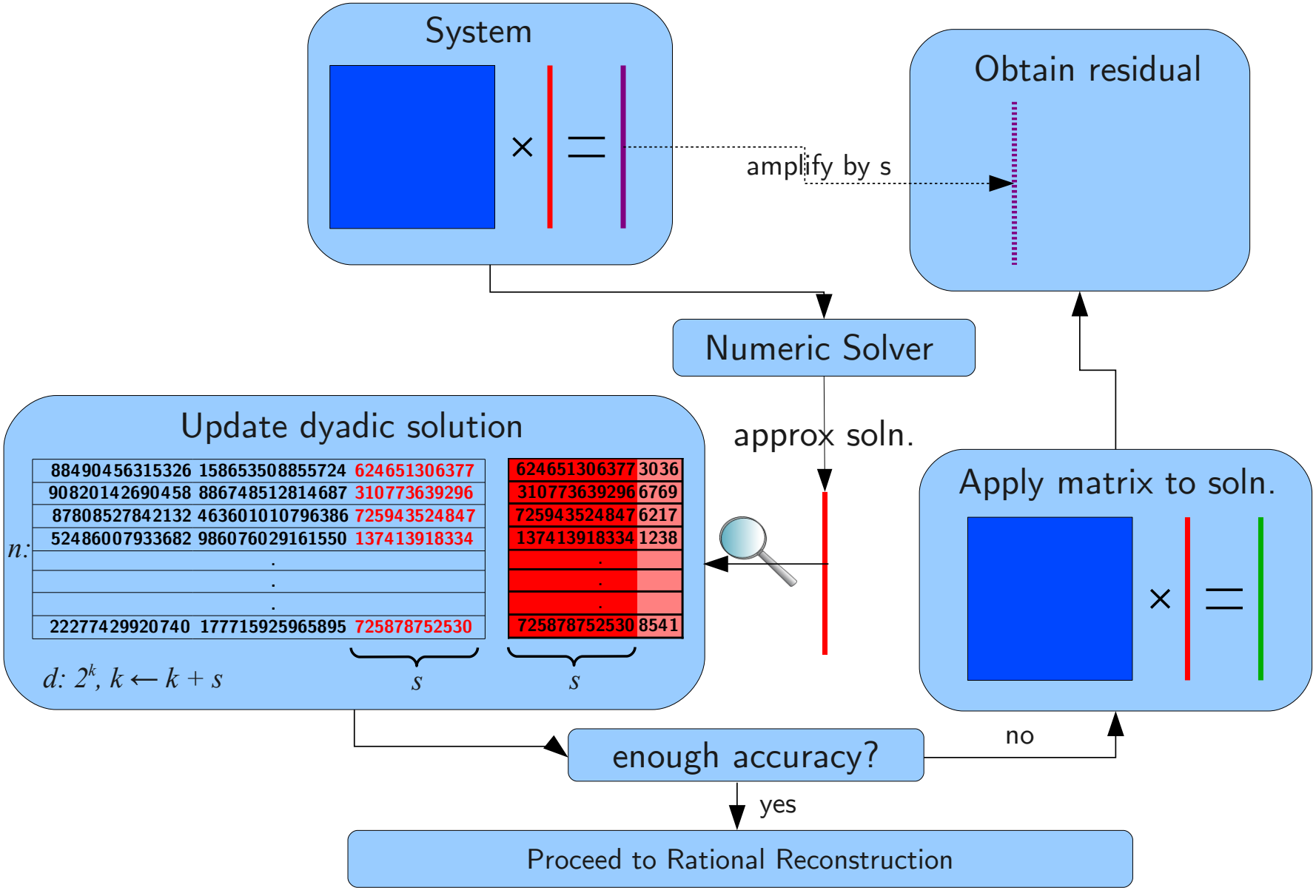

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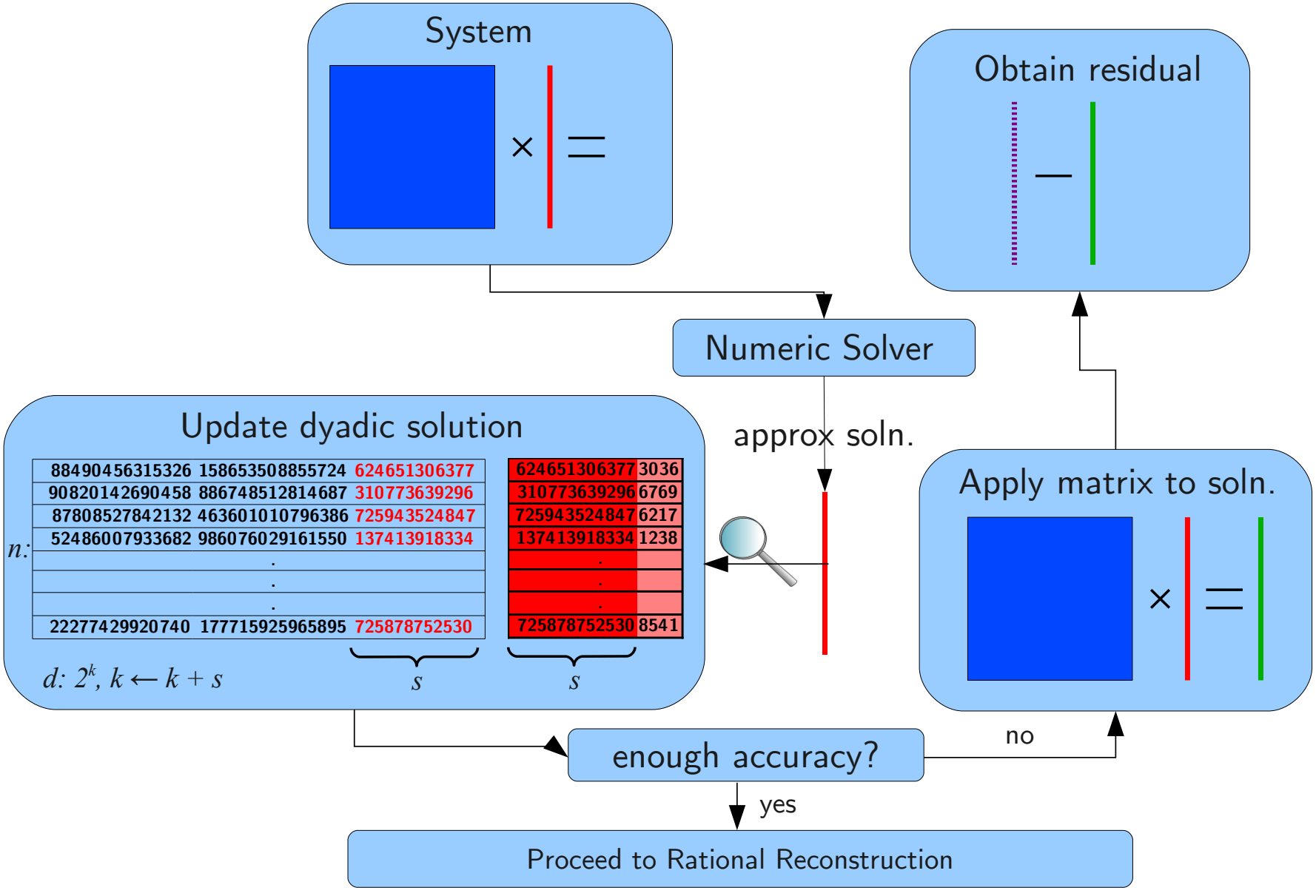

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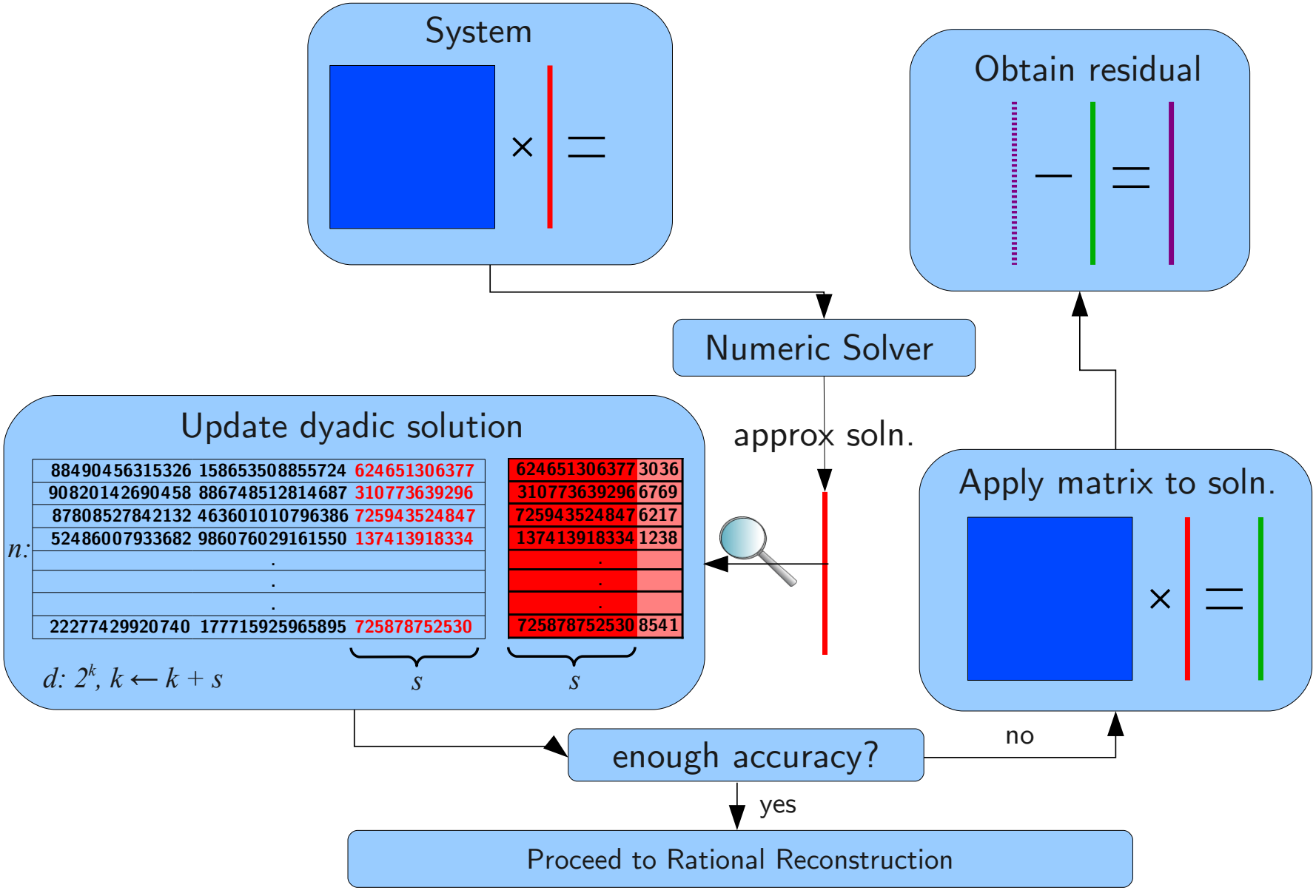

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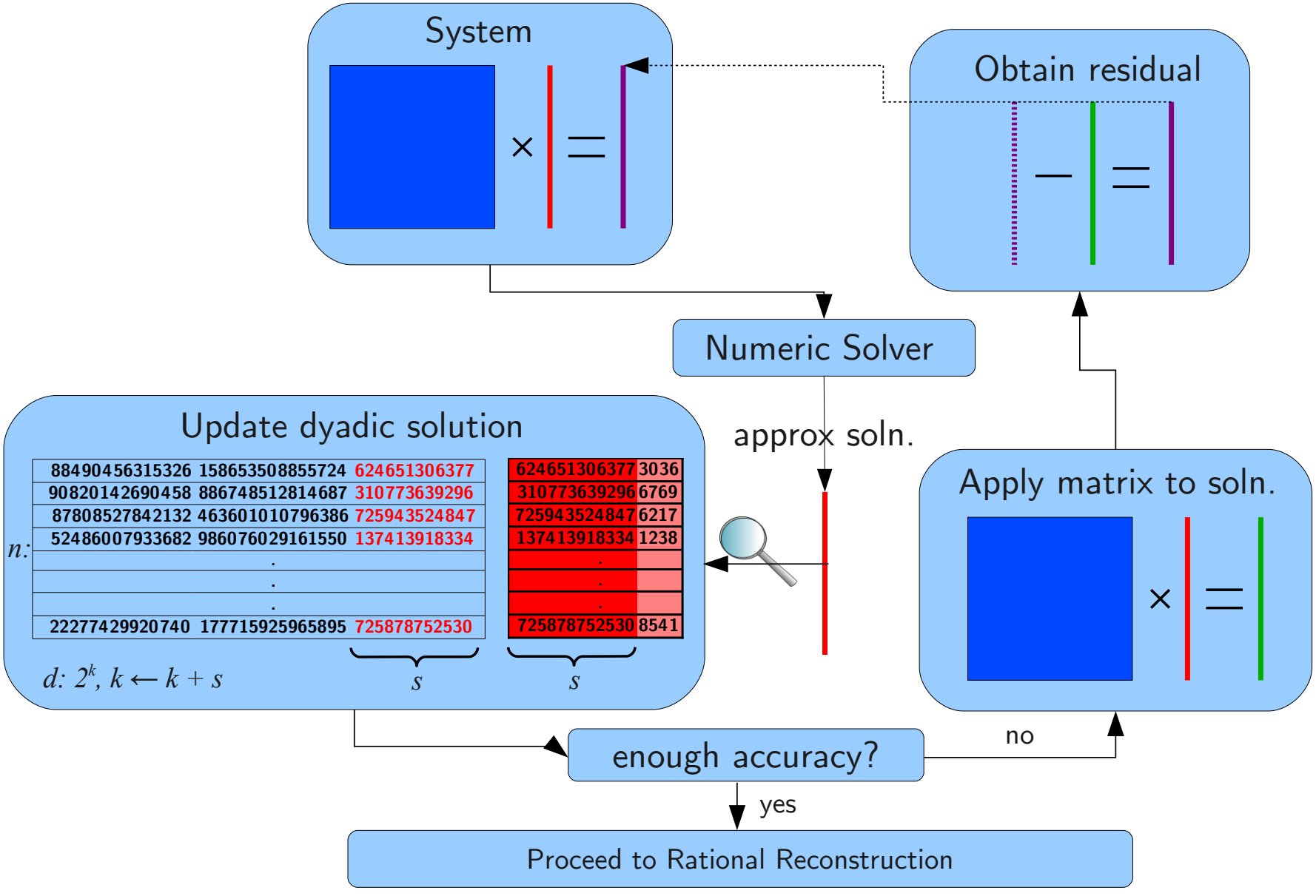

Rational Reconstruction

- Reconstruct from p-adic digits $p_0, p_1, \ldots p_k$
  $n/d = \sum_{i=i_0}^{k} x_i p^i + \epsilon$
  where $p^{k+1} | \epsilon$, i.e. error is small in the p-adic sense.
- Reconstruct from dyadic digits $x_0, x_1, \ldots x_k$
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- P-adically a remainder sequence entry is (r_i, s_i, t_i), where $r_i = s_i x + t_i p^k$, the remainder r_i is a *numerator candidate* and coefficient s_i of the given residue x is a denominator candidate.

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- Dyadically a remainder sequence entry is (r_i, s_i, t_i), where $r_i = s_i x + t_i 2^k$, the remainder r_i is a *measure of error* and coefficient s_i of the given dyadic approximation x is a denominator candidate.

Output sensitivity (early termination)

- No random choice of prime, therefore no probabilistic early termination.
- But *guaranteed* early termination in some cases:

Example

Solving $Ax = b$, suppose:

- $x_1 = 9/17$
- the Hadamard bound for $\det(A)$ is $h = 2^8 = 256$.

In general we need $2\lg(h) = 16$ bits of approximation.

Output sensitivity (example)

```
    |\^/|     Maple 12 (IBM INTEL LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2008
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> #convergents of approximation to 7 bits.
> convert(68/128, confrac, 'cv'): cv;
                                              17
                            [0, 1, 1/2, 8/15, --]
                                              32
> #unique close small denominator convergent is wrong.
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> #convergents of approximation to 8 bits.
> convert(135/256, confrac, 'cv'): cv;
                                    10  19  29  135
                  [0, 1, 1/2, 9/17, --, --, --, ---]
                                    19  36  55  256
> #correct small rational is among several possibilities.
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> #convergents of approximation to 9 bits.
> convert(271/512, confrac, 'cv'): cv;
                                          271
                          [0, 1, 1/2, 9/17, ---]
                                          512
> #unique close small denominator convergent is right, but we can't be certain.
```

Output sensitivity (example)

```
> #convergents of approximation to 12 bits.
> convert(2168/4096, confrac, 'cv'): cv;
                                                271
                              [0, 1, 1/2, 9/17, ---]
                                                512

> #guaranteed early termination: small denominator convergent is only possible solution.
```

Output sensitivity (example)

```
> #convergents of approximation to 12 bits.
> convert(2168/4096, confrac, 'cv'): cv;
                                          271
                          [0, 1, 1/2, 9/17, ---]
                                          512

> #guaranteed early termination: small denominator convergent is only possible solution.


> #convergents of approximation to 16 bits.
> convert((32*1084)/65536, confrac, 'cv'): cv;
                                          271
                          [0, 1, 1/2, 9/17, ---]
                                          512

> #Get same result from full approximation, not using output sensitivity.
```

Vector rational reconstruction

Suppose solution to Ax = b is (9/17, 3/11, 7/187) and we have computed the approximation to 12 bits: $\frac{x_1=2168, x_2=1117, x_3=153}{4096}$

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- x_2: Try division once $(1117 * 17 = 5 * 4096 - 1491)$.
  Try again – success $(1117 * 187 = 51 * 4096 - 17)$
  $n/d = 51/187 = 3/11$
  187 = lcm of denominators so far.

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  Try again – success $(1117 * 187 = 51 * 4096 - 17)$
  $n/d = 51/187 = 3/11$
  187 = lcm of denominators so far.
- x_3: $|n/187 - 153/4096|$ should be less than $1/8192$.
  One division with remainder yields $153 * 187 = 7 * 4096 - 61$.
  n = 7 and small enough remainder.

Data

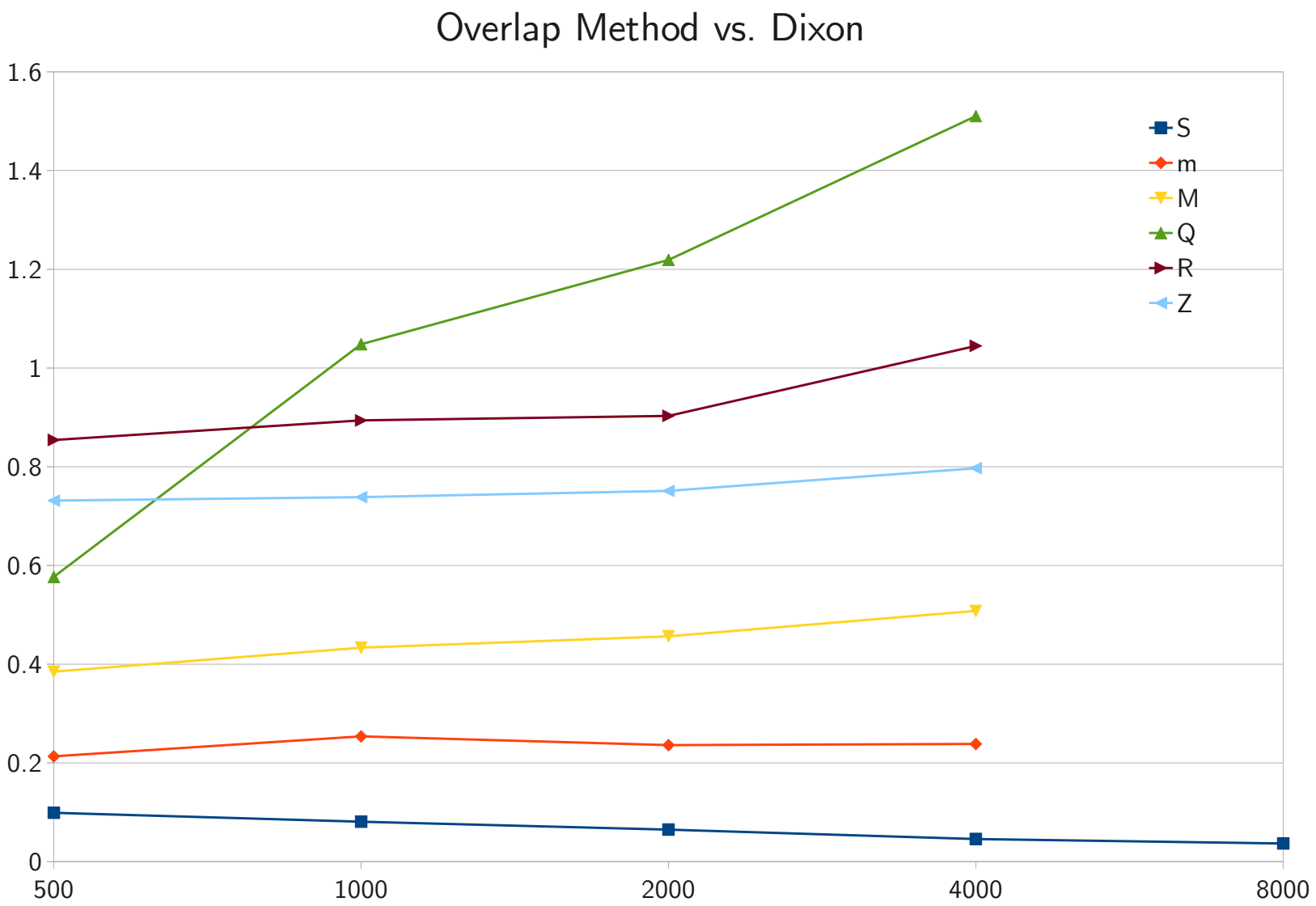
matrix	Dixon	Wan	Overlap	Ov-ET
S_{512}	0.728	0.711	0.0723	**0.0721**
S_{1024}	4.58	4.75	0.380	**0.371**
S_{2048}	32.3	36.6	**2.08**	2.10
S_{4096}	255	297	**11.7**	**11.7**
S_{8192}	2240	2517	**77.6**	82.6
M_{500}	1.46	fail	1.06	**0.562**
M_{1000}	10.2	fail	8.56	**4.42**
M_{2000}	82.8	fail	75.0	**37.8**
M_{4000}	628	fail	658	**319**
M_{8000}	mem	fail	6170.6	**3049**
Z_{500}	0.793	0.864	0.584	**0.580**
Z_{1000}	6.04	6.38	**4.41**	4.46
Z_{2000}	45.4	52.15	35.8	**34.1**
Z_{4000}	340	439	318	**271**
Z_{8000}	mem	5771	2584	**2474**

$$S_1 = (1), S_{2n} = \begin{pmatrix} S_n & S_n \\ S_n & -S_n \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 2 & 3 & \cdots \\ 3 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Z_n is an $n \times n$ $\{0, 1\}$-matrix, with p $= 1/2$ for a given position containing a 1.

Relative Performance



The End