On Optimal Quasi-Orthogonal Space–Time Block Codes With Minimum Decoding Complexity

Haiquan Wang, *Member, IEEE*, Dong Wang, *Member, IEEE*, and Xiang-Gen Xia, *Fellow, IEEE*

***Abstract*—Orthogonal space–time block codes (OSTBC) from orthogonal designs have both advantages of complex symbol-wise maximum-likelihood (ML) decoding and full diversity. However, their symbol rates are upper bounded by 3/4 for more than two antennas for complex symbols. To increase the symbol rates, they have been generalized to quasi-orthogonal space–time block codes (QOSTBC) in the literature but the diversity order is reduced by half and the complex symbol-wise ML decoding is significantly increased to complex symbol pair-wise (pair of complex symbols) ML decoding. The QOSTBC has been modified by rotating half of the complex symbols for achieving the full diversity while maintaining the complex symbol pair-wise ML decoding. The optimal rotation angles for any signal constellation of any finite symbols located on both square lattices and equal-literal triangular lattices have been found by Su-Xia, where the optimality means the optimal diversity product (or product distance). QOSTBC has also been modified by Yuen–Guan–Tjhung by rotating information symbols in another way such that it has full diversity and in the meantime it has real symbol pair-wise ML decoding (the same complexity as complex symbol-wise decoding) and the optimal rotation angle for square and rectangular QAM constellations has been found. In this paper, we systematically study general linear transformations of information symbols for QOSTBC to have both full diversity and real symbol pair-wise ML decoding. We present the optimal transformation matrices (among all possible linear transformations not necessarily symbol rotations) of information symbols for QOSTBC with real symbol pair-wise ML decoding such that the optimal diversity product is achieved for both *general* square QAM and *general rectangular* QAM signal constellations. Furthermore, our newly proposed optimal linear transformations for QOSTBC also work for general QAM constellations in the sense that QOSTBC have full diversity with good diversity product property and real symbol pair-wise ML decoding. Interestingly, the optimal diversity products for square QAM constellations from the optimal linear transformations of information symbols found in this paper coincide with the ones presented by Yuen–Guan–Tjhung by using their optimal rotations. However, the optimal diversity products for (nonsquare) rectangular QAM constellations from the optimal linear transformations of information symbols found in this paper are better than the ones presented by Yuen–Guan–Tjhung by using their optimal rotations. In this paper, we also present the optimal transformations for the co-ordinate interleaved orthogonal designs (CIOD) proposed by Khan-Rajan for rectangular QAM constellations.**

***Index Terms*—Complex symbol-wise decoding, Hurwitz–Radon family, linear transformations of information symbols, optimal product diversity, quasi-orthogonal space–time block codes, real symbol pair-wise decoding.**

I. Introduction

ORTHOGONAL space–time block codes (OSTBC) from orthogonal designs have attracted considerable attention [3]–[20] since Alamouti code [3] was proposed. OSTBC have two advantages, namely they have fast maximum-likelihood (ML) decoding, i.e., complex symbol-wise decoding, and they have full diversity. However, the symbol rates of OSTBC for more than two antennas are upper bounded by $3/4$ for most complex information symbol constellations no matter how large a time delay is or/and no matter whether a linear processing is used [15]. To increase the symbol rates for complex symbols, OSTBC have been generalized to quasi-OSTBC (QOSTBC) in Jafarkhani [21], Tirkkonen–Boarin–Hottinen [22] and Papadias–Foschini [23], [24], also in Mecklenbrauker—Rupp [25], [26] under the name "extended Alamouti codes," by relaxing the orthogonality between all columns of a matrix. The relaxation of the orthogonality in QOSTBC increases the ML decoding complexity and in fact, the ML decoding of QOSTBC is in general complex symbol pair-wise (two complex symbols as a pair) decoding. Moreover, the original QOSTBC in [21]–[26] do not have full diversity. For example, the diversity order of QOSTBC for four antennas is only 2, which is half of the full diversity order 4. The idea of rotating information symbols in a QOSTBC to achieve full diversity and maintain the complex symbol pair-wise ML decoding has appeared independently in [28]–[31]. Furthermore, the optimal rotation angles $\pi/4$ and $\pi/6$ of the above mentioned information symbols for any signal constellations on square lattices and equal-literal triangular lattices, respectively, have been obtained in Su-Xia [30] in the sense that the diversity products are maximized. In this approach, half of the complex information symbols are rotated. It has been shown in [30] that the QOSTBC with the optimal rotations of the complex symbols have achieved the maximal diversity products among all possible linear transformations of all complex symbols. A different rotation method for QOSTBC has been proposed in Yuen-Guan-Tjhung [37]–[39] such that the QOSTBC has full diversity and its ML decoding becomes real symbol pair-wise decoding that has the same complexity as

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H. Wang is now with the College of Communications Engineering, Hangzhou Dianzi University. He was with the Department of Electrical and Computer Engineering, University of Delaware, Newark, DE 19716 USA (e-mail: wanghq33@gmail.com).

D. Wang was with the Department of Electrical and Computer Engineering, University of Delaware, Newark, DE 19716 USA. He is now with the Wireless Communications and Networking Department, Philips Research North America, Briarcliff Manor, NY 10510 USA (e-mail: dong.wang@philips.com).

X.-G. Xia is with the Department of Electrical and Computer Engineering, University of Delaware, Newark, DE 19716 USA (e-mail: xxia@ee.udel.edu).

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the complex symbol-wise decoding. Furthermore, the optimal rotation angle has been found to be $\arctan(1/2)/2 = 13.29°$ in [37]–[39] when the signal constellations are square or rectangular QAM. What is gained with this type of rotations is that the complex symbol pair-wise ML decoding is reduced to the real symbol pair-wise ML decoding and what is sacrificed is that the optimal diversity product obtained in [30] from the complex symbol rotations is reduced but the diversity product reduction is not significant. As a remark, for rectangular QAM signal constellations and OSTBC, the complex symbol-wise ML decoding can be reduced to real symbol-wise decoding that reaches the minimum decoding complexity. For QOSTBC, it is not hard to show that the real symbol pair-wise decoding has already reached the minimum decoding complexity and it can not be reduced to real symbol-wise decoding due to the non-existence of rate 1 complex orthogonal designs [14] as we shall see later.

A different method from the QOSTBC to increase symbol rates in OSTBC has been proposed in Khan–Rajan [32]–[36] by placing OSTBC on diagonal and jointly selecting information symbols across all the OSTBC on the diagonal. This scheme is called coordinate interleaved orthogonal design (CIOD or CID) in [32]–[36], where it was shown that CIOD can also achieve full diversity and have the real symbol pair-wise ML decoding similar to QOSTBC. All the results are only for square QAM signal constellations.

In this paper, we systematically study general linear transformations (not only limited to rotations) of information symbols (their real and imaginary parts are separately treated) for QOSTBC to have both full diversity and real symbol pair-wise ML decoding. We first present necessary and sufficient conditions on a general linear transformation of symbols for QOSTBC such that it has a real symbol pair-wise ML decoding. We then present the optimal transformation matrices (among all possible linear transformations that are not necessarily rotations or orthogonal transforms) of information symbols for QOSTBC with real symbol pair-wise ML decoding such that the optimal diversity products are achieved. The optimal transformation matrices are obtained for both *general* square QAM and *general rectangular* QAM signal constellations. By applying the optimal linear transformations for rectangular QAM signal constellations to any QAM signal constellations, we find that the QOSTBC also have full diversity, good diversity products, and the real symbol pair-wise ML decoding. Interestingly, the optimal diversity products for square QAM signal constellations from the optimal linear transformations of information symbols found in this paper coincide with the ones presented in [37]–[39], which means that the optimal rotation of two real parts and two imaginary parts obtained in [37]–[39] already achieves the optimal diversity products. However, the optimal diversity products for (non-square) rectangular QAM signal constellations from the optimal linear transformations of information symbols found in this paper are better than the ones with the optimal rotations presented in [37]–[39]. Also note that, since a general linear transformation does not require the orthogonality, our study covers signal constellations on not only square lattices but also other lattices as we discuss in Section III-D.

In this paper, we also present the optimal linear transformations of symbols and the optimal diversity products for CIOD studied in [32]–[36] for general rectangular QAM signal constellations that can be treated as a generalization of the results for square QAM signal constellations presented in [32]–[36]. We compare QOSTBC using optimal symbol linear transformations with CIOD also using optimal symbol linear transformations. It turns out that these two schemes perform the same in terms of both the ML decoding complexity and the diversity product, but the peak-to-average power ratio (PAPR) of the QOSTBC is better than that of the CIOD as what has also been pointed out in [37], [39].

This paper is organized as follows. In Section II, we describe the problem of interest in more details. In Section III-A, we present necessary and sufficient conditions on general linear transformations for QOSTBC to have real symbol pair-wise ML decoding. In Sections III-B and E, we present the optimal linear transformations for QOSTBC for both general square and rectangular QAM signal constellations and the optimal linear transformations of symbols for CIOD for rectangular QAM signal constellations, respectively. In Section III-D, we investigate optimal linear transformations for arbitrary QAM constellations on general lattices. In Section IV, we set up a more general problem in terms of generalized Hurwitz–Radon families for QOSTBC with fast ML decoding. In Section V, we present some numerical simulation results. Most of the proofs are in Appendix.

Some Notations: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of all integers, all real numbers, and all complex numbers, respectively. Capital English letters, such as A, B, C, denote matrices and small case English letters, such as r, s, p, q, denote scalars unless otherwise specified. I_m denotes the identity matrix of size $m \times m$. $A^\dagger$, A^t and A^* denote the conjugate transpose, transpose, and conjugate of matrix A, respectively. $\mathrm{tr}(A)$ denotes the trace of matrix A.

II. Motivation and Problem Description

In this section, we describe the problem in more details. Consider a quasi-static and flat Rayleigh-fading channel with n transmit and m receive antennas:

$$Y = \sqrt{\frac{\rho}{n}} CH + W \tag{1}$$

where $C \in \mathcal{C}$ is a transmitted signal matrix, i.e., a space–time codeword matrix, of size $T \times n$, T is the time delay, $\mathcal{C}$ is a space–time code, H is the channel coefficient matrix of size $n \times m$, and Y, W are received signal matrix and AWGN noise matrix, respectively, of size $T \times m$, and ρ is the SNR at each receiver. Assume that the entries h_{ij} of H are independent, zero-mean complex Gaussian random variables of variance $1/2$ per dimension and they are constant in each block of size T. Also assume that the entries w_{ij} of W are independent, zero-mean Gaussian random variables of variances $1/2$ per dimension. Assume at the receiver, channel H is known. Then, the ML decoding is

$$\arg\min_{C \in \mathcal{C}} \left\| Y - \sqrt{\frac{\rho}{n}} CH \right\|_F^2, \tag{2}$$

where $\|\cdot\|_F$ denotes the Frobenious norm. Based on the pairwise symbol error probability analysis for the above ML decoding, the following rank and diversity product criteria were proposed in [1], [2] for the design of a space–time code $\mathcal{C}$: The minimum rank of difference matrix $C - \tilde{C}$ over all pairs of distinct codeword matrices C and $\tilde{C}$ is as large as possible; The minimum of the product of all the nonzero eigenvalues of matrix $(C-\tilde{C})^\dagger(C-\tilde{C})$ over all pairs of distinct codeword matrices C and $\tilde{C}$ is as large as possible.

Clearly, when a space–time code $\mathcal{C}$ has full diversity (or full rank), i.e., any difference matrix of any two distinct codeword matrices in $\mathcal{C}$ has full rank, the product of all the nonzero eigenvalues of matrix $(C-\tilde{C})^\dagger(C-\tilde{C})$ in the diversity product criterion is the same as the determinant $\det((C-\tilde{C})^\dagger(C-\tilde{C}))$. Since, in this paper we are only interested in full diversity space–time codes, in what follows we use the following diversity product definition as commonly used in the literature:

$$\zeta \triangleq \frac{1}{2\sqrt{T}} \min_{C\neq\tilde{C}\in\mathcal{C}} |\det((C-\tilde{C})^\dagger(C-\tilde{C}))|^{\frac{1}{2n}} \tag{3}$$

and it is desired that the diversity product of a space–time code is maximized for a given size, which in fact covers the full rank criterion since if $C-\tilde{C}$ is not full rank then the determinant $\det((C-\tilde{C})^\dagger(C-\tilde{C}))$ is always 0.

Notice that a space–time codeword matrix C has T rows and n columns and the full rankness forces that $T \geq n$ for a fixed number n of transmit antennas. This means that, for R bits/channel use (or R bits/s/Hz), the size of a space–time code $\mathcal{C}$ has to be at least 2^{Rn} for n transmit antennas while it is only 2^R in single antenna systems. Thus, in general the complexity of the ML decoding in (2) increases exponentially in terms of n, the number of transmit antennas, if there is no structure on $\mathcal{C}$ is used. Orthogonal space–time block codes (OSTBC) from orthogonal designs first studied in [3] and [4] do have simplified ML decoding as we can briefly review below.

A. Orthogonal Space–Time Block Codes

A complex orthogonal design (COD) in complex variables $z_1, z_2, \ldots, z_k$ is a $T \times n$ matrix $G(z_1,\ldots,z_k)$ such that

i) any entry of $G(z_1,\ldots,z_k)$ is a complex linear combination of $z_1, z_2, \ldots, z_k, z_1^*, z_2^*, \ldots, z_k^*$;

ii) G satisfies the orthogonality

$$(G(z_1,\ldots,z_k))^\dagger G(z_1,\ldots,z_k) = \left(|z_1|^2+|z_2|^2+\cdots+|z_k|^2\right) I_n \tag{4}$$

for all complex values $z_1, z_2, \ldots, z_k$.

From a COD $G(z_1,\ldots,z_k)$, an OSTBC can be formed by using it and restricting all the complex variables z_i in finite signal constellations $\mathcal{S}_i$: $\mathcal{C} = \{G(z_1,\ldots,z_k) : z_i \in \mathcal{S}_i,\ 1 \leq i \leq k\}$. With the orthogonality ii) and the linearity i), the ML decoding (2) can be simplified as shown in (5)–(7) at the bottom of the page where $f_i(z_i)$ is a quadratic form of the only complex variable z_i. From (7), one can see that the original k-tuple complex symbol ML decoding

$$\min_{z_1,\ldots,z_k} \left\| Y - \sqrt{\frac{\rho}{n}} G(z_1,\ldots,z_k)H \right\|_F^2 = \min_{(z_1,\ldots,z_k)\in\mathcal{S}_1\times\cdots\times\mathcal{S}_k} \left\| Y - \sqrt{\frac{\rho}{n}} G(z_1,\ldots,z_k)H \right\|_F^2$$

is reduced to k independent complex symbol-wise decodings: $\min_{z_i\in\mathcal{S}_i} f_i(z_i)$ for $1 \leq i \leq k$. For convenience, we call the decoding in (7) complex symbol-wise decoding. What we want to emphasize here is that in the above complexity reduction, the finite complex signal constellations $\mathcal{S}_i$ can be any sets of finite complex numbers and do not have to be rectangular or square such as 16-QAM. With the properties i)–ii) in a COD, it is easy to check that $\mathcal{C}$ has full diversity.

When signal constellations $\mathcal{S}_i$ are not arbitrary but have rectangular shapes, the above complex symbol-wise decoding can

$$\begin{aligned}
\min_{C\in\mathcal{C}} \left\| Y - \sqrt{\frac{\rho}{n}} CH \right\|_F^2 &= \min_{C\in\mathcal{C}} \left\{ \mathrm{tr}(Y^\dagger Y) - \sqrt{\frac{\rho}{n}} \mathrm{tr}(Y^\dagger CH + H^\dagger C^\dagger Y) + \frac{\rho}{n}\mathrm{tr}(H^\dagger C^\dagger CH) \right\} &(5)\\
&= \min_{z_1,\ldots,z_k} \left\| Y - \sqrt{\frac{\rho}{n}} G(z_1,\ldots,z_k)H \right\|_F^2 \\
&= \min_{z_1,\ldots,z_k} \Bigg\{ \mathrm{tr}(Y^\dagger Y) - \sqrt{\frac{\rho}{n}} \mathrm{tr}(Y^\dagger G(z_1,\ldots,z_k)H + H^\dagger (G(z_1,\ldots,z_k))^\dagger Y) \\
&\qquad + \frac{\rho}{n}\mathrm{tr}(H^\dagger (G(z_1,\ldots,z_k))^\dagger G(z_1,\ldots,z_k)H) \Bigg\} \\
&= \min_{z_1,\ldots,z_k} \Bigg\{ \mathrm{tr}\Bigg((Y^\dagger Y) - \sqrt{\frac{\rho}{n}} \mathrm{tr}(Y^\dagger G(z_1,\ldots,z_k)H + H^\dagger (G(z_1,\ldots,z_k))^\dagger Y) \\
&\qquad + \frac{\rho}{n}\mathrm{tr}(H^\dagger H)\left(|z_1|^2+\cdots+|z_k|^2\right) \Bigg) \Bigg\} &(6)\\
&= \min_{z_1\in\mathcal{S}_1} f_1(z_1) + \cdots + \min_{z_k\in\mathcal{S}_k} f_k(z_k) &(7)
\end{aligned}$$

be further reduced as follows. Let $\mathbf{j} = \sqrt{-1}$. A signal constellation $\mathcal{S}$ is called rectangular QAM denoted as RQAM if

$$\mathcal{S} = \left\{ z \mid z = \frac{n_1 d}{2} + \mathbf{j}\frac{n_2 d}{2} : n_i \in \mathcal{N}_i \text{ for } i = 1, 2 \right\} \tag{8}$$

where

$$\mathcal{N}_i \triangleq \{-(2N_i - 1), -(2N_i - 3), \ldots, 2N_i - 3, 2N_i - 1\} \tag{9}$$

where N_1 and N_2 are two positive integers and d is a real positive constant that is used to adjust the total signal energy. For convenience, we assume all k signal constellations $\mathcal{S}_i$ for information symbols z_i are the same, $\mathcal{S}$. When $\mathcal{S}$ is an RQAM, by noting $|z_i|^2 = \frac{d^2}{4}n_{i1}^2 + \frac{d^2}{4}n_{i2}^2$, (7) becomes

$$\begin{aligned} &\min_{C \in \mathcal{C}} \left\| Y - \sqrt{\frac{\rho}{n}} CH \right\|_F^2 \\ &= \min_{n_{11} \in \mathcal{N}_1} f_{11}(n_{11}) + \min_{n_{12} \in \mathcal{N}_2} f_{12}(n_{12}) \\ &\quad + \min_{n_{21} \in \mathcal{N}_1} f_{21}(n_{21}) + \min_{n_{22} \in \mathcal{N}_2} f_{22}(n_{22}) \\ &\quad + \cdots \\ &\quad + \min_{n_{k1} \in \mathcal{N}_1} f_{k1}(n_{k1}) + \min_{n_{k2} \in \mathcal{N}_2} f_{k2}(n_{k2}) \end{aligned} \tag{10}$$

where f_{i1} and f_{i2} are independent quadratic forms of integer variables n_{i1} and n_{i2}, respectively. If each complex variable z_i in (7) is treated as a pair of real numbers, the complex symbol-wise decoding in (7) has the same complexity of the real symbol pair-wise decoding, i.e., two real symbols are searched jointly. From (10), one can see that if the signal constellation $\mathcal{S}$ is an RQAM, the complex symbol-wise (or real symbol pair-wise) decoding (7) can be reduced to the real symbol-wise decoding in (10) that has the minimal decoding complexity. Unfortunately, the symbol rates k/T for an OSTBC or COD are upper bounded by $3/4$ for more than two transmit antennas, i.e., $n > 2$, and for most complex signal constellations no matter how large a time delay T is [15].

B. Quasi Orthogonal Space–Time Block Codes

In order to increase the symbol rates of OSTBC, quasi-orthogonal space–time block codes (QOSTBC) from quasi-orthogonal designs have been proposed by Jafarkhani [21], Tirkkonen–Boarin–Hottinen [22], Papadias–Foschini [23], [24], and also Mecklenbrauker–Rupp [25], [26]. For four transmit antennas, let A and B be two Alamouti codes, i.e.

$$A = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, \quad B = \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix}.$$

Then, the QOSTBC by Jafarkhani [21] and by Tirkkonen–Boarin–Hottinen [22] are

$$C_J = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \text{ and } C_{TBH} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

respectively. Similar constructions were also presented in [24], [25], [27]. Although their forms are different, their performances are identical. One can see that the symbol rates in both schemes are 1. However, their rank is only 2 that is only half of the full rank 4. Furthermore, due to the first two columns and the last two columns are not orthogonal each other, the complex symbol-wise decoding (7) does not hold in general. However, since the first two columns are orthogonal and the last two columns are also orthogonal, the original 4-tuple ML decoding can be reduced into the following complex symbol pair-wise decoding:

$$\begin{aligned} &\min_{C_J \in \mathcal{C}_J} \left\| Y - \sqrt{\frac{\rho}{n}} C_J H \right\|_F^2 \\ &\quad = \min_{z_1, z_4} g_{14}(z_1, z_4) + \min_{z_2, z_3} g_{23}(z_2, z_3) \\ &\min_{C_{TBH} \in \mathcal{C}_{TBH}} \left\| Y - \sqrt{\frac{\rho}{n}} C_{TBH} H \right\|_F^2 \\ &\quad = \min_{z_1, z_3} g_{13}(z_1, z_3) + \min_{z_2, z_4} g_{24}(z_2, z_4) \end{aligned}$$

where g_{ij} is a quadratic form of complex variables z_i and z_j. As mentioned before, although the symbol rates are increased from $3/4$ to 1 in the above QOSTBC, their diversity order is only 2. To have full diversity for QOSTBC, the idea of rotating symbols z_3 and z_4 in C_J and rotating symbols z_3 and z_4 in C_{TBH} have been independently proposed in [28]–[31]. Furthermore, the optimal rotation angles for arbitrary signal constellations located on both square lattices and equal-literal triangular lattices have been obtained in [30] such that the optimal possible diversity products for the QOSTBC are achieved. Since C_J and C_{TBH} have the same performance, for convenience, in what follows we only consider C_{TBH}, i.e., the QOSTBC appeared in [22] as follows.

Let $G_n(z_1, \ldots, z_k)$ be a $T \times n$ complex orthogonal design in complex variables $z_1, \ldots, z_k$ (for its designs, see for example [11]–[13]). Let $A = G_n(z_1, \ldots, z_k)$ and $B = G_n(z_{k+1}, \ldots, z_{2k})$. We consider the following quasi-orthogonal design (QCOD) $Q_{2n}(z_1, \ldots, z_{2k})$:

$$Q_{2n}(z_1, \ldots, z_{2k}) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{11}$$

Then

$$Q_{2n}^\dagger Q_{2n} = \begin{pmatrix} aI_n & bI_n \\ bI_n & aI_n \end{pmatrix}. \tag{12}$$

where

$$a = \sum_{i=1}^{2k} |z_i|^2 \text{ and } b = \sum_{i=1}^{k} \left(z_i z_{k+i}^* + z_{k+i} z_i^* \right). \tag{13}$$

From this equation and (5), the ML decoding becomes

$$\begin{aligned} &\min_{C \in \mathcal{C}} \left\| Y - \sqrt{\frac{\rho}{n}} CH \right\|_F^2 \\ &\quad = \min_{z_1, z_{k+1}} g_1(z_1, z_{k+1}) + \cdots + \min_{z_k, z_{2k}} g_k(z_k, z_{2k}). \end{aligned} \tag{14}$$

where g_i is a quadratic form of z_i and z_{k+i}, which is called complex symbol pair-wise decoding. By rotating z_{k+i} as $z_{k+i} \in \mathcal{S}_{k+i} = \{e^{\mathbf{j}\theta} s : s \in \mathcal{S}_i\}$ from z_i, it is shown in [28]–[31], QOSTBC with rotated symbols can achieve full diversity. Since z_i and z_{k+i} are jointly decoded anyway, the symbol rotation does not change the complex symbol-wise decoding.

One can see that the above complex symbol pair-wise decoding holds for QOSTBC for any signal constellations $z_i \in \mathcal{S}_i$.

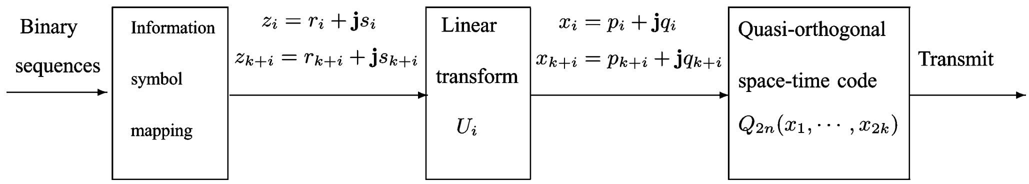

Fig. 1. Encoding of an QOSTBC.

Similar to OSTBC studied before, when signal constellations $\mathcal{S}_i$ are RQAM in (8), it can be further reduced as follows.

When symbols $z_1, \ldots, z_{2k}$ are all taken from RQAM, i.e., z_i can be written as $z_i = r_i + \mathbf{j}s_i$, where r_i and s_i are independent real numbers. Then

$$a = \sum_{i=1}^{2k} \left(|r_i|^2 + |s_i|^2\right), \quad b = \sum_{i=1}^{k} 2\left(r_i r_{k+i} + s_i s_{k+i}\right) \quad (15)$$

and the ML decoding (14) becomes

$$\begin{aligned} &\min_{C \in \mathcal{C}} \left\| Y - \sqrt{\frac{\rho}{n}} CH \right\|_F^2 \\ &\quad = \min_{r_1, r_{k+1}} f_{11}(r_1, r_{k+1}) + \min_{s_1, s_{k+1}} f_{12}(s_1, s_{k+1}) \\ &\quad\quad + \min_{r_2, r_{k+2}} f_{21}(r_2, r_{k+2}) + \min_{s_2, s_{k+2}} f_{22}(s_1, s_{k+1}) \\ &\quad\quad + \cdots \\ &\quad\quad + \min_{r_k, r_{2k}} f_{k1}(r_k, r_{2k}) + \min_{s_k, s_{2k}} f_{k2}(s_k, s_{2k}) \end{aligned} \quad (16)$$

where f_{ij}, $i = 1, 2, \ldots, k$, $j = 1, 2$, are independent of each other and also quadratic forms of independent variables r_i, r_{k+i} when $j = 1$ and of independent variables s_i, s_{k+i} when $j = 2$. The decoding in (16) is real symbol pair-wise decoding that has the same complexity as complex symbol-wise decoding in (7). Note that the decomposition (16) is due to the form of QOSTBC and its properties (11)–(15), and the independence of all the real and imaginary parts of all information symbols. As we mentioned earlier, QOSTBC (11) has only diversity order 2, i.e., half of the full diversity 4. The *question* now is whether we can rotate the information symbols in such a way that the QOSTBC has full diversity and in the meantime, a real symbol pair-wise ML decoding similar to (16) also holds. This problem has been studied by Yuen–Guan–Tjhung [37]–[39] and in [38], they proposed to rotate (r_i, r_{k+i}) into (p_i, p_{k+i}) and (s_i, s_{k+i}) into (q_i, q_{k+i}), while in [37], [39], they proposed to rotate (r_i, s_i) into (p_i, q_i) and (r_{k+1}, s_{k+i}) into (p_{k+i}, q_{k+i}), which is similar to the idea of rotating complex symbols in (14). This type of rotations has been proposed earlier in co-ordinate interleaved orthogonal designs (CIOD or CID) by Khan–Rajan in [32]–[36], which is a different approach than QOSTBC and shall be compared in more details later. Furthermore, the optimal rotation angle for square QAM and rectangular QAM signal constellations in QOSTBC have been obtained in [37]–[39].

C. General Symbol Transformation Formulation

One can see that the reason why the ML decoding of a QOSTBC can be reduced to the real symbol pair-wise decoding in (16) is due to properties (12) and (15). From (15), one can see that a real symbol pair-wise ML decoding exists as long as the quadratic formulas a and b in (15) of four real symbols p_i, p_{k+i}, q_i, q_{k+i} can be decomposed into sums of two independent forms each of which has two real variables and the two forms have disjoint variables, respectively. This motivates the following general linear transformation formulation of symbols such that a real symbol pair-wise ML decoding of QOSTBC is maintained.

For convenience, all information symbol constellations $\mathcal{S}_i$ are assumed the same, $\mathcal{S}$, that is a finite set of at least four points on the integer/square lattice

$$\mathbb{Z}[\mathbf{j}] = \{z = n_1 + \mathbf{j}n_2 : n_1, n_2 \in \mathbb{Z}\}. \quad (17)$$

We assume that $\mathcal{S}$ is not equivalent to any PAM constellation, i.e., not all points in $\mathcal{S}$ are collinear (on a single straight line). Then, the detailed *encoding* is as follows and shown in Fig. 1.

- A binary information sequence is mapped to points z_i in $\mathcal{S}$ as $z_i = r_i + \mathbf{j}s_i$ for $1 \le i \le 2k$.
- For each i, $1 \le i \le k$, take a predesigned real linear transform U_i and the real vector $(r_i, s_i, r_{k+i}, s_{k+i})^t$ of dimension 4 is transformed to another real vector $(p_i, q_i, p_{k+i}, q_{k+i})^t$ of dimension 4:
$$(p_i, q_i, p_{k+i}, q_{k+i})^t = U_i(r_i, s_i, r_{k+i}, s_{k+i})^t \quad (18)$$
where U_i is non-singular.
- Form complex variables $x_i = p_i + \mathbf{j}q_i$ for $1 \le i \le 2k$.
- With these complex variables x_i, form a QOSTBC $Q_{2n}(x_1, x_2, \ldots, x_{2k})$ that is used as a space–time block code and transmitted through $2n$ transmit antennas.

The question now is how to design a real linear transformations U_i of size 4×4 for a QOSTBC to possess a real symbol pair-wise ML decoding and to have full diversity (or optimal diversity product). In order to study a real symbol pair-wise ML decoding, let us study a and b in (15). Let

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) \triangleq p_i^2 + q_i^2 + p_{k+i}^2 + q_{k+i}^2 \quad (19)$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) \triangleq 2(p_i p_{k+i} + q_i q_{k+i}). \quad (20)$$

To possess a real symbol pair-wise ML decoding, linear transformation U_i in (18) needs to be chosen such that one of the following three cases holds

Case 1) Functions g_i and f_i can be separated as

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) = g_{i1}(r_i, s_i) + g_{i2}(r_{k+i}, s_{k+i}),$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) = f_{i1}(r_i, s_i) + f_{i2}(r_{k+i}, s_{k+i}).$$

Case 2) Functions g_i and f_i can be separated as

$$g_i(p_i, q_i, p_{k+i}, q_{k+i}) = g_{i1}(r_i, r_{k+i}) + g_{i2}(s_i, s_{k+i}),$$
$$f_i(p_i, q_i, p_{k+i}, q_{k+i}) = f_{i1}(r_i, r_{k+i}) + f_{i2}(s_i, s_{k+i}).$$

Case 3) Functions g_i and f_i can be separated as

$$g_i(p_i,q_i,p_{k+i},q_{k+i}) = g_{i1}(r_i,s_{k+i}) + g_{i2}(s_i,r_{k+i})$$
$$f_i(p_i,q_i,p_{k+i},q_{k+i}) = f_{i1}(r_i,s_{k+i}) + f_{i2}(s_i,r_{k+i}).$$

With the above encoding and properties, the ML decoding can be similarly described below. For the illustration purpose, assume Case 1 holds for a design of U_i. Then, the ML decoding becomes

$$\begin{aligned}&\min_{(z_1,\ldots,z_{2k})\in\mathcal{S}^{2k}} \left\| Y - \sqrt{\frac{\rho}{n}} Q_{2n}(x_1,\ldots,x_{2k})H \right\|_F^2 \\ &\quad = \min_{r_1,s_1} v_1(r_1,s_1) + \cdots + \min_{r_{2k},s_{2k}} v_{2k}(r_{2k},s_{2k}) \end{aligned} \tag{21}$$

where we notice that $z_i = r_i + \mathbf{j}s_i$. Real symbol pair-wise ML decodings for other two cases can be similarly derived.

For Case 1, the ith real and imaginary parts of the ith complex symbol z_i are decoded jointly, i.e., it is the same as the complex symbol z_i symbol-wise decoding. Since its real and imaginary parts are not separated in the decoding, the signal constellations $\mathcal{S}_i$ for z_i can be any constellations.

For Case 2, the real parts of the ith and the $(k+i)$th complex symbols z_i and z_{k+i} are decoded jointly and the imaginary parts of the ith and the $(k+i)$th complex symbols z_i and z_{k+i} are decoded jointly. In this case, real parts and imaginary parts of complex symbols are required to be independent. Thus, signal constellations $\mathcal{S}_i$ and $\mathcal{S}_{k+i}$ for z_i and z_{k+i} have to be square or rectangular QAM.

For Case 3, the real part of the ith complex symbol z_i and the imaginary part of the $(k+i)$th complex symbol z_{k+i} are decoded jointly and the imaginary part of the ith symbol z_i and the real part of the $(k+i)$th complex symbol z_{k+i} are decoded jointly. In this case, similar to Case 2, signal constellations $\mathcal{S}_i$ and $\mathcal{S}_{k+i}$ for z_i and z_{k+i} have to be square or rectangular QAM, too.

It is not hard to see that in terms of real symbol pair-wise decoding, the above three cases are the only possibilities. One might want to ask why a transformation U_i is only used for four variables in (18) but not for more variables. The answer to this question is rather simple and it is because of the particular forms of the quadratic forms a and b in (15) and (13) that appear in the ML decoding of a QOSTBC due to the structure of a QOSTBC (12). To include more than four variables does not do anything better in terms of real symbol pair-wise ML decoding and diversity product.

The main goal of the remaining of this paper is to investigate how to design a linear transformations U_i in (18) such that the above encoded QOSTBC $Q_{2n}(x_1,\ldots,x_{2k})$ has a real symbol pair-wise ML decoding in terms of real information symbols r_i and s_i for $1 \le i \le 2k$ in each of the above three cases, and full diversity and furthermore the diversity product is maximized.

III. Designs of Linear Transformation Matrices U_i

In this section, we first characterize all linear transformation matrices U_i in (18) for Cases 1–3, i.e., for QOSTBC to possess a real symbol pair-wise ML decoding. We then present the optimal U_i in the sense that the diversity products of the QOSTBC are maximized.

A. Necessary and Sufficient Conditions on U_i for Real Symbol Pair-Wise ML Decoding

First of all, the two quadratic forms of p_i and q_i and p_{k+i} and q_{k+i} in (19)–(20) can be formulated as

$$\begin{aligned} &g_i(p_i,q_i,p_{k+i},q_{k+i}) \\ &\quad = p_i^2 + q_i^2 + p_{i+1}^2 + q_{i+1}^2 \\ &\quad = (p_i,q_i,p_{k+i},q_{k+i}) \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} p_i \\ q_i \\ p_{k+i} \\ q_{k+i} \end{pmatrix} \end{aligned} \tag{22}$$

$$\begin{aligned} &f_i(p_i,q_i,p_{k+i},q_{k+i}) \\ &\quad = 2(p_ip_{k+i} + q_iq_{k+i}) \\ &\quad = (p_i,q_i,p_{k+i},q_{k+i}) \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} p_i \\ q_i \\ p_{k+i} \\ q_{k+i} \end{pmatrix}. \end{aligned} \tag{23}$$

In terms of the information symbols r_i, s_i, r_{k+i}, s_{k+i} through linear transformations (18), these quadratic forms can be further expressed as

$$\begin{aligned} &\bar{g}_i(r_i,s_i,r_{k+i},s_{k+i}) \\ &\quad \triangleq g_i(p_i,q_i,p_{k+i},q_{k+i}) \\ &\quad = (r_i,s_i,r_{k+i},s_{k+i})U_i^t \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} U_i \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix} \end{aligned} \tag{24}$$

$$\begin{aligned} &\bar{f}_i(r_i,s_i,r_{k+i},s_{k+i}) \\ &\quad \triangleq f_i(p_i,q_i,p_{k+i},q_{k+i}) \\ &\quad = (r_i,s_i,r_{k+i},s_{k+i})U_i^t \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} U_i \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix}. \end{aligned} \tag{25}$$

We now have the following necessary and sufficient conditions.

Theorem 1: Let U_i be a 4×4 non-singular matrix with all real entries used in (18). Then, we have the following results.

i) Case 1 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} \tag{26}$$

where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are 2×2 matrices of real entries, $R_{i1}^2 = I_2$ and $R_{i2}^2 = I_2$, and

$$R_{i1}^tU_{i1}^tU_{i2} + U_{i1}^tU_{i2}R_{i2} = 0. \tag{27}$$

ii) Case 2 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} P_1 \tag{28}$$

where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are the same as in (i) for Case 1 and

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

iii) Case 3 holds if and only if U_i can be written as

$$U_i = \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} P_2 \tag{29}$$

where $U_{i1}, U_{i2}, R_{i1}, R_{i2}$ are the same as in i) for Case 1 and

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its proof is in Appendix. From what was discussed in Section II-C, if U_i in (18) satisfies Theorem 1, then the QOSTBC has a real symbol pair-wise ML decoding. Although this is the case, as what was explained in Section II-C, different cases may require information signal constellations $\mathcal{S}_i$ differently. If U_i satisfies i) in Theorem 1, then information signal constellations $\mathcal{S}_i$ can be any QAM and not necessarily square or rectangular QAM. On the other hand, if U_i satisfies (ii) or (iii) in Theorem 1, then information signal constellations $\mathcal{S}_i$ have to be square or rectangular QAM for a real symbol pair-wise ML decoding. When U_i satisfies i) in Theorem 1, the quadratic objective functions $\bar{g}_i(r_i, s_i, r_{k+i}, s_{k+i})$ and $\bar{f}_i(r_i, s_i, r_{k+i}, s_{k+i})$ in (24) and (25) can be rewritten as follows:

$$\begin{aligned} &\bar{g}_i(r_i, s_i, r_{k+i}, s_{k+i}) \\ &\quad = (r_i, s_i)\left(U_{i1}^t U_{i1} + R_{i1}^t U_{i1}^t U_{i1} R_{i1}\right)\begin{pmatrix} r_i \\ s_i \end{pmatrix} \\ &\qquad + (r_{k+i}, s_{k+i})\left(U_{i2}^t U_{i2} + R_{i2}^t U_{i2}^t U_{i2} R_{i2}\right)\begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix} \end{aligned} \tag{30}$$

$$\begin{aligned} &\bar{f}_i(r_i, s_i, r_{k+i}, s_{k+i}) \\ &\quad = (r_i, s_i)\left(R_{i1}^t U_{i1}^t U_{i1} + U_{i1}^t U_{i1} R_{i1}\right)\begin{pmatrix} r_i \\ s_i \end{pmatrix} \\ &\qquad + (r_{k+i}, s_{k+i})\left(R_{i2}^t U_{i2}^t U_{i2} + U_{i2}^t U_{i2} R_{i2}\right)\begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix}. \end{aligned} \tag{31}$$

In what follows, we only consider linear transformations U_i that satisfy Theorem 1. From Theorem 1, one can see that there may be infinitely many options of such linear transformations U_i satisfying Theorem 1, i.e., there are infinitely many options of U_i in (18) such that the QOSTBC has a real symbol pair-wise ML decoding. The question now is which U_i is optimal in the sense that U_i satisfies Theorem 1 and the diversity product of the QOSTBC is maximized when the mean transmission signal power is fixed. In what follows, we present solutions for U_i of the following optimal linear transformation problem:

$$\max_{U_i \text{ satisfies Theorem 1}} \zeta(Q_{2n}(x_1, \ldots, x_{2k})), \tag{32}$$

when the mean transmission signal power is fixed, where Q_{2n} is a QOSTBC, $\zeta(Q_{2n}(x_1, \ldots, x_{2k}))$ is the diversity product of QOSTBC Q_{2n} as defined in (3), and x_i are defined in the encoding in Section II-C as shown in Fig. 1.

Before we come to the solution of (32), let us see the rotations proposed in [37]–[39]. In [38], only rotations for (r_i, r_{k+i}) and (s_i, s_{k+i}) are used

$$(p_i, p_{k+i})^t = R(r_i, r_{k+i})^t \text{ and } (q_i, q_{k+i})^t = R(s_i, s_{k+i})^t, \tag{33}$$

where R is a 2×2 rotation. This rotation only corresponds to Case 2, and the corresponding U_i in (18) in terms of the form in (ii) in Theorem 1 for the rotation in [38] can be written as

$$\begin{aligned} &U_{i1} = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) \\ 0 & 0 \end{pmatrix}, \quad U_{i2} = \begin{pmatrix} 0 & 0 \\ \cos(\gamma) & \sin(\gamma) \end{pmatrix} \\ &R_{i1} = R_{i2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

In [37], [39], rotations are done individually for each complex symbol as follows:

$$(p_i, q_i)^t = R(r_i, s_i)^t \text{ and } (p_{k+i}, q_{k+i})^t = R(r_{k+i}, s_{k+i})^t \tag{35}$$

where R is a 2×2 rotation. This rotation corresponds to Case 1 and the corresponding U_i in (18) in terms of the form in i) in Theorem 1 for the rotation in [37] can be written as

$$\begin{aligned} &U_{i1} = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) \\ 0 & 0 \end{pmatrix}, \quad U_{i2} = \begin{pmatrix} 0 & 0 \\ \cos(\gamma) & \sin(\gamma) \end{pmatrix} \\ &R_{i1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R_{i2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{36}$$

It has been found in [37]–[39] that the optimal rotation angle $\gamma_{\text{optimal}} = \frac{1}{2}\arctan\left(\frac{1}{2}\right)$ in the sense that the diversity product of the QOSTBC is maximized among all different rotation angles γ for square and rectangular QAM constellations.

Another remark we want to make here is that the optimal diversity product achieved by the optimal complex symbol rotations for a QOSTBC in Su-Xia [30] has been shown in [30] optimal among all possible linear transformations U_i without any restriction. We may expect that the optimal diversity product of a QOSTBC with the optimal solution of U_i in (32) for the real symbol pair-wise ML decoding may be smaller than the optimal one obtained in [30] with the complex symbol pair-wise ML decoding as we shall see in Section III-C.

B. Optimal Linear Transformations U_i for Square QAM

In this subsection, we consider square QAM, i.e., M^2-QAM for any M. We present a solution of linear transformations U_i for the optimization problem (32), which turns out to be independent of the size M^2 of the M^2-QAM constellations and is different from other QAM constellations, such as rectangular QAM as we shall see in later subsections.

From a QOSTBC form (11) and the designs of complex orthogonal space–time block codes [11]–[13], one can see that in most COD designs, the complex symbols $\pm x_i$ or their complex conjugates $\pm x_i^*$ or zeros are directly placed in an COD $G_n(x_1, \ldots, x_k)$, i.e., no linear processing of these symbols is used, therefore $\pm x_i$ or $\pm x_i^*$, or 0 are directly transmitted. Thus, the mean transmission signal power is determined by the mean power of complex symbols x_i. Even for a COD $G_n(x_1, \ldots, x_k)$ with linear processing of symbols $\pm x_i$ or $\pm x_i^*$, as long as the QOSTBC pattern Q_{2n} is fixed, the mean transmission signal power is also determined by the mean power of complex symbols x_i. From the encoding in Section II-C, the signal powers of $x_i = p_i + \mathbf{j}q_i$ are determined by U_i in (18) and the information symbols $z_i = r_i + \mathbf{j}s_i$ where r_i and s_i are integers. Since U_i are not necessarily orthogonal transforms, the problem of the mean signal power of x_i can be formulated as the real lattice packing problem [53] as follows. The $4k$ dimensional lattice of p_i and q_i, $i = 1, 2, \ldots, 2k$, for the transmission signal can be formulated in terms of the $4k$ information integer lattice of r_i and s_i, $i = 1, 2, \ldots, 2k$

$$\begin{pmatrix} p_1 \\ q_1 \\ p_{k+1} \\ q_{k+1} \\ \vdots \\ p_k \\ q_k \\ p_{2k} \\ q_{2k} \end{pmatrix} = \begin{pmatrix} U_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & U_k \end{pmatrix} \begin{pmatrix} r_1 \\ s_1 \\ r_{k+1} \\ s_{k+1} \\ \vdots \\ r_k \\ s_k \\ r_{2k} \\ s_{2k} \end{pmatrix} \tag{37}$$

where $r_i, s_i, i = 1, 2, \ldots, 2k$, are from the integer set $\mathbb{Z}$ and U_i are 4×4 nonsingular real matrices. Then, the mean signal power of the transmission signal lattice p_i, q_i is reciprocal to the packing density of the $4k$ dimensional real lattice, which is determined by the determinant of the lattice generating matrix:

$$\frac{1}{\prod_{i=1}^{k} |\det(U_i)|}.$$

Therefore, the normalized diversity product of the QOSTBC Q_{2n} in (11) in terms of the mean signal power becomes

$$\bar{\zeta}(Q_{2n}) \triangleq \frac{\zeta(Q_{2n})}{\left(\prod_{i=1}^{k} |\det(U_i)|\right)^{1/(4k)}} \tag{38}$$

where $\zeta(Q_{2n})$ is the diversity product (3) of QOSTBC $Q_{2n}(x_1, \ldots, x_{2k})$. Thus, the optimization problem (32) becomes: to find a non-singular linear transformation U_i that satisfies Theorem 1 such that the above normalized diversity product $\bar{\zeta}(Q_{2n})$ is maximized. For this optimization problem, we have the following solution.

Theorem 2: Let

$$\alpha = \arctan(2) \text{ and } R = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \tag{39}$$

and P_1 and P_2 be the 4×4 matrices defined in ii) and iii) in Theorem 1, respectively. For the three cases in Section II-C, we have the following results, respectively.

i) For Case 1, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix}. \tag{40}$$

Then, the above orthogonal matrices U_i satisfy i) for Case 1 in Theorem 1, i.e., the quadratic forms f_i in (19) and g_i in (20) of four variables can be separated as Case 1, and furthermore, U_i are optimal in the sense that the normalized diversity product $\bar{\zeta}(Q_{2n})$ in (38) is maximized among all other nonsingular linear transformations U_i that satisfy i) in Theorem 1 and

$$\max_{U_i \text{ in i) Theorem 1}} \bar{\zeta}(Q_{2n}) = \frac{1}{2\sqrt{2T}} \left(\frac{4}{5}\right)^{1/4}. \tag{41}$$

ii) For Case 2, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix} P_1. \tag{42}$$

Then, the above orthogonal matrices U_i satisfy (ii) for Case 2 in Theorem 1 and they are optimal, and the same maximum normalized diversity product in (41) is achieved.

iii) For Case 3, let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ R & -R \end{pmatrix} P_2. \tag{43}$$

Then, the above orthogonal matrices U_i satisfy iii) for Case 3 in Theorem 1 and they are optimal, and the same maximum normalized diversity product in (41) is achieved.

Its proof is in Appendix. As one can see, in this subsection, instead of using the actual mean transmission signal power of x_i, we use the packing density in $4k$ dimensional real lattices in the study. As we shall see in the next subsection, the results in Theorem 2 are indeed true for any finite size square QAM, when the actual mean transmission signal power is used. In other words, when the actual mean transmission power is fixed, the maximal diversity products are achieved by the optimal linear transformations presented in Theorem 2. This is due to the square size of a square QAM that can be well represented by a $4k$ dimensional real lattice. The optimality result is also independent of the size of a square QAM signal constellation.

Interestingly, although the optimal linear transformations U_i found in ii) for Case 2 in Theorem 2 are not the same as the optimal rotations (34) obtained in [38], the optimal diversity products for a QOSTBC with the two different optimal transformations are the same for *square* QAM signal constellations as we shall see later in the next subsection. The same conclusion can be drawn for the optimal rotations (36) obtained in [37], [39] and the optimal linear transformations U_i found in i) for Case 1. The optimal rotation can be similarly obtained for iii) for Case 3. This means that in square QAM case, considering the rotations as in (33)–(36) is sufficient in the sense that a QOSTBC has a real symbol pair-wise ML decoding and in the meantime, the diversity product is maximized among all linear transformations U_i that satisfy the conditions in Theorem 1. Note that since rotations in (33) are only for the real parts of the ith and $(k+i)$th complex symbols, and the imaginary parts of the ith and $(k+i)$th complex symbols, separately, the rotations in (33) do not apply to Case 1 and Case 3.

It should be emphasized here that Case 1 is of particular interest. It is because the real symbol pair-wise decoding is not for two real or two imaginary parts but for the real and the imaginary parts of a single complex information symbol z_i. In this case, the signal constellation $\mathcal{S}_i$ of z_i can be any QAM. When we consider (i) for Case 1 for an arbitrary QAM, although we are not able to prove that the above U_i is optimal, the obtained QOSTBC still has full diversity with good diversity product property. The details about this issue shall be discussed in Section III-D later.

For rectangular QAM constellations, the situation is different and we use a different approach and find that the optimal linear transformations U_i depends on the sizes/shapes of rectangular QAM constellations as we shall see in the next subsection.

C. Optimal Linear Transformations U_i for Rectangular QAM (RQAM)

In this subsection, we consider RQAM signal constellations in (8)–(9) with total N_1N_2 points. It is not hard to see that the total energy $\mathcal{E}$ of the RQAM constellation in (8)–(9) is

$$\mathcal{E} = \frac{2}{3}N_1N_2\left(2N_1^2+2N_2^2-1\right)d^2. \tag{44}$$

For convenience, we assume the total energy of an RQAM constellation is normalized to 1, i.e., $\mathcal{E}=1$. Therefore, the distance d between two nearest neighboring points in the constellation becomes

$$d=\sqrt{\frac{3}{2N_1N_2\left(2N_1^2+2N_2^2-1\right)}}. \tag{45}$$

Since there exist $2k$ variables in a QOSTBC $Q_{2n}(z_1,\ldots,z_{2k})$, the total energy of all these information complex symbols $z_1,\ldots,z_{2k}$ is $2k$.

Let us consider Case 1 and the other two cases can be similarly done. Linear transformations U_i in (18) transform an information signal constellation RQAM for information symbols $z_i=r_i+\mathbf{j}s_i$ and $z_{k+i}=r_{k+i}+\mathbf{j}s_{k+i}$ into another one for transmitted symbols $x_i=p_i+\mathbf{j}q_i$ and $x_{k+i}=p_{k+i}+\mathbf{j}q_{k+i}$, $1\le i\le k$, in $Q_{2n}(x_1,\ldots,x_{2k})$. For convenience, we require that the total energies of these two signal constellations (before and after transformations U_i) are the same, i.e., U_i, $1\le i\le k$, are total-energy-invariant. The total-energy-invariance implies

$$\sum_{i=1}^{k}\sum_{p_i+\mathbf{j}q_i\in\mathcal{D}_i}\left(p_i^2+q_i^2\right) + \sum_{i=1}^{k}\sum_{p_{k+i}+\mathbf{j}q_{k+i}\in\mathcal{D}_{k+i}}\left(p_{k+i}^2+q_{k+i}^2\right)=2k \tag{46}$$

where, for $i=1,2,\ldots,2k$,

$$\mathcal{D}_i \triangleq \{p_i+\mathbf{j}q_i : z_i=r_i+\mathbf{j}s_i\in \mathrm{RQAM} \text{ and } z_{(k+i)\mathrm{mod}2k}=r_{(k+i)\mathrm{mod}2k}+\mathbf{j}s_{(k+i)\mathrm{mod}2k}\in\mathrm{RQAM}\}.$$

Since U_i satisfy i) in Theorem 1, from (30)

$$\begin{aligned} p_i^2+q_i^2+p_{k+i}^2+q_{k+i}^2 &= (r_i,s_i)\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right)\begin{pmatrix} r_i\\ s_i\end{pmatrix}\\ &\quad+(r_{k+i},s_{k+i})\left(U_{i2}^tU_{i2}+R_{i2}^tU_{i2}^tU_{i2}R_{i2}\right)\begin{pmatrix} r_{k+i}\\ s_{k+i}\end{pmatrix}.\end{aligned}$$

Thus, the total-energy-invariance (46) becomes (47) shown at the bottom of the page.

Under the above energy invariance/normalization requirement, we have the following results.

Theorem 3: Regarding to Cases 1)–3), we have the following results.

i) For Case 1 and an RQAM in (8)–(9) with total energy 1, i.e., (45). Let $\varepsilon_1=\frac{4N_1^2-1}{2\left(2N_1^2+2N_2^2-1\right)}$, $\varepsilon_2=\frac{4N_2^2-1}{2\left(2N_1^2+2N_2^2-1\right)}$, $\alpha=\arctan(2)$, $\rho=\sqrt{\frac{5}{12(1+\varepsilon_1\varepsilon_2)}}$, and

$$R_1=\begin{pmatrix}\cos(\alpha) & \sin(\alpha)\\ \sin(\alpha) & -\cos(\alpha)\end{pmatrix},\quad P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\Sigma=\begin{pmatrix}1+\varepsilon_1 & 1-2\varepsilon_1\\ 1-2\varepsilon_1 & 2-\varepsilon_1\end{pmatrix}. \tag{48}$$

Denote a diagonalization of symmetric matrix Σ as $\Sigma=V^tDV$, where $D=\mathrm{diag}(\lambda_1,\lambda_2)$, λ_1, λ_2 are the eigenvalues of Σ and V is an orthogonal matrix. Let

$$\begin{aligned} U_{i1}&=\rho V^t\begin{pmatrix}\sqrt{\lambda_1}&0\\0&\sqrt{\lambda_2}\end{pmatrix}V\\ U_{i2}&=\rho V^t\begin{pmatrix}\sqrt{\lambda_2}&0\\0&\sqrt{\lambda_1}\end{pmatrix}VP\\ R_2&=-PR_1P.\end{aligned} \tag{49}$$

Then

$$U_i=\begin{pmatrix}U_{i1}&U_{i2}\\U_{i1}R_1&U_{i2}R_2\end{pmatrix},\quad i=1,2,\ldots,k \tag{50}$$

satisfy i) for Case 1 in Theorem 1, and are optimal in the sense that the diversity product of the QOSTBC is

$$\begin{aligned}2k=\sum_{i=1}^{k}\Bigg(&\sum_{r_i+\mathbf{j}s_i\in\mathrm{RQAM}}(r_i,s_i)\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right)\begin{pmatrix}r_i\\s_i\end{pmatrix}\\ &+\sum_{r_{k+i}+\mathbf{j}s_{k+i}\in\mathrm{RQAM}}(r_{k+i},s_{k+i})\left(U_{i2}^tU_{i2}+R_{i2}^tU_{i2}^tU_{i2}R_{i2}\right)\begin{pmatrix}r_{k+i}\\s_{k+i}\end{pmatrix}\Bigg).\end{aligned} \tag{47}$$

maximized among all U_i under i) in Theorem 1 and the optimal diversity product is

$$\zeta_{\text{optimal,WWX}} = \frac{1}{2\sqrt{2T}}\frac{d}{\sqrt[4]{1+\varepsilon_1\varepsilon_2}} = \frac{1}{2\sqrt{2T}}\sqrt{\frac{3}{N_1N_2}} \times \frac{1}{\sqrt[4]{16N_1^4+16N_2^4+48N_1^2N_2^2-20N_1^2-20N_2^2+5}}. \quad (51)$$

ii) For Case 2, U_iP_1 with P_1 in Theorem 1 is optimal, where U_i is defined in (50).

iii) For Case 3, U_iP_2 with P_2 in Theorem 1 is optimal, where U_i is defined in (50).

Its proof is in Appendix. A QOSTBC $Q_{2n}(x_1,\ldots,x_{2k})$ from a $T\times n$ COD $G_n(x_1,\ldots,x_k)$ has size $2T\times 2n$ and thus in the optimal diversity product formula in (51), there is $2T$ instead of T as in the diversity product definition (3). Note that the optimal diversity products in Theorem 3 are not in terms of the normalized diversity products as used in the previous subsection for square QAM constellations.

Since an RQAM covers a square QAM, i.e., $N_1 = N_2 = N$ in (8)–(9). In this case, $\varepsilon_1=\varepsilon_2=1/2$ and $\rho=\sqrt{\frac{1}{3}}$, $\Sigma=\frac{3}{2}I_2$. Thus, $\lambda_1=\lambda_2=\frac{3}{2}$ and $U_{i1}=U_{i2}=\frac{1}{\sqrt{2}}I_2$. Therefore, the optimal U_i in Theorem 3 coincides with the one obtained in Theorem 2. Thus, we have following corollary.

Corollary 1: For a square QAM of size N^2, the optimal diversity product of a QOSTBC among all different linear transformations U_i in (18) that satisfy Theorem 1 is

$$\zeta_{\text{optimal,WWX}} = \frac{1}{2\sqrt{2T}}\sqrt{\frac{3}{\sqrt{5}N^2(4N^2-1)}}. \quad (52)$$

As mentioned in Section II-B that although the linear transformation U_i for Case 1 and Case 2 in Theorems 2 and 3 are different from the optimal rotations in (36) obtained in [37], [39] and the optimal rotations in (34) obtained in [38], respectively, the corresponding optimal diversity products are the same, i.e., (52), for a square QAM signal constellation of size N^2.

From Theorem 2 and Theorem 3, one can see that, although we use two different energy normalization methods and different proofs, the optimal linear transformations U_i are the same for square QAM constellations. In this regard, Theorem 3 covers Theorem 2.

From Theorem 3, one can see that the optimal linear transformations U_i in Theorem 3 depend on the size and the shape of an RQAM, i.e., N_1 and N_2, which is not what a rotation in (34) and a rotation in (36) can achieve. In fact, with the optimal rotation in (36) obtained in [37], [39] and the optimal rotation in (34) obtained in [38], the optimal diversity product for an RQAM in (8)–(9) is

$$\zeta_{\text{optimal,YGT}} = \frac{1}{2\sqrt{2T}}\left(\frac{4}{5}\right)^{1/4} d = \frac{1}{2\sqrt{2T}}\sqrt{\frac{3}{\sqrt{5}N_1N_2(2N_1^2+2N_2^2-1)}}. \quad (53)$$

It is not hard to see that

$$\zeta_{\text{optimal,WWX}} = \frac{1}{2\sqrt{2T}}\left(\frac{1}{1+\varepsilon_1\varepsilon_2}\right)^{1/4} d \geq \frac{1}{2\sqrt{2T}}\left(\frac{4}{5}\right)^{1/4} d = \zeta_{\text{optimal,YGT}} \quad (54)$$

where the equality "=" holds if and only if $\varepsilon_1=\varepsilon_2=1/2$, i.e., $N_1=N_2$ or square QAM. This concludes the following result.

Theorem 4: The diversity product of a QOSTBC with the optimal linear transformation U_i in Theorem 3 is greater than the one with the optimal rotation in (34) for any nonsquare RQAM.

Comparing with the optimal diversity product $\zeta_{\text{optimal,SX}}$ in [30] of a QOSTBC among all possible linear transformations without any restriction in Theorem 1 for RQAM, we have

$$\zeta_{\text{optimal,SX}} = \frac{1}{2\sqrt{2T}}d_{\min} = \frac{1}{2\sqrt{2T}}d > \frac{1}{2\sqrt{2T}}\left(\frac{1}{1+\varepsilon_1\varepsilon_2}\right)^{1/4} d = \zeta_{\text{optimal,WWX}} \quad (55)$$

where $d_{\min}$ is the minimum Euclidean distance of the signal constellation $\mathcal{S}$ and in the above RQAM case, $d_{\min}=d$. Note that in [30], it was shown that $\frac{1}{2\sqrt{2T}}d_{\min}$ is an upper bound for the diversity product of QOSTBC for any kind of linear transformations of information symbols and any signal constellation and furthermore the one from the optimal rotation of the half complex information symbols reaches the upper bound.

As an example, for $32(8\times 4)$-RQAM, i.e., $N_1=4$ and $N_2=2$, and a 4×4 QOSTBC, i.e., $T=2$, we have

$$\zeta_{\text{optimal,SX}} = \frac{1}{4(43264)^{1/4}} > \zeta_{\text{optimal,WWX}} = \frac{1}{4(49984)^{1/4}} > \frac{1}{4(54080)^{1/4}} = \zeta_{\text{optimal,YGT}}.$$

D. Linear Transformations U_i for Arbitrary QAM on Any Lattice

In this subsection, we first present the optimal linear transformations U_i for a rectangular QAM on an arbitrary lattice and then investigate them for an arbitrary QAM on an arbitrary lattice.

As we explained before, when two real parts or two imaginary parts of two complex information symbols, such as $\{r_i, r_{k+i}\}$ or $\{s_i, s_{k+i}\}$, or one real part of one complex symbol and one imaginary part of another complex symbol, such as r_i of z_i and s_{k+i} of z_{k+i}, are jointly decoded but independently among the pairs, the real and the imaginary parts of a complex number have to be independent each other.

This requirement forces that a signal constellation that a complex symbol belongs to has to be RQAM on a square lattice. Thus, some commonly used signal constellations, such as the

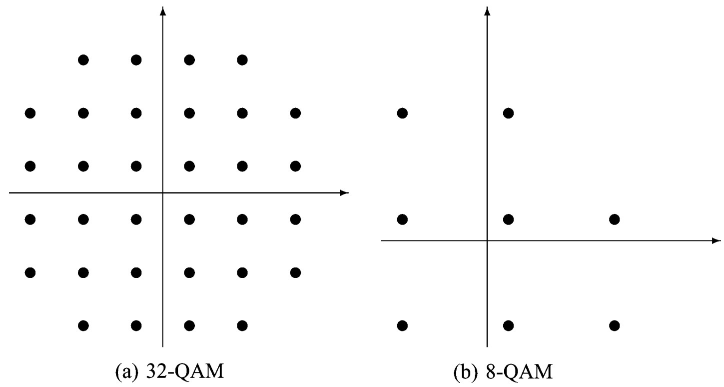

Fig. 2. Nonrectangular QAM.

32-QAM shown in Fig. 2(a), and also equal-literal triangular lattices are excluded for the fast decoding. Therefore, Case 2 and Case 3 in Section II-C do not apply to an arbitrary QAM (nonrectangular QAM), such as the ones in Fig. 2, that may have better energy compactness than RQAM, or other lattices than a square lattice. On the other hand, if the real part and the imaginary part of a complex symbol are not separated into two independent decodings, such as in Case 1, then the real part and the imaginary part of a complex symbol do not have to be independent for a real symbol pair-wise decoding for QOSTBC. In this case, a signal constellation can be any set of finite complex numbers. For QOSTBC with a real symbol pair-wise ML decoding, Case 1 is the only case for linear transformations U_i.

We first consider a rectangular QAM on any lattice (not necessarily square/integer) in the complex plane. Any two real-dimensional lattice on the complex plane can be generated from the square integer lattice through a 2×2 nonsingular real matrix Λ

$$(\bar{r}, \bar{s})^t = \Lambda(r, s)^t \tag{56}$$

where r and s are integers and it is simply called lattice Λ. A rectangular shape (or for simplicity QAM) constellation $\bar{\mathcal{S}}$ on lattice Λ corresponds to an RQAM $\mathcal{S} = \{(r,s)^t : (\bar{r}, \bar{s})^t = \Lambda(r,s)^t \in \bar{\mathcal{S}}\}$ on a square/integer lattice. With this relationship, one might think that what we have obtained previously could be applied to any rectangular QAM on any lattice Λ by absorbing the matrix Λ into the linear transformation U_i as $U_i \mathrm{diag}(\Lambda^{-1}, \Lambda^{-1})$ and then converting the problem to an RQAM on a square/integer lattice. This is correct and incorrect. For a rectangular QAM $\bar{\mathcal{S}}$ on a general lattice Λ, the real and the imaginary parts $\bar{r}$ and $\bar{s}$ with $\bar{z} = \bar{r} + \mathbf{j}\bar{s} \in \bar{\mathcal{S}}$ may not be independent. Thus, the results for Case 2 or Case 3 obtained previously do not apply to a rectangular QAM on an arbitrary lattice. Since in Case 1, the independence of the real and the imaginary parts of a complex symbol is not necessary, all the results for Case 1 apply to an rectangular QAM on any lattice Λ for a non-singular generating matrix Λ. This gives us the following corollary.

Corollary 2: For a rectangular QAM $\bar{\mathcal{S}}$ on lattice Λ with a nonsingular 2×2 real matrix Λ, let $\bar{U}_i = U_i \mathrm{diag}(\Lambda^{-1}, \Lambda^{-1})$ where U_i is defined in i) for Case 1 in Theorem 3. Let information symbols $\bar{z}_i = \bar{r}_i + \mathbf{j}\bar{s}_i$ be randomly taken in $\bar{\mathcal{S}}$ and then follow the encoding procedure in Section II-C by replacing r_i, s_i, z_i, p_i, q_i, x_i, and U_i with $\bar{r}_i, \bar{s}_i, \bar{z}_i, \bar{p}_i, \bar{q}_i, \bar{x}_i$, and $\bar{U}_i$, respectively. Then, $\bar{U}_i$ is optimal in the sense that the QOSTBC has a real symbol pair-wise ML decoding and the diversity product is maximized.

For an arbitrary QAM on an arbitrary lattice we have the following result.

Theorem 5: Let $\bar{U}_i$ be defined in Corollary 2 and U_i be the linear transformation defined in i) in Theorem 3. Then, the QOSTBC with this linear transformation has full diversity and the real symbol pair-wise ML decoding (21) for any QAM signal constellation on any lattice.

Proof: From the discussion of Corollary 2, without loss of generality, we only need to consider a square lattice, i.e., $\Lambda = I_2$. In this case, let $\mathcal{S}$ be an arbitrary set of finite points on the square lattice. Then, by adding proper points to it, constellation $\mathcal{S}$ can be made up into a RQAM $\mathcal{S}_1$ of size $N_1 \times N_2$. Clearly, if a QOSTBC has full diversity for a larger constellation $\mathcal{S}_1$, it also has full diversity for a smaller constellation $\mathcal{S}$. For RQAM $\mathcal{S}_1$, we apply the result in i) for Case 1 in Theorem 3 and know that the QOSTBC has the optimal diversity product expressed in (51) that is not zero. This proves Theorem 5. **Q.E.D.**

Although we are not able to prove the optimality of $\bar{U}_i$ in Theorem 5 in terms of the optimal diversity, we have the following conjecture.

1) Conjecture 1: For an arbitrary QAM $\bar{\mathcal{S}}$ on an arbitrary lattice, there exists a tightest $N_1 \times N_2$ RQAM $\bar{\mathcal{S}}_1$ in the sense that the Euclidean distance between $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}_1$ is minimized. With this RQAM $\bar{\mathcal{S}}_1$, let $\bar{U}_i$ be defined in Corollary 2. We conjecture that $\bar{U}_i$ is optimal in the sense that the QOSTBC reaches the optimal diversity product.

Although we are not able to prove the optimality of U_i in i) in Theorem 3 for an arbitrary QAM on a square lattice, we can calculate the diversity product of the QOSTBC when this U_i is used

$$\zeta_{\mathrm{WWX}} = \frac{1}{2\sqrt{2T}} \left(\frac{1}{1+\varepsilon_1\varepsilon_2} \right)^{1/4} d \tag{57}$$

where ε_1 and ε_2 are defined in Theorem 3, N_1 and N_2 are defined in Conjecture 1 and d is the Euclidean distance of the two nearest neighboring points in constellation $\mathcal{S}$ with normalized total energy 1. When $N_1 = N_2$, i.e., $\mathcal{S}$ is approximated by a tightest square QAM, the diversity product of the QOSTBC with the above U_i is

$$\zeta_{\mathrm{WWX}} = \frac{1}{2\sqrt{2T}}\left(\frac{4}{5}\right)^{1/4} d \tag{58}$$

where d is as before. The optimal diversity of the QOSTBC among all possible linear transformations is [30]

$$\zeta_{\mathrm{optimal,SX}} = \frac{1}{2\sqrt{2T}} d \tag{59}$$

where d is the same as before, which can be achieved by optimally rotating half of the complex symbols [30].

As an example, let us consider 32-QAM in Fig. 2(a). By using the transform U_i in i) in Theorem 3, the diversity product $\zeta_{\mathrm{WWX}} = 0.0187$ while the optimal diversity product in [30] is $\zeta_{\mathrm{optimal,SX}} = 0.0198$. Note that for the $32(8\times 4)$-RQAM, by using the optimal transform U_i in (i) in Theorem 3, the diversity product $\zeta_{\mathrm{optimal,WWX}} = 1/(4(49984)^{1/4}) = 0.0167$ while the optimal diversity in [30] is $\zeta_{\mathrm{optimal,SX}} = 1/(4(43264)^{1/4}) = 0.0173$ as calculated at the end of Section III-C.

E. Optimal Transformations for Co-Ordinate Interleaved Orthogonal Designs (CIOD) for RQAM and Some Comparisons

To increase the symbol rates of COD, a different approach has been proposed by Khan-Rajan [32]–[36], where they propose to place a COD on diagonal repeatedly with different information symbols and then these different information symbols are interleaved in such a way that the final overall design has full diversity, which is called a co-ordinate interleaved orthogonal design (CIOD). Its definition is given below.

Let $G_n(x_1,\ldots,x_k)$ be a $T\times n$ COD with complex variables $x_1,\ldots,x_k$ as before. For $x_i = p_i + \mathbf{j}q_i$, $1\le i\le 2k$, define $\tilde{x}_i = p_i + \mathbf{j}q_{(k+i)\mathrm{mod}2k}$, $1\le i\le 2k$, as interleaved variables of x_i. A $(2k,2T,2n)$ CIOD $\tilde{Q}_{2n}(\tilde{x}_1,\ldots,\tilde{x}_{2k})$ is a $2T\times 2n$ matrix and defined by

$$\tilde{Q}_{2n}(\tilde{x}_1,\ldots,\tilde{x}_{2k}) = \begin{pmatrix} G(\tilde{x}_1,\ldots,\tilde{x}_k) & 0_{T\times n} \\ 0_{T\times n} & G(\tilde{x}_{k+1},\ldots,\tilde{x}_{2k}) \end{pmatrix}. \tag{60}$$

By rotating a signal constellation properly, it has been shown in [32]–[36] that the above CIOD can achieve full diversity. The encoding is similar to QOSTBC shown in Fig. 1. Let $\mathcal{S}$ be a signal constellation of finite complex numbers. The encoding for a CIOD scheme is as follows.

- Map a binary information sequence into symbols $z_i = r_i + \mathbf{j}s_i$ in $\mathcal{S}$, $1\le i\le 2k$.
- Rotate the mapped complex symbols $z_i = r_i + \mathbf{j}s_i$ into $x_i = p_i + \mathbf{j}q_i$:

$$(p_i,q_i)^t = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} (r_i,s_i)^t. \tag{61}$$

- Define $\tilde{x}_i = p_i + \mathbf{j}q_{(k+i)\mathrm{mod}2k}$, $1\le i\le 2k$.
- Transmit CIOD matrix $\tilde{Q}_{2n}(\tilde{x}_1,\ldots,\tilde{x}_{2k})$.

With this scheme, it has been shown in [32]–[36] that the CIOD possesses a real symbol pair-wise ML decoding where (r_i,s_i) are jointly decoded but independently in terms of index i. Therefore, it is similar to Case 1 in our study in Section II-C. Thus, for the real symbol pair-wise ML decoding, the original signal constellation $\mathcal{S}$ does not have to be square or rectangular QAM. Also, the rotation in (61) is not necessary and any 2×2 real linear transform U does not change the real symbol pair-wise ML decoding property. Regarding to diversity product property, the following result was obtained in [35].

Proposition 1: Let $\mathcal{S}$ be a square QAM on a square lattice, i.e., an RQAM in (8)–(9) with $N_1 = N_2$. Then, $\theta = \arctan(2)/2$ is the optimal rotation angle in terms of the optimal diversity product for the rotation (61) and the optimal diversity product with this optimal rotation angle is

$$\zeta_{\mathrm{optimal,KRL}} = \frac{1}{2\sqrt{2T}}\left(\frac{4}{5}\right)^{1/4} d \tag{62}$$

where d is as before.

From the above result, one can see that the optimal diversity products in both QOSTBC and CIOD for *square QAM* constellations on square lattices are the same when a real symbol pair-wise ML decoding is imposed

$$\zeta_{\mathrm{optimal,KRL}} = \zeta_{\mathrm{optimal,YGT}} = \zeta_{\mathrm{optimal,WWX}}.$$

For a nonsquare RQAM constellation, the optimal rotations have not appeared so far. We next present an optimality result for a nonsquare RQAM constellation.

Theorem 6: Let information signal constellation $\mathcal{S}$ be an RQAM defined in (8)–(9), and ε_1, ε_2 be defined as in Theorem 3. Define $\alpha = \arctan\left(\frac{1}{\sqrt{\varepsilon_1\varepsilon_2}}\right)$, $\theta_1 = \arctan\left(\frac{\sqrt{5}-1}{2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$, $\theta_2 = \alpha - \theta_1$, and

$$U = \begin{pmatrix} \frac{\cos(\theta_1)}{\sqrt{2\varepsilon_1}} & \frac{\sin(\theta_1)}{\sqrt{2\varepsilon_2}} \\ \frac{-\sin(\theta_2)}{\sqrt{2\varepsilon_1}} & \frac{\cos(\theta_2)}{\sqrt{2\varepsilon_2}} \end{pmatrix}.$$

Replace the rotation in (61) by the above transform U. Then, the above transformation U is optimal for a CIOD in terms of optimal diversity product among all nonsingular linear transformations and the optimal diversity product with this transformation is

$$\zeta_{\mathrm{optimal,WWX,CIOD}} = \frac{1}{2\sqrt{2T}}\left(\frac{1}{1+\varepsilon_1\varepsilon_2}\right)^{1/4} d \tag{63}$$

where d is the same as before.

Its proof is in Appendix. From this result and Theorem 3, one can see that the optimal diversity products for QOSTBC and CIOD for any RQAM on a square lattice are the same, i.e.,

$$\zeta_{\mathrm{optimal,WWX,CIOD}} = \zeta_{\mathrm{optimal,WWX}}.$$

Thus, QOSTBC and CIOD with optimal linear transformations of complex symbols perform identically in terms of both decoding complexity and diversity product property, which shall be verified via numerical simulations in terms of symbol error rates vs. SNR for 4-QAM in Section V. However, since half of the entries are 0 in a CIOD, for a fixed mean transmission signal power, PAPR for QOSTBC is better than that for CIOD as pointed out in [37], [39].

As a remark, the optimal linear transformation U in Theorem 6 is a rotation, i.e., orthogonal, if and only if $N_1 = N_2$, i.e., the RQAM is square. Then, in this case, $\varepsilon_1 = \varepsilon_2 = 1/2$, $\alpha = \arctan(2)$ and $\theta_1 = \theta_2 = \alpha/2$, and therefore it coincides with the optimal rotation in Proposition 1 obtained by Khan–Rajan–Lee [35] and the optimal transformation becomes

$$U = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}.$$

IV. General Setting With Generalized Hurwitz-Radon Families

In this section, we discuss some generalizations of QOSTBC and CIOD we have previously studied. It is not hard to see that, by splitting real and imaginary parts of complex symbols, both QOSTBC and CIOD of k complex variables[1] can be represented by a general linear dispersive form [45], [46] of $2k$ real variables

$$C(s_1, s_2, \ldots, s_{2k}) = s_1 A_1 + s_2 A_2 + \cdots + s_{2k} A_{2k} \quad (64)$$

where A_i are $T \times n$ constant matrices of complex entries and s_i are real variables. For convenience, we assume information signal constellations are all RQAM. Then, all the real and imaginary parts are independent when an i.i.d. information sequence is mapped to complex symbols in the RQAMs. Let $\mathcal{S}_i$ be the signal constellation for the ith variable s_i for $1 \le i \le 2k$. Then, it is not hard to see that using the above C as a space–time codeword matrix, it has a real symbol κ-tuple ML decoding (regardless of a permutation of symbols) if and only if

$$\begin{aligned}&(C(s_1, \ldots, s_{2k}))^\dagger C(s_1, \ldots, s_{2k}) \\ &= f_1(s_1, \ldots, s_\kappa) B_1 + \cdots + f_\tau(s_{2k-\kappa+1}, \ldots, s_{2k}) B_\tau \quad (65)\end{aligned}$$

where $\tau = 2k/\kappa$, B_i, $1 \le i \le \tau$, are $n \times n$ constant matrices, and f_i, $1 \le i \le \tau$, are independent functions of constant coefficients. If (65) holds for all real values of variables s_i, when $\kappa = 1$, it is not hard to see that (65) is equivalent to

$$A_i^\dagger A_j + A_j^\dagger A_i = 0 \text{ for } 1 \le i \ne j \le 2k, \quad (66)$$

which is equivalent to that $S(s_1, \ldots, s_{2k})$ is a kind[2] of COD (when it has full diversity that implies $A_i^\dagger A_i$ has full rank) as also mentioned in [32]–[36] and [37]–[39]. This also implies that it is impossible for QOSTBC or CIOD to have full diversity and real symbol-wise ML decoding unless its size is 2×2 by applying the COD symbol rate upper bound for a Hurwitz family in [15]. In other words, real symbol pair-wise decoding is the lowest complexity decoding of a QOSTBC or CIOD can have, i.e., it is already the minimum complexity decoding for QOSTBC or CIOD, for more than two transmit antennas.

For a general κ, it is related to a generalized Hurwitz–Radon family. Let Ω_i, $1 \le i \le \tau = 2k/\kappa$, be a partition of the index set $\Omega = \{1, 2, \ldots, 2k\}$, i.e., none of Ω_i is empty, $\Omega_i \cap \Omega_j = \phi$, empty, for $i \ne j$, and the union of all Ω_i is the whole set Ω. A set of constant matrices A_i, $1 \le i \le 2k$, of size $T \times n$ and complex entries are called a κ-tuple Hurwitz–Radon family if

$$A_{i_1}^\dagger A_{j_1} + A_{j_1}^\dagger A_{i_1} = 0, \quad (67)$$

for any $i_1 \in \Omega_i$ and $j_1 \in \Omega_j$ with $1 \le i \ne j \le \tau$. Clearly, (67) is equivalent to (66) when $\kappa = 1$. With the above generalized Hurwitz–Radon family, it is not hard to see that if the matrices A_i in $C(s_1, \ldots, s_{2k})$ in (64) form a κ-tuple Hurwitz–Radon family, then the linear dispersive code $C(s_1, \ldots, s_{2k})$ has a real symbol κ-tuple ML decoding.

Corresponding to the real symbol pair-wise ML decoding studied in Sections II and III, if we let $\kappa = 2$, $\tau = 2k$ and $\Omega_i = \{2i-1, 2i\}$ for $1 \le i \le 2k$, then QOSTBC or CIOD has a real symbol pair-wise ML decoding if its constant matrices A_i, $1 \le i \le 4k$, form a 2-tuple Hurwitz–Radon family. From Section III and [32]–[39], a 2-tuple Hurwitz–Radon family of $4k$ many $2T \times 2n$ matrices exists for $n = T = k = 2$.

[1]For convenience, we consider k complex variables instead of $2k$ complex variables as previously used.

[2]Strictly speaking, it may not be a COD due to that $A_i^\dagger A_i$ may not be I but has full rank.

TABLE I
Diversity Product Comparison

Constellations	4-QAM	8-QAM	32-QAM
Su-Xia [30]	0.1768	0.0801	0.0198
Case 1 (QAM in Fig. 2)	0.1672	0.0757	0.0187
CIOD	0.1672	0.0757	0.0187
RQAM with new optimal transform	0.1672	0.0699	0.0167
RQAM with optimal rotation [37][39]	0.1672	0.0683	0.0164

V. Some Simulation Results

In this section, we present some simulation results to verify the theoretical results obtained in the preceding sections. We consider four transmit and one receive antennas with quasi-static Rayleigh fadings. We simulate QOSTBC and CIOD of rate 1 for a square QAM (4-QAM), and QOSTBC of rate 1 for two nonsquare QAMs (8-QAM and 32-QAM). For 8-QAM and 32-QAM, two different cases are tested: one is RQAM with $N_1 = 2$, $N_2 = 1$ and $N_1 = 4$, $N_2 = 2$, respectively, and the other is non-RQAM shown in Fig. 2(a) and (b), respectively. For nonsquare RQAM, we compare our newly obtained optimal transformation U_i in Theorem 3 with the optimal rotation in [37], [39] that is not optimal among all linear transformations in terms of diversity product. For the two non-square QAMs in Fig. 2, we also apply the optimal transformation for Case 1 in Theorem 2 or Theorem 3 for square QAM or RQAM. The corresponding diversity products are listed in Table I. In Table I, we also listed the optimal diversity products using the optimal rotations of half complex symbols obtained in Su–Xia [30] where only complex symbol pair-wise ML decoding is possible. The numbers of trials of their ML decodings are listed in Table II. One can clearly see that for the same decoding complexity of real symbol pair-wise ML decoding, the diversity products for non-RQAM in Fig. 2 using Case 1 transformation are better than the others since the constellations in Fig. 2 are better compacted than the corresponding RQAMs.

Fig. 3 shows the block error rates vs. SNR at the receiver for the square QAM, 4-QAM, for QOSTBC without any rotation of symbols (diversity order 2), with the optimal rotations obtained in [37], [39], with the optimal transformations in Theorems 2–3,

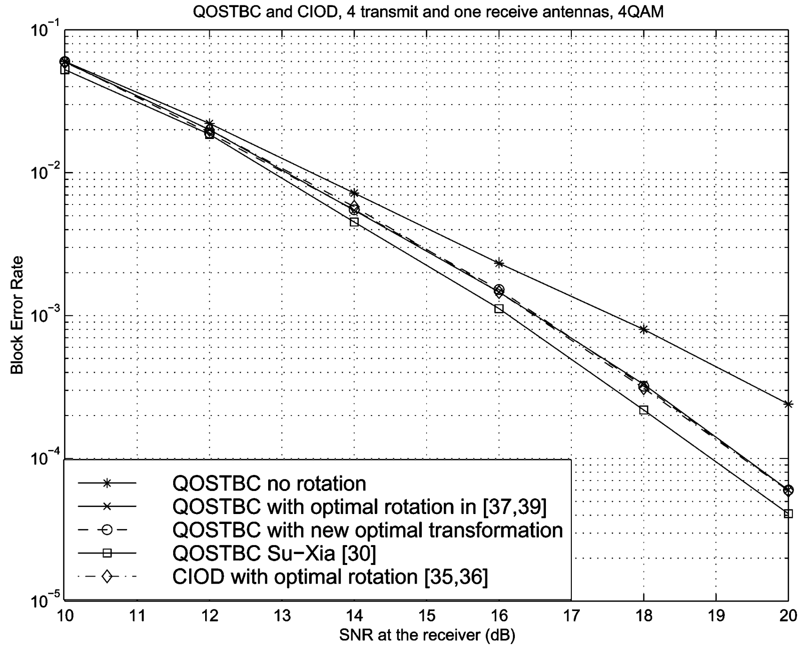

Fig. 3. Simulation results for schemes with the transmission rate of 2 bits/channel use.

TABLE II
ML DECODING COMPLEXITY COMPARISON: NUMBER OF TRIALS

Constellations	4-QAM	8-QAM	32-QAM
Su-Xia (complex symbol pair-wise)[30]	32	128	2048
Case 1 (real symbol pair-wise)	16	32	128
CIOD (real symbol pair-wise)	16	32	128
RQAM (real symbol pair-wise)	16	32	128

and for CIOD with the optimal rotation in [32]–[36], and finally for the optimal rotation obtained in [30] with the complex symbol pair-wise ML decoding. All other methods but the one in [30] are real symbol pair-wise ML decoding that is the same as the complex symbol-wise decoding. One can see that three curves, QOSTBC with the optimal rotations obtained in [37], [39], with the optimal transformations in Theorems 2–3, and CIOD with the optimal rotation in [32]–[36] coincide, which verifies what we mentioned at the end of Section III-E, i.e., they perform identically in terms of the performance and the decoding complexity.

The aim of Figs. 4 and 5 is to compare the performance between the newly obtained optimal transformations U_i in Theorem 3 for both RQAM and non-RQAM (Case 1) with the optimal rotation obtained in [37], [39] for RQAM. The block error rates vs. SNR at the receiver for 8-QAM (3 bits/s/Hz) and 32-QAM (5 bits/s/Hz) are shown in Figs. 4 and 5, respectively. One can see that newly obtained optimal transformations U_i in Theorem 3 for RQAM are better than the optimal rotation obtained in [37], [39] for RQAM, and the newly obtained optimal transformation U_i for Case 1 in Theorems 2–3 for the non-RQAM constellations in Fig. 2 have the best performance when real symbol pair-wise ML decoding is imposed. In Figs. 4 and 5, we also compare with the optimal rotation obtained in [30] with the complex symbol pair-wise ML decoding. All these results have confirmed the theoretical results obtained and discussed in Sections II–III. Note that, for the QAM constellations generated from *square* QAM as shown in Fig. 2, the optimal rotation obtained in [37], [39] can achieve the same performance as the optimal rotations for Case 1 in Theorems 2–3 do.

VI. CONCLUSION

In this paper, we systematically studied general linear transformations of information symbols for QOSTBC to have both full diversity and real symbol pair-wise ML decoding. We presented necessary and sufficient conditions on the linear transformations for a QOSTBC to possess a real symbol pair-wise ML decoding. We then presented the optimal transformation matrices (among all possible linear transformations not necessarily symbol rotations) of information symbols for QOSTBC with real symbol pair-wise ML decoding such that the optimal diversity product is achieved for both *general* square QAM and *general rectangular* QAM signal constellations. We showed that with our newly proposed optimal linear transformations for QOSTB for RQAM in one of the three cases, i.e., Case 1, QOSTB has full diversity and good diversity product property and real symbol pair-wise ML decoding for any finite signal constellation on any lattice. Interestingly, the optimal diversity products for square QAM constellations from the optimal linear transformations of information symbols found in this paper coincide with the ones presented by Yuen-Guan-Tjhung by using their optimal rotations. However, the optimal diversity products for non-square RQAM constellations from the optimal linear transformations of information symbols found in this paper are better than the ones presented by Yuen–Guan–Tjhung by using their optimal rotations. We also presented the optimal transformations for the coordinate interleaved orthogonal designs (CIOD) proposed by Khan–Rajan for rectangular QAM constellations.

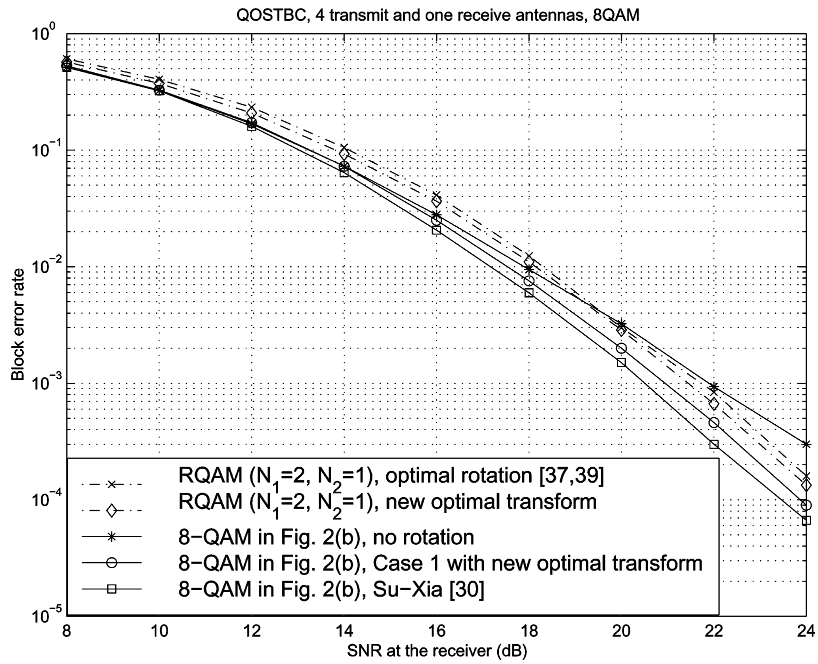

Fig. 4. Simulation results for schemes with the transmission rate of 3 bits/channel use.

As a remark, in this paper, we assume MIMO channels are uncorrelated and no any feedback is used. When MIMO channels are correlated, studies on QOSTBC can be found in, for example, [47]. When a limited feedback is available, studies on OSTBC and QOSTBC can be found in, for example, [48]–[52]. Further recent developments on fast decoding and linear transformations can be found in, for example, [41]–[44].

APPENDIX

In this Appendix, we prove Theorems 1, 2, 3, and 6. To do so, let us first see a lemma.

Lemma 1: Let A be a real $4 \times 4\backslash$ symmetry matrix and $\mathcal{S} \subset \mathbb{R}^2$ be a subset of two dimensional real space $\mathbb{R}^2$. Assume that there are at least four points in $\mathcal{S}$ such that they are not collinear. If for any real number pairs $(x_1, y_1), (x_2, y_2) \in \mathcal{S}$

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 \end{pmatrix} A \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = 0$$

then, $A = 0$.

Proof: Assume $(x_i, y_i) \in \mathcal{S}$, $i = 1, 2, 3, 4$, are not collinear. Since A is a real symmetric matrix, A has a diagonalized form $A = V^t D V$, where V is an orthogonal matrix and $D = \mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4)$ with real μ_i. To prove Lemma 1, it is enough to prove that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$.

From the condition of the lemma,

$$(x_i, y_i, x_j, y_j) V^t D V (x_i, y_i, x_j, y_j)^t = 0$$

for any $1 \le i, j \le 4$. Thus, if there exist i, j such that the first component of vector $V(x_i, y_i, x_j, y_j)^t$ is not 0, then $\mu_1 = 0$ is proved. To show so, let us use the contradiction method. Assume that for any i, j, $1 \le i, j \le 4$, the first component of $V(x_i, y_i, x_j, y_j)^t$ is 0. Denote the first row of matrix V as (v_1, v_2, v_3, v_4). Then

$$(v_1, v_2, v_3, v_4)(x_i, y_i, x_j, y_j)^t = 0.$$

Considering $i = 1$ and $j = 1$, we have

$$(v_1, v_2, v_3, v_4)(x_1, y_1, x_1, y_1)^t = 0.$$

Considering $i = 2$ and $j = 1$, we have

$$(v_1, v_2, v_3, v_4)(x_2, y_2, x_1, y_1)^t = 0.$$

Therefore

$$(v_1, v_2, v_3, v_4)(x_1 - x_2, y_1 - y_2, 0, 0)^t = 0$$

i.e.,

$$v_1(x_1 - x_2) + v_2(y_1 - y_2) = 0.$$

Similarly, we have

$$v_1(x_1 - x_3) + v_2(y_1 - y_3) = 0$$

and

$$v_1(x_1 - x_4) + v_2(y_1 - y_4) = 0.$$

If $(v_1, v_2) \neq 0$, the above three equations imply that the four points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) are collinear, which contradicts with the assumption in the beginning of the proof. Thus, we have proved $(v_1, v_2) = 0$. Similarly, we can prove $v_3 = v_4 = 0$. This means that the first row of orthogonal matrix V is the all zero vector, which is impossible. Therefore, we have proved that there exist i, j such that the first component

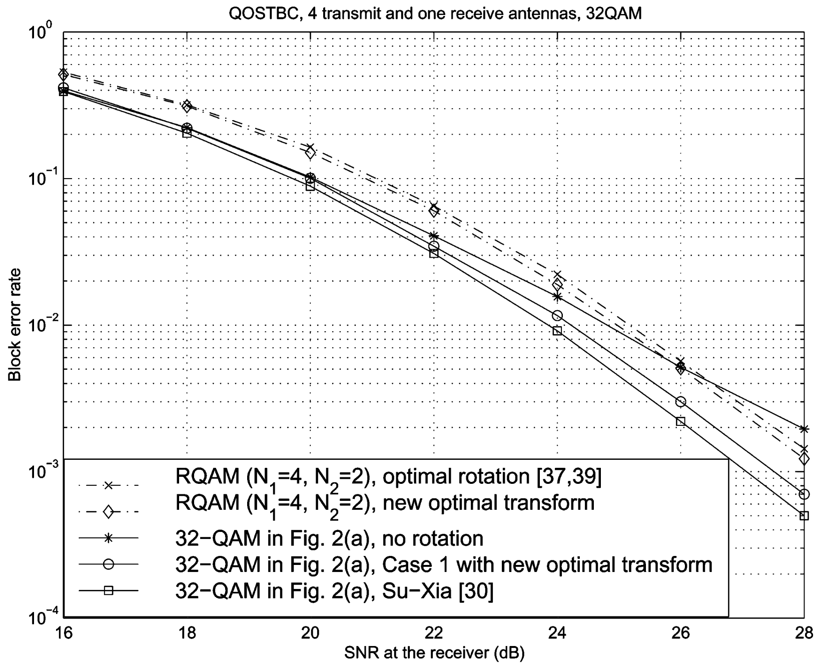

Fig. 5. Simulation results for schemes with the transmission rate of 5 bits/channel use.

of $V(x_i,y_i,x_j,y_j)^t$ is not 0. Hence, $\mu_1=0$. Similarly, we can prove $\mu_2=\mu_3=\mu_4=0$. **Q.E.D.**

Proof of Theorem 1: We first consider Case 1. Consider f_i in (20) or its representation $\bar{f}_i$ in (25). Clearly, $\bar{f}_i$ is a quadratic form of the variables r_i, s_i, r_{k+i}, s_{k+i} and hence if $\bar{f}_i$ can be separated as

$$\bar{f}_i(r_i,s_i,r_{k+i},s_{k+i}) = f_{i1}(r_i,s_i)+f_{i2}(r_{k+i},s_{k+i})$$

then, f_{i1} and f_{i2} are also quadratic forms of r_i, s_i and r_{k+i}, s_{k+i}, respectively. Therefore, there exist two real 2×2 symmetric matrices A_{i1} and A_{i2}, such that

$$f_{i1}(r_i,s_i)=(r_i,s_i)A_{i1}\begin{pmatrix} r_i \\ s_i\end{pmatrix} \tag{68}$$

and

$$f_{i2}(r_{k+i},s_{k+i})=(r_{k+i},s_{k+i})A_{i2}\begin{pmatrix} r_{k+i} \\ s_{k+i}\end{pmatrix}. \tag{69}$$

Combining with (25), we have

$$(r_i,s_i,r_{k+i},s_{k+i})U_i^t\begin{pmatrix} 0 & I_2 \\ I_2 & 0\end{pmatrix}U_i\begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i}\end{pmatrix}$$
$$=(r_i,s_i)A_{i1}\begin{pmatrix} r_i \\ s_i\end{pmatrix}+(r_{k+i},s_{k+i})A_{i2}\begin{pmatrix} r_{k+i} \\ s_{k+i}\end{pmatrix}. \tag{70}$$

Letting $U_i=\begin{pmatrix} U_{i1} & U_{i2} \\ U_{i3} & U_{i4}\end{pmatrix}$ and rewriting (70), we have

$$(r_i,s_i,r_{k+i},s_{k+i})M\begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i}\end{pmatrix}=0, \tag{71}$$

where

$$M=\begin{pmatrix} U_{i1}^tU_{i3}+U_{i3}^tU_{i1}-A_{i1} & U_{i3}^tU_{i2}+U_{i1}^tU_{4i} \\ (U_{i3}^tU_{i2}+U_{i1}^tU_{i4})^t & U_{i2}^tU_{i4}+U_{i4}^tU_{i2}-A_{i2}\end{pmatrix}.$$

Therefore, by Lemma 1, $\bar{f}_i$ can be separated as a sum of two functions as Case 1 if and only if

$$U_{i3}^tU_{i2}+U_{i1}^tU_{i4}=0. \tag{72}$$

Doing the similar calculations for function g_i in (19) or $\bar{g}_i$ in (24), we have that functions $\bar{g}_i$ can be separated as a sum of two functions as Case 1 if and only if

$$U_{i1}^tU_{i2}+U_{i3}^tU_{i4}=0. \tag{73}$$

Thus,

$$\begin{pmatrix} U_{i2} \\ U_{i4}\end{pmatrix}^t\begin{pmatrix} U_{i1} \\ U_{i3}\end{pmatrix}=0 \text{ and } \begin{pmatrix} U_{i2} \\ U_{i4}\end{pmatrix}^t\begin{pmatrix} U_{i3} \\ U_{i1}\end{pmatrix}=0. \tag{74}$$

Since U_i is non-singular and thus, the column vectors of matrix $\begin{pmatrix} U_{i2} \\ U_{i4}\end{pmatrix}$ are linearly independent and so are true for matrices $\begin{pmatrix} U_{i1} \\ U_{i3}\end{pmatrix}$ and $\begin{pmatrix} U_{i3} \\ U_{i1}\end{pmatrix}$. The first equation in (74) tells us that the two dimensional real space spanned by the column vectors of $\begin{pmatrix} U_{i1} \\ U_{i3}\end{pmatrix}$ is the complimentary space of the one spanned by the column vectors of $\begin{pmatrix} U_{i2} \\ U_{i4}\end{pmatrix}$ in the four dimensional real space. The second equation in (74) also tells us that the two dimensional real space spanned by the column vectors of $\begin{pmatrix} U_{i3} \\ U_{i1}\end{pmatrix}$ is the complimentary space of the one spanned by the column vectors of $\begin{pmatrix} U_{i2} \\ U_{i4}\end{pmatrix}$ in the four dimensional real space. Thus, the two dimensional real space spanned by the column vectors of

$\begin{pmatrix} U_{i1} \\ U_{i3} \end{pmatrix}$ is the same as the one spanned by the column vectors of $\begin{pmatrix} U_{i3} \\ U_{i1} \end{pmatrix}$. This means that there exists a 2×2 (real) non-singular matrix R_{i1} such that

$$\begin{pmatrix} U_{i3} \\ U_{i1} \end{pmatrix} = \begin{pmatrix} U_{i1} \\ U_{i3} \end{pmatrix} R_{i1}$$

i.e.

$$U_{i3} = U_{i1}R_{i1} \text{ and } U_{i1} = U_{i3}R_{i1}. \tag{75}$$

Therefore,

$$\begin{pmatrix} U_{i1} \\ U_{i3} \end{pmatrix} = \begin{pmatrix} U_{i3} \\ U_{i1} \end{pmatrix} R_{i1} = \begin{pmatrix} U_{i1}R_{i1} \\ U_{i3}R_{i1} \end{pmatrix} R_{i1} = \begin{pmatrix} U_{i1} \\ U_{i3} \end{pmatrix} R_{i1}^2.$$

Again, by the linear independence of the two columns of matrix $\begin{pmatrix} U_{i1} \\ U_{i3} \end{pmatrix}$, we have

$$R_{i1}^2 = I_2. \tag{76}$$

Similarly, we can obtain the results for $\begin{pmatrix} U_{i2} \\ U_{i4} \end{pmatrix}$. Thus, we have proved (i) for Case 1 in Theorem 1.

For (ii) for Case 2, it is enough to note that

$$\begin{aligned} \bar{f}_i(r_i, s_i, r_{k+i}, s_{k+i}) &= (r_i, r_{k+i}, s_i, s_{k+i})(U_iP_1)^t \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \\ &\quad \times (U_iP_1) \begin{pmatrix} r_i \\ r_{k+i} \\ s_i \\ s_{k+i} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \bar{g}_i(r_i, s_i, r_{k+i}, s_{k+i}) &= (r_i, r_{k+i}, s_i, s_{k+i}) \\ &\quad \times (U_iP_1)^t \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} (U_iP_1) \begin{pmatrix} r_i \\ r_{k+i} \\ s_i \\ s_{k+i} \end{pmatrix}. \end{aligned}$$

Then, all results can be derived if U_i is replaced by U_iP_1 as in i) for Case 1.

The proof of iii) for Case 3 is similar to the one of ii) for Case 2. **q.e.d.**

Proof of Theorem 2: Let $r_i + \mathbf{j}s_i$ and $r_i' + \mathbf{j}s_i'$, $i = 1, 2, \ldots, k, k+1, \ldots, 2k$, be symbols taken from a square-QAM and denote $\Delta r_i = r_i - r_i'$ and $\Delta s_i = s_i - s_i'$. The transmitted symbols are obtained through the transformations (18):

$$\begin{aligned} (p_i, q_i, p_{k+i}, q_{k+i})^t &= U_i(r_i, s_i, r_{k+i}, s_{k+i})^t \\ \left(p_i', q_i', p_{k+i}', q_{k+i}'\right)^t &= U_i\left(r_i', s_i', r_{k+i}', s_{k+i}'\right)^t. \end{aligned} \tag{77}$$

Denote $\Delta p_i = p_i - p_i'$ and $\Delta q_i = q_i - q_i'$. Then

$$\begin{aligned} &(\Delta p_i, \Delta q_i, \Delta p_{k+i}, \Delta q_{k+i})^t \\ &\qquad = U_i(\Delta r_i, \Delta s_i, \Delta r_{k+i}, \Delta s_{k+i})^t \end{aligned} \tag{78}$$

and

$$\det((\Delta C)^\dagger(\Delta C)) = \det \begin{pmatrix} (\Delta a)I_n & (\Delta b)I_n \\ (\Delta b)I_n & (\Delta a)I_n \end{pmatrix}$$

where $\Delta a = \sum_{i=1}^{2k}(\Delta p_i)^2 + \sum_{i=1}^{k}(\Delta q_i)^2$ and $\Delta b = 2\sum_{i=1}^{k}(\Delta p_i \Delta p_{k+i} + \Delta q_i \Delta q_{k+i})$. And

$$\begin{aligned} \det \begin{pmatrix} (\Delta a)I_n & (\Delta b)I_n \\ (\Delta b)I_n & (\Delta a)I_n \end{pmatrix} &= [(\Delta a)^2 - (\Delta b)^2]^n \\ &= (\Delta a + \Delta b)^n(\Delta a - \Delta b)^n. \end{aligned} \tag{79}$$

In the rest of this proof, since a QOSTBC is linear in terms of variables r_i, s_i, p_i, q_i and their differences are also integers, for simplicity we omit the notation Δ. For convenience, we also denote

$$D_{\min} \triangleq \min_{(x_1,\ldots,x_{2k}) \neq \left(x_1',\ldots,x_{2k}'\right)} \det((\Delta C)^\dagger(\Delta C))$$

where $x_i = p_i + \mathbf{j}q_i$ and $x_i' = p_i' + \mathbf{j}q_i'$. Thus, the diversity product of a QOSTC Q_{2n} can be expressed as

$$\zeta(Q_{2n}) = \frac{1}{2\sqrt{2T}}(D_{\min})^{\frac{1}{4n}}.$$

Let us first consider i) for Case 1 in Theorem 2. By the definitions of Δa and Δb, we have

$$\begin{aligned} \Delta a + \Delta b &= \sum_{i=1}^{k}(p_i, q_i, p_{k+i}, q_{k+i}) \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} p_i \\ q_i \\ p_{k+i} \\ q_{k+i} \end{pmatrix} \\ &= \sum_{i=1}^{k}(r_i, s_i, r_{k+i}, s_{k+i})U_i^t \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} U_i \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix} \\ &= \sum_{i=1}^{k}(r_i, s_i, r_{k+i}, s_{k+i}) \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix}^t \\ &\quad \times \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} U_{i1} & U_{i2} \\ U_{i1}R_{i1} & U_{i2}R_{i2} \end{pmatrix} \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix} \\ &= \sum_{i=1}^{k}(r_i, s_i, r_{k+i}, s_{k+i}) \begin{pmatrix} K_{i1} & K_{i2} \\ K_{i2}^t & K_{i3} \end{pmatrix} \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix} \end{aligned} \tag{80}$$

where

$$\begin{aligned} K_{i1} &= U_{i1}^tU_{i1} + R_{i1}^tU_{i1}^tU_{i1} + U_{i1}^tU_{i1}R_{i1} + R_{i1}^tU_{i1}^tU_{i1}R_{i1} \\ K_{i2} &= U_{i1}^tU_{i2} + R_{i1}^tU_{i1}^tU_{i2} + U_{i1}^tU_{i2}R_{i2} + R_{i1}^tU_{i1}^tU_{i2}R_{i2} \\ K_{i3} &= U_{i2}^tU_{i2} + R_{i2}^tU_{i2}^tU_{i2} + U_{i2}^tU_{i2}R_{i2} + R_{i2}^tU_{i2}^tU_{i2}R_{i2}. \end{aligned}$$

By (27) and $R_{i1}^2 = R_{i2}^2 = I_2$, we know that $K_{i2} = 0$. Therefore

$$\Delta a + \Delta b = \sum_{i=1}^{k} (r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} + \sum_{i=1}^{k} (r_{k+i}, s_{k+i}) K_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix}. \quad (81)$$

For any i, $i = 1, 2 \ldots, k$

$$(r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} = \frac{1}{2} (r_i, s_i, r_i, s_i) \times \begin{pmatrix} U_{i1} & U_{i1}R_{i1} \\ U_{i1}R_{i1} & U_{i1} \end{pmatrix}^t \begin{pmatrix} U_{i1} & U_{i1}R_{i1} \\ U_{i1}R_{i1} & U_{i1} \end{pmatrix} \begin{pmatrix} r_i \\ s_i \\ r_i \\ s_i \end{pmatrix}.$$

Hence

$$(r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \geq 0$$

and K_{i1}, $i = 1, 2 \ldots, k$, are nonnegative definite. Similarly, K_{i3}, $i = 1, 2 \ldots, k$, are also nonnegative definite. Therefore, for any fixed index i, $i \in \{1, 2, \ldots, k\}$

$$\Delta a + \Delta b \geq (r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \quad (82)$$

and

$$\Delta a + \Delta b \geq (r_{k+i}, s_{k+i}) K_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix}. \quad (83)$$

Similarly

$$\begin{aligned} \Delta a - \Delta b &= \sum_{i=1}^{k} (r_i, s_i, r_{k+i}, s_{k+i}) \begin{pmatrix} \bar{K}_{i1} & 0 \\ 0 & \bar{K}_{i3} \end{pmatrix} \begin{pmatrix} r_i \\ s_i \\ r_{k+i} \\ s_{k+i} \end{pmatrix} \\ &= \sum_{i=1}^{k} (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} + \sum_{i=1}^{k} (r_{k+i}, s_{k+i}) \bar{K}_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix} \end{aligned} \quad (84)$$

where

$$\begin{aligned} \bar{K}_{i1} &= U_{i1}^t U_{i1} - R_{i1}^t U_{i1}^t U_{i1} - U_{i1}^t U_{i1} R_{i1} + R_{i1}^t U_{i1}^t U_{i1} R_{i1} \\ \bar{K}_{i3} &= U_{i2}^t U_{i2} - R_{i2}^t U_{i2}^t U_{i2} - U_{i2}^t U_{i2} R_{i2} + R_{i2}^t U_{i2}^t U_{i2} R_{i2}. \end{aligned}$$

Also for any i, $i = 1, 2, \ldots, k$,

$$(r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} = \frac{1}{2} (r_i, s_i, r_i, s_i) \begin{pmatrix} U_{i1} & -U_{i1}R_{i1} \\ -U_{i1}R_{i1} & U_{i1} \end{pmatrix}^t \times \begin{pmatrix} U_{i1} & -U_{i1}R_{i1} \\ -U_{i1}R_{i1} & U_{i1} \end{pmatrix} \begin{pmatrix} r_i \\ s_i \\ r_i \\ s_i \end{pmatrix}.$$

Hence

$$(r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \geq 0$$

and $\bar{K}_{i1}$, $i = 1, 2 \ldots, k$, are nonnegative definite. Similarly, $\bar{K}_{i3}$, $i = 1, 2 \ldots, k$, are nonnegative definite too. Therefore, for any fixed index i, $i \in \{1, 2, \ldots, k\}$

$$\Delta a - \Delta b \geq (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \quad (85)$$

and

$$\Delta a - \Delta b \geq (r_{k+i}, s_{k+i}) \bar{K}_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix}. \quad (86)$$

Summarizing the above results, we have

$$\det \begin{pmatrix} (\Delta a) I_n & (\Delta b) I_n \\ (\Delta b) I_n & (\Delta a) I_n \end{pmatrix} \geq \left\{ (r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \right\}^n \quad (87)$$

$$\det \begin{pmatrix} (\Delta a) I_n & (\Delta b) I_n \\ (\Delta b) I_n & (\Delta a) I_n \end{pmatrix} \geq \left\{ (r_{k+i}, s_{k+i}) K_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix} \times (r_{k+i}, s_{k+i}) \bar{K}_{i3} \begin{pmatrix} r_{k+i} \\ s_{k+i} \end{pmatrix} \right\}^n \quad (88)$$

for any index i, $i = 1, 2, \ldots, k$, which gives lower bounds of the determinant. Since r_i, s_i, $i = 1, \ldots, 2k$, are independent integers, these lower bounds can be achieved by letting $r_t = s_t = 0$ for $t \neq i$. Hence, the minimum of (87) or (88) for $0 \neq (r_i, s_i) \in \mathbb{Z} \times \mathbb{Z}$ is the minimum of the determinant, i.e., see (89) at the bottom of the page where $(S_i, \bar{S}_i)$ can be $(K_{i1}, \bar{K}_{i1})$ or $(K_{i3}, \bar{K}_{i3})$ that are nonnegative definite.

We next want to have an upper bound for $D_{\min}$. To do so, we need the following lemma, which can be found in pp. 83, [54].

Lemma 2: Let $\xi = \alpha x + \beta y, \eta = \gamma x + \delta y$, and $|\alpha\delta - \beta\gamma|^2 = \Delta > 0$, where x and y are integers. Then, for the indefinite binary quadratic form $f = \xi\eta$ with discriminant Δ, there exists a point $(x, y) \neq 0$ such that

$$|\xi\eta| = |f(x, y)| \leq \sqrt{\frac{\Delta}{5}}. \quad (90)$$

To use this lemma, we need to change

$$(r_i, s_i) S_i \begin{pmatrix} r_i \\ s_i \end{pmatrix} (r_i, s_i) \bar{S}_i \begin{pmatrix} r_i \\ s_i \end{pmatrix} \quad (91)$$

$$D_{\min} = \min_{i \in \{1,2,\ldots,k\}} \min_{0 \neq (r_t, s_t) \in \mathbb{Z} \times \mathbb{Z}} \left\{ (r_t, s_t) S_i \begin{pmatrix} r_t \\ s_t \end{pmatrix} (r_t, s_t) \bar{S}_i \begin{pmatrix} r_t \\ s_t \end{pmatrix} \right\}^n \quad (89)$$

into a product of two linear forms. To do so, it is enough to consider the case when $S_i = K_{i1}$ and $\bar{S}_i = \bar{K}_{i1}$ and the other case is the same.

From $R_{i1}^2 = I_2$, we have $R_{i1} = \pm I_2$ or

$$R_{i1} = \begin{pmatrix} a_{i1} & b_{i1} \\ c_{i1} & -a_{i1} \end{pmatrix}$$

where a_{i1}, b_{i1}, c_{i1} are real numbers and subject to the following condition:

$$a_{i1}^2 + b_{i1}c_{i1} = 1. \tag{92}$$

We next make some assumptions to partially prove the result we wanted, and then we fully prove the result by removing the assumptions.

First, we assume Assumption 1: $R_{i1} \neq \pm I_2$. Let

$$R_{i1}^t U_{i1}^t U_{i1} + U_{i1}^t U_{i1} R_{i1} \triangleq \begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix}.$$

Then, we have (93) shown at the bottom of the page. Note that K_{i1} is a nonnegative definite (symmetry) matrix. Thus, $(1 + a_{i1})k_{i1} + c_{i1}k_{i2} \geq 0$, and

$$(1 + a_{i1})k_{i2} + c_{i1}k_{i3} = b_{i1}k_{i1} + (1 - a_{i1})k_{i2}. \tag{94}$$

Assume Assumption 2: $(1+a_{i1})k_{i1}+c_{i1}k_{i2} > 0$. Then, using (92) and (94), we have

$$(r_{i1}, s_{i1})K_{i1}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} = \left(\sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}\,r_{i1} + \frac{(1 + a_{i1})k_{i2} + c_{i1}k_{i3}}{\sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}}s_{i1}\right)^2. \tag{95}$$

Similarly, for $\bar{K}_{i1}$ we have (96) shown at the bottom of the page. Also note that $\bar{K}_{i1}$ is a nonnegative definite (symmetry) matrix and thus $(a_{i1} - 1)k_{i1} + c_{i1}k_{i2} \geq 0$.

Assume Assumption 3: $(a_{i1} - 1)k_{i1} + c_{i1}k_{i2} > 0$. Then

$$(r_{i1}, s_{i1})\bar{K}_{i1}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} = \left(\sqrt{(a_{i1} - 1)k_{i1} + c_{i1}k_{i2}}\,r_{i1} + \frac{(a_{i1} - 1)k_{i2} + c_{i1}k_{i3}}{\sqrt{(a_{i1} - 1)k_{i1} + c_{i1}k_{i2}}}s_{i1}\right)^2. \tag{97}$$

Therefore, (91) can be rewritten into the following squared product of two linear forms as shown in (98) at the bottom of the page. Let

$$\alpha = \sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}, \quad \beta = \frac{(1 + a_{i1})k_{i2} + c_{i1}k_{i3}}{\sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}}$$

$$\begin{aligned}
(r_{i1}, s_{i1})K_{i1}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} &= (r_{i1}, s_{i1})\left(U_{i1}^tU_{i1} + R_{i1}^tU_{i1}^tU_{i1}R_{i1} + R_{i1}^tU_{i1}^tU_{i1} + U_{i1}^tU_{i1}R_{i1}\right)\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} \\
&= (r_{i1}, s_{i1})\left(R_{i1}^t(R_{i1}^tU_{i1}^tU_{i1} + U_{i1}^tU_{i1}R_{i1}) + R_{i1}^tU_{i1}^tU_{i1} + U_{i1}^tU_{i1}R_{i1}\right)\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} \\
&= (r_{i1}, s_{i1})\left(\begin{pmatrix} a_{i1} & b_{i1} \\ c_{i1} & -a_{i1} \end{pmatrix}^t\begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix} + \begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix}\right)\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} \\
&= (r_{i1}, s_{i1})\begin{pmatrix} (1 + a_{i1})k_{i1} + c_{i1}k_{i2} & (1 + a_{i1})k_{i2} + c_{i1}k_{i3} \\ b_{i1}k_{i1} + (1 - a_{i1})k_{i2} & b_{i1}k_{i2} + (1 - a_{i1})k_{i3} \end{pmatrix}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix}.
\end{aligned} \tag{93}$$

$$\begin{aligned}
(r_{i1}, s_{i1})\bar{K}_{i1}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} &= (r_{i1}, s_{i1})\left(U_{i1}^tU_{i1} + R_{i1}^tU_{i1}^tU_{i1}R_{i1} - R_{i1}^tU_{i1}^tU_{i1} - U_{i1}^tU_{i1}R_{i1}\right)\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} \\
&= (r_{i1}, s_{i1})\left(\begin{pmatrix} a_{i1} & b_{i1} \\ c_{i1} & -a_{i1} \end{pmatrix}^t\begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix} - \begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix}\right)\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} \\
&= (r_{i1}, s_{i1})\begin{pmatrix} (a_{i1} - 1)k_{i1} + c_{i1}k_{i2} & (a_{i1} - 1)k_{i2} + c_{i1}k_{i3} \\ b_{i1}k_{i1} - (a_{i1} + 1)k_{i2} & b_{i1}k_{i2} - (a_{i1} + 1)k_{i3} \end{pmatrix}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix}.
\end{aligned} \tag{96}$$

$$\begin{aligned}
(r_{i1}, s_{i1})K_{i1}\,(\,r_{i1}s_{i1}\,)\,(r_{i1}, s_{i1})\bar{K}_{i1}\begin{pmatrix} r_{i1} \\ s_{i1} \end{pmatrix} &= \left(\sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}\;r_{i1} + \frac{(1 + a_{i1})k_{i2} + c_{i1}k_{i3}}{\sqrt{(1 + a_{i1})k_{i1} + c_{i1}k_{i2}}}\;s_{i1}\right)^2 \\
&\times\left(\sqrt{(a_{i1} - 1)k_{i1} + c_{i1}k_{i2}}\,r_{i1} + \frac{(a_{i1} - 1)k_{i2} + c_{i1}k_{i3}}{\sqrt{(a_{i1} - 1)k_{i1} + c_{i1}k_{i2}}}s_{i1}\right)^2.
\end{aligned} \tag{98}$$

$$\gamma = \sqrt{(a_{i1}-1)k_{i1}+c_{i1}k_{i2}}, \quad \delta = \frac{(a_{i1}-1)k_{i2}+c_{i1}k_{i3}}{\sqrt{(a_{i1}-1)k_{i1}+c_{i1}k_{i2}}}.$$

By Assumptions 2 and 3, $(1+a_{i1})k_{i1}+c_{i1}k_{i2}>0$ and $(a_{i1}-1)k_{i1}+c_{i1}k_{i2}>0$, and using (92), we conclude that $c_{i1}\neq 0$. Thus, by using (92) and (94), the discriminant is

$$\begin{aligned}\Delta_i &= (\alpha\delta-\beta\gamma)^2\\ &= 4\left(k_{i2}+\frac{a_{i1}+1}{c_{i1}}k_{i1}\right)\left(k_{i2}+\frac{a_{i1}-1}{c_{i1}}k_{i1}\right)\\ &> 0.\end{aligned} \tag{99}$$

By Lemma 2, we have (100) shown at the bottom of the page.

We next begin to remove the three assumptions: Assumption 1, $R_{i1}\neq \pm I_2$; Assumption 2, $(1+a_{i1})k_{i1}+c_{i1}k_{i2}>0$; and Assumption 3, $(a_{i1}-1)k_{i1}+c_{i1}k_{i2}>0$.

For Assumption 1, if $R_{i1}=I$, then $\bar{K}_{i1}=0$ from the definition of $\bar{K}_{i1}$. Using (89), we know that $D_{\min}=0$. So (100) is true. If $R_{i1}=-I_2$, then $K_{i1}=0$ and (100) is also true.

For Assumption 2, if $(1+a_{i1})k_{i1}+c_{i1}k_{i2}=0$, then from (93)

$$(r_{i1},s_{i1})K_{i1}\begin{pmatrix} r_{i1}\\ s_{i1}\end{pmatrix} = (r_{i1},s_{i1}) \times \begin{pmatrix} 0 & (1+a_{i1})k_{i2}+c_{i1}k_{i3}\\ b_{i1}k_{i1}+(1-a_{i1})k_{i2} & b_{i1}k_{i2}+(1-a_{i1})k_{i3}\end{pmatrix}\begin{pmatrix} r_{i1}\\ s_{i1}\end{pmatrix}.$$

Letting $r_{i1}=1$ and $s_{i1}=0$, we have

$$(r_{i1},s_{i1})K_{i1}\begin{pmatrix} r_{i1}\\ s_{i1}\end{pmatrix}=0.$$

Going back to (89), we have $D_{\min}=0$. Hence, (100) is still true. The same method can be used to remove Assumption 3. Therefore, the upper bound formula (100) is always true.

To derive an upper bound of the normalized diversity product $\bar{\zeta}(Q_{2n})$ in (38), we need to calculate the determinant

$$\det(U_1)\det(U_2)\cdots\det(U_k).$$

Let us see $\det(U_i)$ as shown in the last equation at the bottom of the page, where the third equality comes from (27) and $R_{i1}^2=I_2$ and in the last equality, we denote that

$$\begin{pmatrix} l_{i1} & l_{i2}\\ l_{i2} & l_{i3}\end{pmatrix} \triangleq R_{i2}^tU_{i2}^tU_{i2}+U_{i2}^tU_{i2}R_{i2}.$$

For l_{ij} and R_{i2}, similar properties as (92), (94), and (100) also hold. Again by $R_{i1}^2=R_{i2}^2=I_2$, we have

$$|\det(R_{i1})|=|\det(R_{i2})|=1.$$

Therefore

$$(\det(U_i))^2 = \left|k_{i1}k_{i3}-k_{i2}^2\right|\left|l_{i1}l_{i3}-l_{i2}^2\right|. \tag{101}$$

and

$$\left|\prod_{i=1}^k \det(U_i)\right| = \sqrt{\prod_{i=1}^k |k_{i1}k_{i3}-k_{i2}^2|\prod_{i=1}^k |l_{i1}l_{i3}-l_{i2}^2|}. \tag{102}$$

We assume that

$$\left|k_{i_01}k_{i_03}-k_{i_02}^2\right| = \min_{i=1,2,\ldots,k}\left\{\left|k_{i1}k_{i3}-k_{i2}^2\right|,\left|l_{i1}l_{i3}-l_{i2}^2\right|\right\}. \tag{103}$$

Then

$$\left|\prod_{i=1}^k \det(U_i)\right| \ge \left|k_{i_01}k_{i_03}-k_{i_02}^2\right|^k. \tag{104}$$

By the definition of $\bar{\zeta}(Q_{2n})$ in (38), (100) and (104) for k_{ij} or l_{ij}, we obtain the following upper bound as shown in (105) at the bottom of the following page. On the other hand, from (92) and (94), it is not hard to see that

$$\left|k_{i_01}k_{i_03}-k_{i_02}^2\right| = \left|\left(k_{i_02}+\frac{a_{i_01}+1}{c_{i_01}}k_{i_01}\right)\left(k_{i_02}+\frac{a_{i_01}-1}{c_{i_01}}k_{i_01}\right)\right|.$$

$$\begin{aligned}D_{\min} &\le \min_{i\in\{1,2,\ldots,k\}}\left(\frac{\Delta_i}{5}\right)^n\\ &= \min_{i\in\{1,\ldots,k\}}\left\{\frac{4}{5}\left(k_{i2}+\frac{a_{i1}+1}{c_{i1}}k_{i1}\right)\left(k_{i2}+\frac{a_{i1}-1}{c_{i1}}k_{i1}\right)\right\}^n.\end{aligned} \tag{100}$$

$$\begin{aligned}(\det(U_i))^2 &= \det\left(U_i^tU_i\right) = \det\left(\begin{pmatrix} U_{i1} & U_{i2}\\ U_{i1}R_{i1} & U_{i2}R_{i2}\end{pmatrix}^t\begin{pmatrix} U_{i1} & U_{i2}\\ U_{i1}R_{i1} & U_{i2}R_{i2}\end{pmatrix}\right)\\ &= \det\begin{pmatrix} U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1} & 0\\ 0 & U_{i2}^tU_{i2}+R_{i2}^tU_{i2}^tU_{i2}R_{i2}\end{pmatrix}\\ &= \det\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right)\det\left(U_{i2}^tU_{i2}+R_{i2}^tU_{i2}^tU_{i2}R_{i2}\right)\\ &= \det\left(R_{i1}^t\right)\det\begin{pmatrix} k_{i1} & k_{i2}\\ k_{i2} & k_{i3}\end{pmatrix}\det\left(R_{i2}^t\right)\det\begin{pmatrix} l_{i1} & l_{i2}\\ l_{i2} & l_{i3}\end{pmatrix}\end{aligned}$$

Thus, we obtain the following upper bound on $\bar{\zeta}(Q_{2n})$:

$$\bar{\zeta}(Q_{2n}) \leq \frac{1}{2\sqrt{2T}} \left(\frac{4}{5}\right)^{1/4}. \tag{106}$$

Next, we show that if we let $U_{i1} = U_{i2} = \frac{1}{\sqrt{2}} I_2$ and

$$R_{i1} = -R_{i2} = R = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

where $\theta = \arctan(2)$, then $\bar{\zeta}(Q_{2n})$ achieves the above upper bound.

By letting $a_{i1} = \cos(\theta)$, $b_{i1} = c_{i1} = \sin(\theta)$, we know that U_i are orthogonal matrices. By the definitions in (80) and (84), we know that $K_{i1} = I_2 + R$ and $\bar{K}_{i1} = I_2 - R$. Thus, from (89), we have (107) shown at the bottom of the page, where the last equality comes from $\tan(\theta) = 2$, i.e., $\sin(\theta) = 2\cos(\theta)$.

Since r and s are integers and $(r, s) \neq 0$, we have

$$[r^2 - rs - s^2]^2 \geq 1.$$

If we let $r = 1$ and $s = 0$, we have

$$[r^2 - rs - s^2]^2 = 1.$$

Therefore

$$D_{\min} = (\sin^2(\theta))^n.$$

But $\sin^2(\theta) = \frac{4}{5}$, and hence

$$D_{\min} = \left(\frac{4}{5}\right)^n. \tag{108}$$

On the other hand, since U_i are orthogonal matrices, we have $|\det(U_i)| = 1$. Therefore, we obtain

$$\bar{\zeta}(Q_{2n}) = \frac{1}{2\sqrt{2T}} \left(\frac{4}{5}\right)^{1/4}. \tag{109}$$

This proves i) for Case 1 in Theorem 2.

If in the above proof, we replace U_i by $U_i P_1$, we can similarly prove ii) for Case 2, and if we replace U_i by $U_i P_2$, we can prove iii) for Case 3, in Theorem 2. **Q.E.D**.

Proof of Theorem 3: We inherit all the notations in the above proof of Theorem 2. Similar to the proof of Theorem 2, we evaluate $D_{\min}$ and obtain an upper bound, and then show that this upper bound can be achieved. Unfortunately, the result for indefinite binary quadratic forms in Lemma 2 can not be applied here for finite signal constellations.

By checking formulas from (77) to (89) and from (91) to (98), we find that these formulas are all true under the conditions of Theorem 3. Especially, (89) and (98) are true and the three assumptions in the proof of Theorem 2 can be removed too. Let us recalculate

$$(r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \tag{110}$$

under the conditions of Theorem 3. Continuing with (98) and using $a_{i1}^2 + b_{i1} c_{i1} = 1$ and $c_{i1} k_{i3} = b_{i1} k_{i1} - 2 a_{i1} k_{i2}$ from (94), we have

$$\begin{aligned} &(r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \\ &\quad = \left(k_{i2}^2 + 2\frac{a_{i1}}{c_{i1}} k_{i1} k_{i2} - \frac{b_{i1}}{c_{i1}} k_{i1}^2\right) \\ &\quad\quad \times \left(c_{i1} r_i^2 - 2 a_{i1} r_{i1} s_{i1} - b_{i1} s_i^2\right)^2. \end{aligned} \tag{111}$$

Since

$$\begin{aligned} k_{i2}^2 - k_{i1} k_{i3} &= k_{i2}^2 - k_{i1} \frac{1}{c_{i1}} (b_{i1} k_{i1} - 2 a_{i1} k_{i2}) \\ &= k_{i2}^2 + 2 \frac{a_{i1}}{c_{i1}} k_{i1} k_{i2} - \frac{b_{i1}}{c_{i1}} k_{i1}^2 \end{aligned}$$

we have

$$\begin{aligned} &(r_i, s_i) K_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} (r_i, s_i) \bar{K}_{i1} \begin{pmatrix} r_i \\ s_i \end{pmatrix} \\ &\quad = \left(k_{i1}^2 - k_{i1} k_{i3}\right) \left(c_{i1} r_i^2 - 2 a_{i1} r_{i1} s_{i1} - b_{i1} s_i^2\right)^2. \end{aligned} \tag{112}$$

On the other hand

$$\begin{aligned} k_{i1} k_{i3} - k_{i2}^2 &= \det \begin{pmatrix} k_{i1} & k_{i2} \\ k_{i2} & k_{i3} \end{pmatrix} \\ &= \det \left(R_{i1}^t U_{i1}^t U_{i1} + U_{i1}^t U_{i1} R_{i1}\right) \end{aligned}$$

$$\bar{\zeta}(Q_{2n}) \leq \frac{1}{2\sqrt{2T}} \left(\frac{4}{5}\right)^{\frac{1}{4}} \left(\frac{\left|\left(k_{i_0 2} + \frac{a_{i_0 1}+1}{c_{i_0 1}} k_{i_0 1}\right)\left(k_{i_0 2} + \frac{a_{i_0 1}-1}{c_{i_0 1}} k_{i_0 1}\right)\right|}{\left|k_{i_0 1} k_{i_0 3} - k_{i_0 2}^2\right|}\right)^{1/4}. \tag{105}$$

$$\begin{aligned} D_{\min} &= \min_{0 \neq (r,s) \in \mathbb{Z} \times \mathbb{Z}} \left((r,s)(I_2 + R)(r,s)^t (r,s)(I_2 - R)(r,s)^t\right)^n \\ &= \min_{0 \neq (r,s) \in \mathbb{Z} \times \mathbb{Z}} \left[(1 + \cos(\theta)) r^2 + 2\sin(\theta) rs + (1 - \cos(\theta)) s^2\right]^n \left[(1 - \cos(\theta)) r^2 - 2\sin(\theta) rs + (1 + \cos(\theta)) s^2\right]^n \\ &= \min_{0 \neq (r,s) \in \mathbb{Z} \times \mathbb{Z}} \left(4[\cos(\theta/2) r + \sin(\theta/2) s]^2 [\sin(\theta/2) r - \cos(\theta/2) s]^2\right)^n \\ &= \min_{0 \neq (r,s) \in \mathbb{Z} \times \mathbb{Z}} \left([\sin(\theta) r^2 - 2\cos(\theta) rs - \sin(\theta) s^2]^2\right)^n \\ &= (\sin^2(\theta))^n \min_{0 \neq (r,s) \in \mathbb{Z} \times \mathbb{Z}} \left([r^2 - rs - s^2]^2\right)^n \end{aligned} \tag{107}$$

$$= \det\left(R_{i1}^t\right)\det\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right)$$
$$= -\det\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right) \quad (113)$$

where the second equality is from the definition of k_{ij} in the proof of Theorem 2, and the last equality is from the facts that

$$R_{i1}=\begin{pmatrix} a_{i1} & b_{i1} \\ c_{i1} & -a_{i1}\end{pmatrix} \text{ and } a_{i1}^2+b_{i1}c_{i1}=1.$$

Consequently, (112) can be changed into

$$(r_i,s_i)K_{i1}\begin{pmatrix} r_i \\ s_i\end{pmatrix}(r_i,s_i)\bar{K}_{i1}\begin{pmatrix} r_i \\ s_i\end{pmatrix}$$
$$= \det\left(U_{i1}^tU_{i1}+R_{i1}^tU_{i1}^tU_{i1}R_{i1}\right) \times \left(c_{i1}r_i^2-2a_{i1}r_{i1}s_{i1}-b_{i1}s_i^2\right)^2. \quad (114)$$

Note that the above calculations are also true for $(K_{i3},\bar{K}_{i3})$, and thus (114) holds if the subscript 1 in (114) is replaced by 2. Hence, going back to (89), we obtain (115) shown at the bottom of the page, where

$$\Delta\text{RQAM} \triangleq \{(r_1-r_2,s_1-s_2)\mid r_1+\mathbf{j}s_1\in\text{RQAM}; r_2+\mathbf{j}s_2\in\text{RQAM}\}.$$

From the form of an RQAM in (8)–(9), ones can see that for $(r,s)\in\Delta\text{RQAM}$, r and s must be integer multiples of d.

Let us first evaluate determinant

$$\det\left(U_{ij}^tU_{ij}+R_{ij}^tU_{ij}^tU_{ij}R_{ij}\right).$$

We now need the energy constraint (47). Without loss of generality, we may assume that

$$\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}}(r_u,s_u)\left(U_{u1}^tU_{u1}+R_{u1}^tU_{u1}^tU_{u1}R_{u1}\right)\begin{pmatrix} r_u \\ s_u\end{pmatrix} \le \sum_{r_i+\mathbf{j}s_i\in\text{RQAM}}(r_i,s_i)\left(U_{ij}^tU_{ij}+R_{ij}^tU_{ij}^tU_{ij}R_{ij}\right)\begin{pmatrix} r_i \\ s_i\end{pmatrix}$$

where $1\le i\le k$, $j=1,2$. Then, from (47), we have

$$0\le \sum_{r_u+\mathbf{j}s_u\in\text{RQAM}}(r_u,s_u)\left(U_{u1}^tU_{u1}+R_{u1}^tU_{u1}^tU_{u1}R_{u1}\right)\times\begin{pmatrix} r_u \\ s_u\end{pmatrix} \le 1. \quad (116)$$

From the definition

$$\begin{pmatrix} k_{u1} & k_{u2} \\ k_{u2} & k_{u3}\end{pmatrix} = R_{u1}^tU_{u1}^tU_{u1}+U_{u1}^tU_{u1}R_{u1}$$

and $R_{u1}^2=I_2$, we have

$$U_{u1}^tU_{u1}+R_{u1}^tU_{u1}^tU_{u1}R_{u1} = R_{u1}^t\left(R_{u1}^tU_{u1}^tU_{u1}+U_{u1}^tU_{u1}R_{u1}\right) = \begin{pmatrix} a_{u1} & c_{u1} \\ b_{u1} & -a_{u1}\end{pmatrix}\begin{pmatrix} k_{u1} & k_{u2} \\ k_{u2} & k_{u3}\end{pmatrix} = \begin{pmatrix} a_{u1}k_{u1}+c_{u1}k_{u2} & a_{u1}k_{u2}+c_{u1}k_{u3} \\ b_{u1}k_{u1}-a_{u1}k_{u2} & b_{u1}k_{u2}-a_{u1}k_{u3}\end{pmatrix}. \quad (117)$$

Note that the above matrix is symmetric, we have $a_{u1}k_{u2}+c_{u1}k_{u3}=b_{u1}k_{u1}-a_{u1}k_{u2}$, and (116) can be changed into

$$(a_{u1}k_{u1}+c_{u1}k_{u2})\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} r_u^2 + 2(a_{u1}k_{u2}+c_{u1}k_{u3})\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} r_us_u + (b_{u1}k_{u2}-a_{u1}k_{u3})\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} s_u^2 \le 1. \quad (118)$$

Because

$$\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} r_u^2 = N_2\left(\frac{4N_1^3}{3}-\frac{N_1}{3}\right)d^2 = \frac{4N_1^2-1}{2\left(2N_1^2+2N_2^2-1\right)} \triangleq \varepsilon_1 \quad (119)$$

$$\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} s_u^2 = N_1\left(\frac{4N_2^3}{3}-\frac{N_2}{3}\right)d^2 = \frac{4N_2^2-1}{2\left(2N_1^2+2N_2^2-1\right)} \triangleq \varepsilon_2 \quad (120)$$

and $\sum_{r_u+\mathbf{j}s_u\in\text{RQAM}} r_us_u=0$, (118) becomes

$$0\le\varepsilon_1(a_{u1}k_{u1}+c_{u1}k_{u2})+\varepsilon_2(b_{u1}k_{u2}-a_{u1}k_{u3})\le 1. \quad (121)$$

From $2a_{u1}k_{u2}=b_{u1}k_{u1}-c_{u1}k_{u3}$, the energy constraint (116) becomes

$$0\le \frac{\left(a_{u1}^2+1\right)\varepsilon_1+b_{u1}^2\varepsilon_2}{2a_{u1}}k_{u1}-\frac{c_{u1}^2\varepsilon_1+\left(a_{u1}^2+1\right)\varepsilon_2}{2a_{u1}}k_{u3} \le 1. \quad (122)$$

Now we can evaluate determinant

$$\det\left(U_{u1}^tU_{u1}+R_{u1}^tU_{u1}^tU_{u1}R_{u1}\right).$$

From (113), and $2a_{u1}k_{u2}=b_{u1}k_{u1}-c_{u1}k_{u3}$, we have

$$\det\left(U_{u1}^tU_{u1}+R_{u1}^tU_{u1}^tU_{u1}R_{u1}\right) = k_{u2}^2-k_{u1}k_{u3}$$

$$D_{\min}=\min_{i\in\{1,2,\ldots,k\}}\ \min_{0\ne(r,s)\in\Delta\text{RQAM}}\ \min_{j=1,2}\left\{\det\left(U_{ij}^tU_{ij}+R_{ij}^tU_{ij}^tU_{ij}R_{ij}\right)\left(c_{ij}r^2-2a_{ij}rs-b_{ij}s^2\right)^2\right\}^n \quad (115)$$

$$= \frac{b_{u1}^2 k_{u1}^2 - 2(a_{u1}^2+1)k_{u1}k_{u3} + c_{u1}^2 k_{u3}^2}{4a_{u1}^2} \tag{123}$$

where we have assumed that $a_{u1} \neq 0$. The case when $a_{u1} = 0$ will be considered later.

Using Lagrange multipliers, we can obtain the upper bound of $\det(U_{u1}^t U_{u1} + R_{u1}^t U_{u1}^t U_{u1} R_{u1})$ under condition (122):

$$\begin{aligned} 0 &\leq \det\left(U_{u1}^t U_{u1} + R_{u1}^t U_{u1}^t U_{u1} R_{u1}\right) \\ &= k_{u2}^2 - k_{u1}k_{u3} \\ &\leq \frac{1}{(b_{u1}\varepsilon_2 - c_{u1}\varepsilon_1)^2 + 4\varepsilon_1\varepsilon_2}. \end{aligned} \tag{124}$$

Combining it with (115), we have

$$D_{\min} \leq \min_{0\neq(r,s)\in\Delta\mathrm{RQAM}} \left\{ \frac{\left(c_{u1}r^2 - 2a_{u1}rs - b_{u1}s^2\right)^2}{\left(b_{u1}\varepsilon_2 - c_{u1}\varepsilon_1\right)^2 + 4\varepsilon_1\varepsilon_2} \right\}^n. \tag{125}$$

Since $(d,0) \in \Delta\mathrm{RQAM}$, $(0,d) \in \Delta\mathrm{RQAM}$ and $(d,d) \in \Delta\mathrm{RQAM}$

$$\begin{aligned} &\min_{0\neq(r,s)\in\Delta\mathrm{RQAM}} \left(c_{u1}r^2 - 2a_{u1}rs - b_{u1}s^2\right)^2 \\ &\qquad \leq \min\left\{b_{u1}^2 d^4, c_{u1}^2 d^4, (c_{u1} - 2a_{u1} - b_{u1})^2 d^4\right\}. \end{aligned}$$

Therefore

$$D_{\min} \leq \left\{ \frac{\min\left\{b_{u1}^2 d^4, c_{u1}^2 d^4, (c_{u1} - 2a_{u1} - b_{u1})^2 d^4\right\}}{(b_{u1}\varepsilon_2 - c_{u1}\varepsilon_1)^2 + 4\varepsilon_1\varepsilon_2} \right\}^n. \tag{126}$$

It is not hard to prove that under conditions $a_{u1}^2 + b_{u1}c_{u1} = 1$ and $\varepsilon_1 + \varepsilon_2 = 1$ that is from the definitions of ε_1 and ε_2 in Theorem 3, when $b_{u1}^2 = c_{u1}^2 = (c_{u1} - 2a_{u1} - b_{u1})^2$, i.e., $b_{u1}^2 = c_{u1}^2 = 4a_{u1}^2 = \frac{4}{5}$, the right hand side of (126) achieves the maximum. Therefore, we have

$$D_{\min} \leq \left\{ \frac{d^4}{1 + \varepsilon_1\varepsilon_2} \right\}^n. \tag{127}$$

We now consider the case when $a_{u1} = 0$. If $a_{u1} = 0$, then $b_{u1}c_{u1} = 1$ from the fact that $a_{u1}^2 + b_{u1}c_{u1} = 1$ and the energy constraint (121) becomes

$$(c_{u1}\varepsilon_1 + b_{u1}\varepsilon_2)k_{u2} \leq 1. \tag{128}$$

On the other hand, from (117),

$$U_{u1}^t U_{u1} + R_{u1}^t U_{u1}^t U_{u1} R_{u1} = \begin{pmatrix} c_{u1}k_{u2} & c_{u1}k_{u3} \\ b_{u1}k_{u1} & b_{u1}k_{u2} \end{pmatrix}.$$

Since the left hand side of the above equation is nonnegative definite, we have $c_{u1}k_{u2} \geq 0$ and $b_{u1}k_{u2} \geq 0$.

Since $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $b_{u1}c_{u1} = 1$, we have $c_{u1}\varepsilon_1 + b_{u1}\varepsilon_2 \neq 0$. Hence, combining $c_{u1}k_{u2} \geq 0$ and $b_{u1}k_{u2} \geq 0$ with inequality (128) and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we obtain that the left-hand side of inequality (128) is positive and

$$k_{u2}^2 \leq \left(\frac{1}{c_{u1}\varepsilon_1 + b_{u1}\varepsilon_2} \right)^2.$$

Consequently

$$\begin{aligned} &\det\left(U_{u1}^t U_{u1} + R_{u1}^t U_{u1}^t U_{u1} R_{u1}\right) \\ &= k_{u2}^2 - k_{u1}k_{u3} = k_{u2}^2 - k_{u1}\left(\frac{b_{u1}}{c_{u1}}k_{u1}\right) \\ &= k_{u2}^2 - k_{u1}^2 \frac{b_{u1}c_{u1}}{c_{u1}^2} \\ &\leq k_{u2}^2 \leq \left(\frac{1}{c_{u1}\varepsilon_1 + b_{u1}\varepsilon_2}\right)^2 \\ &\leq \frac{1}{(b_{u1}\varepsilon_2 - c_{u1}\varepsilon_1)^2 + 4\varepsilon_1\varepsilon_2} \end{aligned}$$

where the first equality comes from (123), and the second equality comes from $b_{u1}k_{u1} - c_{u1}k_{u3} = 2a_{u1}k_{u2} = 0$, and the first inequality is from $b_{u1}c_{u1} = 1$, and the last inequality is also due to $b_{u1}c_{u1} = 1$ and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$. Therefore, (124) is also true when $a_{u1} = 0$, and so is the upper bound formula (127).

With the upper bound (127) on $D_{\min}$, we next prove that this upper bound is achievable. To do so, it is enough to construct matrices U_i such that this bound is achieved.

For a fixed RQAM, the numbers N_1, N_2 and $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 + \varepsilon_2 = 1$ are fixed. Let

$$\alpha = \arctan(2) \text{ and } \rho = \sqrt{\frac{5}{12(1+\varepsilon_1\varepsilon_2)}}.$$

Let the matrices R, P, Σ be

$$\begin{aligned} R &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \Sigma &= \begin{pmatrix} 1+\varepsilon_1 & 1-2\varepsilon_1 \\ 1-2\varepsilon_1 & 2-\varepsilon_1 \end{pmatrix}. \end{aligned} \tag{129}$$

Clearly, $R^t = R$, $R^2 = I_2$ and $P^2 = I_2$. Because $0 < \varepsilon_1 < 1$, Σ is symmetric and positive definite. So, Σ has a diagonal decomposition $\Sigma = V^t D V$, where $D = \mathrm{diag}(\lambda_1, \lambda_2)$, λ_1, λ_2 are the eigenvalues of Σ, V is an orthogonal matrix.

Let

$$U_{i1} = \rho V^t \sqrt{D} V$$

for $i = 1, 2, \ldots, k$ and

$$U_{i2} = \rho V^t \mathrm{diag}\left(\sqrt{\lambda_2}, \sqrt{\lambda_1}\right) V P$$

for $i = 1, 2, \ldots, k$. Then

$$U_{i1}^t U_{i1} = \rho^2 \Sigma$$

and

$$U_{i2}^t U_{i2} = \rho^2 P V^t \mathrm{diag}(\lambda_2, \lambda_1) V P.$$

Let $R_{i1} = R$ and $R_{i2} = -PRP$ for $i = 1, 2, \ldots, k$. Then

$$\begin{aligned} R_{i1}^2 &= R^2 = I_2 \\ R_{i2}^2 &= PR^2P = I_2 \end{aligned}$$

and

$$\begin{aligned} &R_{i1}^t U_{i1}^t U_{i2} + U_{i1}^t U_{i2} R_{i2} \\ &\quad = \rho^2 R V^t \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} V V^t \begin{pmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\lambda_1} \end{pmatrix} V P \end{aligned}$$

$$+\rho^2 V^t \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} V V^t \begin{pmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\lambda_1} \end{pmatrix} V P(-PRP)$$
$$= \rho^2 \sqrt{\lambda_1\lambda_2}(RP - RP) = 0.$$

Therefore, the conditions in i) for Case 1 in Theorem 1 are satisfied.

We next calculate the minimum determinant $D_{\min}$ by using (115). Since, for any $i \in \{1, 2, \ldots, k\}$, $j = 1$,

$$\det\left(U_{ij}^t U_{ij} + R_{ij}^t U_{ij}^t U_{ij} R_{ij}\right)\left(c_{ij}r^2 - 2a_{ij}rs - b_{ij}s^2\right)^2$$
$$= \rho^4 \det(\Sigma + R\Sigma R)\left(\frac{4}{5}\right)(r^2 - rs - s^2)^2$$
$$= \frac{(r^2 - rs - s^2)^2}{1 + \varepsilon_1\varepsilon_2}. \tag{130}$$

For $j = 2$, we want to show

$$\det\left(U_{ij}^t U_{ij} + R_{ij}^t U_{ij}^t U_{ij} R_{ij}\right)\left(c_{ij}r^2 - 2a_{ij}rs - b_{ij}s^2\right)^2 = \frac{(-r^2 - rs + s^2)^2}{1 + \varepsilon_1\varepsilon_2}. \tag{131}$$

To do so, we first show

$$\det\left(U_{i2}^t U_{i2} + R_{i2}^t U_{i2}^t U_{ij} R_{i2}\right) = \det\left(U_{i1}^t U_{i1} + R_{i1}^t U_{i1}^t U_{i1} R_{i1}\right). \tag{132}$$

From the definitions of U_{i2} and R_{i2},

$$\det\left(U_{i2}^t U_{i2} + R_{i2}^t U_{i2}^t U_{i2} R_{i2}\right)$$
$$= \det\left(\rho^2 P V^t \mathrm{diag}(\lambda_2, \lambda_1)\right) VP$$
$$+ (-PRP)\left(\rho^2 P V^t \mathrm{diag}(\lambda_1, \lambda_2) VP\right)(-PRP)$$
$$= \rho^4 \det\left(V^t \mathrm{diag}(\lambda_2, \lambda_1)\, V + R V^t \mathrm{diag}(\lambda_2, \lambda_1) V R\right).$$

Note that

$$\det\left(U_{i1}^t U_{i1} + R_{i1}^t U_{i1}^t U_{i1} R_{i1}\right) = \rho^4 \det\left(V^t \mathrm{diag}(\lambda_1, \lambda_2) V + R V^t \mathrm{diag}(\lambda_1, \lambda_2) V R\right).$$

Thus, to prove (132), it is enough to prove

$$\det\left(V^t \mathrm{diag}(\lambda_2, \lambda_1) V + R V^t \mathrm{diag}(\lambda_2, \lambda_1) V R\right) = \det\left(V^t \mathrm{diag}(\lambda_1, \lambda_2) V + R V^t \mathrm{diag}(\lambda_1, \lambda_2) V R\right).$$

Denote the matrices inside the determinant signs of the left- and the right-hand sides of the above equation by M_1 and M_2, respectively. Then

$$M_1 + M_2 = V^t \mathrm{diag}(\lambda_1 + \lambda_2, \lambda_2 + \lambda_1) V$$
$$+ R V^t \mathrm{diag}(\lambda_1 + \lambda_2, \lambda_2 + \lambda_1) V R$$
$$= 2(\lambda_1 + \lambda_2) I_2.$$

If we denote $M_1 = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}$, then

$$\det(M_2)$$
$$= \det(2(\lambda_1 + \lambda_2) I_2 - M_1)$$
$$= \begin{pmatrix} 2(\lambda_1 + \lambda_2) - m_1 & -m_2 \\ -m_2 & 2(\lambda_1 + \lambda_2) - m_3 \end{pmatrix}$$
$$= [2(\lambda_1 + \lambda_2) - m_1][2(\lambda_1 + \lambda_2) - m_3] - m_2^2$$
$$= 4(\lambda_1 + \lambda_2)^2 - 2(m_1 + m_3)(\lambda_1 + \lambda_2) + m_1 m_3 - m_2^2$$
$$= 4(\lambda_1 + \lambda_2)^2 - 2\mathrm{tr}(M_1)(\lambda_1 + \lambda_2) + \det(M_1)$$
$$= 4(\lambda_1 + \lambda_2)^2 - 2(2(\lambda_1 + \lambda_2))(\lambda_1 + \lambda_2) + \det(M_1)$$
$$= \det(M_1),$$

where we use the fact that $\mathrm{tr}(M_1) = 2(\lambda_1 + \lambda_2)$. Thus, we have proved (132). From (132) and (130), we have (131).

Clearly

$$\min_{0 \neq (r,s) \in \Delta\mathrm{RQAM}} (r^2 - rs - s^2)^2 \leq d^4$$

because $(d, 0) \in \Delta\mathrm{RQAM}$. On the other hand, for any $0 \neq (r, s) \in \Delta\mathrm{RQAM}$

$$(r^2 - rs - s^2)^2 = M d^4$$

where M is a nonzero positive integer. Hence

$$\min_{0 \neq (r,s) \in \Delta\mathrm{RQAM}} (r^2 - rs - s^2)^2 = d^4.$$

Similarly

$$\min_{0 \neq (r,s) \in \Delta\mathrm{RQAM}} (-r^2 - rs + s^2)^2 = d^4.$$

Going back to (115), we have

$$D_{\min} = \left\{\frac{d^4}{1 + \varepsilon_1\varepsilon_2}\right\}^n.$$

This proves i) for Case 1 in Theorem 3. Similar to the proof of Theorem 2, the rest two cases ii) and iii) can be proved the same as Case 1 by replacing U_i with $U_i P_1$ and $U_i P_2$, respectively. **q.e.d.**

Proof of Theorem 6: Let us first see the conditions on U to maintain the total signal constellation energy to be 1. Consider a RQAM and let

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

and

$$\mathcal{B} = \{p + \mathbf{j}q : (p, q)^t = U(r, s)^t \text{ for } r + \mathbf{j}s \in \mathrm{RQAM}\}.$$

Thus, see (133) at the top of the following page. We next calculate the determinant of $(\Delta\tilde{Q}_{2n})^\dagger \Delta\tilde{Q}_{2n}$. Denote

$$x_j = p_j + \mathbf{j}q_j, \quad x'_j = p'_j + \mathbf{j}q'_j, \text{ for } j = 1, 2, \ldots, 2k.$$

Then

$$\tilde{x}_j = p_j + \mathbf{j}q_{k+j}, \ \ \tilde{x}'_j = p'_j + \mathbf{j}q'_{k+j}, \quad \text{for } j = 1, 2, \ldots, k$$

and

$$\tilde{x}_{k+j} = p_{k+j} + \mathbf{j}q_j, \ \ \tilde{x}'_{k+j} = p'_{k+j} + \mathbf{j}q'_j, \ \text{ for } j = 1, 2, \ldots, k.$$

Denote

$$\Delta\tilde{x}_j = \tilde{x}_j - \tilde{x}'_j, \text{ and } \Delta p_j = p_j - p'_j, \quad \Delta q_j = q_j - q'_j.$$

$$
\begin{aligned}
1 &= \sum_{(p+\mathbf{j}q)\in\mathcal{B}} (p^2+q^2) = \sum_{(p+\mathbf{j}q)\in\mathcal{B}} (p,q)\begin{pmatrix} p \\ q \end{pmatrix} \\
&= \sum_{(r+\mathbf{j}s)\in\mathrm{RQAM}} (r,s)U^tU\begin{pmatrix} r \\ s \end{pmatrix} \\
&= \sum_{(r+\mathbf{j}s)\in\mathrm{RQAM}} \left\{\left(u_{11}^2+u_{21}^2\right)r^2 + 2(u_{11}u_{12}+u_{21}u_{22})rs + \left(u_{12}^2+u_{22}^2\right)s^2\right\} \\
&= \left(u_{11}^2+u_{21}^2\right)\varepsilon_1 + \left(u_{12}^2+u_{22}^2\right)\varepsilon_2. \qquad (133)
\end{aligned}
$$

$$
\begin{aligned}
D_{\min} &= \min_{0\neq(\Delta x_1,\ldots,\Delta x_{2k})} \det\left(\left(\sqrt{2}\Delta\tilde{Q}_{2n}\right)^\dagger\left(\sqrt{2}\Delta\tilde{Q}_{2n}\right)\right) \\
&= \min_{0\neq(\Delta x_1,\ldots,\Delta x_{2k})} 2^{2n}\left(|\Delta\tilde{x}_1|^2+\cdots+|\Delta\tilde{x}_k|^2\right)^n \\
&\quad \times\left(|\Delta\tilde{x}_{k+1}|^2+\cdots+|\Delta\tilde{x}_{2k}|^2\right)^n \\
&= \min_{\Delta p_i+\mathbf{j}\Delta q_i\in\Delta\mathcal{B}} 2^{2n}\left(|\Delta p_1|^2+|\Delta q_{k+1}|^2+\cdots+|\Delta p_k|^2+|\Delta q_{k+k}|^2\right)^n \\
&\quad \times\left(|\Delta p_{k+1}|^2+|\Delta q_1|^2+\cdots+|\Delta p_{k+k}|^2+|\Delta q_k|^2\right)^n
\end{aligned}
$$

$$
\begin{aligned}
D_{\min} &= \min_{i=1,2,\ldots,2k}\ \min_{0\neq\Delta p_i+\mathbf{j}\Delta q_i\in\Delta\mathcal{B}} (2\Delta p_i\Delta q_i)^{2n} \\
&= \min_{i=1,2,\ldots,2k}\ \min_{0\neq\Delta r_i+\mathbf{j}\Delta s_i\in\Delta\mathrm{RQAM}} \left\{2(u_{11}\Delta r_i+u_{12}\Delta s_i)(u_{21}\Delta r_i+u_{22}\Delta s_i)\right\}^{2n}. \qquad (134)
\end{aligned}
$$

$$
D_{\min} = \min_{i=1,2,\ldots,2k}\quad \min_{r_i+\mathbf{j}s_i\in\Delta\mathrm{RQAM}} \left\{\det(U^tU)\left(cr_i^2-2ar_is_i-bs_i^2\right)^2\right\}^n.
$$

By the linearity of the transformation U

$$(\Delta p_j,\Delta q_j) = U(\Delta r_j,\Delta s_j)^t$$

where

$$
\begin{aligned}
\Delta r_j &= r_j - r_j', \quad \Delta s_j = s_j - s_j' \\
\text{and}(p_j,q_j) &= U(r_j,s_j)^t, \quad (p_j',q_j') = U\left(r_j',s_j'\right)^t.
\end{aligned}
$$

Hence, by the linearity and the orthogonality of $G(x_1,\ldots,x_k)$, we have the second equation shown at the top of the following page where not all $\Delta p_i+\mathbf{j}\Delta q_i = 0$, the second equality is due to the orthogonality of G in (60), and $\sqrt{2}$ in the first equality comes from the consideration on energy because at every transmission, the half of antennas are silent, i.e., 0.

Since $(\Delta p_i,\Delta q_i)$ are independent in terms of i, from the above equation the following holds as shown in (134) at the top of the page. We next simplify $2(u_{11}\Delta r_i+u_{12}\Delta s_i)(u_{21}\Delta r_i+u_{22}\Delta s_i)$ and use the same method as in the proof of Theorem 3. We also omit the notation Δ.

Since U is nonsingular, $\det(U)\neq 0$. Hence

$$
\begin{aligned}
&2(u_{11}r_i+u_{12}s_i)(u_{21}r_i+u_{22}s_i) \\
&\quad = \left(2u_{11}u_{21}r_i^2 + 2(u_{11}u_{22}+u_{12}u_{21})r_is_i + 2u_{12}u_{22}s_i^2\right) \\
&\quad = \sqrt{\det(U^tU)}\left(cr_i^2-2ar_is_i-bs_i^2\right)
\end{aligned}
$$

where

$$
\begin{aligned}
c &\triangleq \frac{2u_{11}u_{21}}{\sqrt{\det(U^tU)}}, \quad a \triangleq -\frac{u_{11}u_{22}+u_{12}u_{21}}{\sqrt{\det(U^tU)}} \\
b &\triangleq -\frac{2u_{12}u_{22}}{\sqrt{\det(U^tU)}} \qquad (135)
\end{aligned}
$$

and obviously, $a^2+bc=1$. Therefore, see the fourth equation at the top of the page. It is not hard to check that, under the constraint (133)

$$\det(U^tU) \leq \frac{1}{(c\varepsilon_1-b\varepsilon_2)^2+4\varepsilon_1\varepsilon_2}.$$

Therefore

$$D_{\min} = \min_{i=1,\ldots,2k}\ \min_{r_i+\mathbf{j}s_i\in\Delta\mathrm{RQAM}} \left\{\frac{(cr_i^2-2ar_is_i-bs_i^2)^2}{(c\varepsilon_1-b\varepsilon_2)^2+4\varepsilon_1\varepsilon_2}\right\}^n. \qquad (136)$$

The same as the proof of Theorem 3, we can prove that $D_{\min}$ is upper bounded by

$$D_{\min} \leq \left\{\frac{d^4}{1+\varepsilon_1\varepsilon_2}\right\}^n. \qquad (137)$$

Furthermore, for a fixed RQAM, define $\alpha = \arctan(\frac{1}{\sqrt{\varepsilon_1\varepsilon_2}})$, $\theta_1 = \arctan\left(\frac{\sqrt{5}-1}{2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$, $\theta_2 = \alpha - \theta_1$, and

$$U = \begin{pmatrix} \frac{\cos(\theta_1)}{\sqrt{2\varepsilon_1}} & \frac{\sin(\theta_1)}{\sqrt{2\varepsilon_2}} \\ -\frac{\sin(\theta_2)}{\sqrt{2\varepsilon_1}} & \frac{\cos(\theta_2)}{\sqrt{2\varepsilon_2}} \end{pmatrix}. \quad (138)$$

Then, this transformation achieves the upper bound (137). **Q.E.D.**

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Haiquan Wang (M'95) received the M.S. degree in Nankai University, China, in 1989, and the Ph.D. degree in Kyoto University, Japan, in 1997, both in mathematics, and the Ph.D. degree in University of Delaware, Newark, in 2005, in electrical Eengineering.

From 1997 to 1998, he was a Postdoctoral Researcher in the Department of Mathematics, Kyoto University, Japan. From 1998 to 2001, he was a Lecturer (part-time) in the Ritsumei University, Japan. From 2001 to 2002, he was a Visiting Scholar in the Department of Electrical and Computer Engineering, University of Delaware, Newark, Delaware. From 2005 to 2008, he was a Postdoctoral Researcher in the Department of Electrical and Computer Engineering, University of Waterloo, Canada. He has joined the College of Communications Engineering, Hangzhou Dianzi University, Hangzhou, China, since July 2008 as a faculty member. His current research interests are space–time code designs for MIMO systems and joint source and channel coding.

Dong Wang (M'99) received the B.S. and M.S. degrees from Zhejiang University, Hangzhou, China, in 1996 and 1999, respectively, and the Ph.D. degree from University of Delaware, Newark, in 2005, all in electrical engineering.

From April 1999 to September 2000, he was employed as a Senior System Engineer at Shanghai No.2 R&D Institute, ZTE Corporation. From October 2000 to June 2002, he was with Philips Research East Asia, Shanghai, China, as a Research Scientist. From May 2005 to December 2005, he worked at Mitsubishi Electrical Research Labs, Cambridge MA. Since 2006, he has been with Philips Research North America, Briarcliff Manor, NY, as a Senior Member Research Staff. His research interests are in the general areas of signal processing and wireless communications. He has authored or coauthored over 20 journal and referred conference papers and has 23 U.S./European patents granted or pending.

Dr. Wang received the Competitive Fellowship Award, in 2004, and the Signal Processing & Communication Faculty Award, in 2005, both from University of Delaware.

Xiang-Gen Xia (M'97–SM'00–F'09) received the B.S. degree in mathematics from Nanjing Normal University, Nanjing, China, and the M.S. degree in mathematics from Nankai University, Tianjin, China, and the Ph.D. degree in electrical engineering from the University of Southern California, Los Angeles, in 1983, 1986, and 1992, respectively.

He was a Senior/Research Staff Member at Hughes Research Laboratories, Malibu, CA, during 1995–1996. In September 1996, he joined the Department of Electrical and Computer Engineering, University of Delaware, Newark, Delaware, where he is the Charles Black Evans Professor. He was a Visiting Professor at the Chinese University of Hong Kong during 2002–2003, where he is an Adjunct Professor. Before 1995, he held visiting positions in a few institutions. His current research interests include space–time coding, MIMO and OFDM systems, and SAR and ISAR imaging. He has over 180 refereed journal articles published and accepted, and seven U.S. patents awarded and is the author of the book *Modulated Coding for Intersymbol Interference Channels* (New York, Marcel Dekker, 2000).

Dr. Xia received the National Science Foundation (NSF) Faculty Early Career Development (CAREER) Program Award in 1997, the Office of Naval Research (ONR) Young Investigator Award in 1998, and the Outstanding Overseas Young Investigator Award from the National Nature Science Foundation of China in 2001. He also received the Outstanding Junior Faculty Award of the Engineering School of the University of Delaware in 2001. He is currently an Associate Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, *Signal Processing (EURASIP)*, and the *Journal of Communications and Networks (JCN)*. He was a guest editor of Space-Time Coding and Its Applications in the *EURASIP Journal of Applied Signal Processing* in 2002. He served as an Associate Editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING during 1996 to 2003, the IEEE TRANSACTIONS ON MOBILE COMPUTING during 2001 to 2004, the IEEE SIGNAL PROCESSING LETTERS during 2003 to 2007, IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY during 2005 to 2008, and the *EURASIP Journal of Applied Signal Processing* during 2001 to 2004. He is also a Member of the Sensor Array and Multichannel (SAM) Technical Committee in the IEEE Signal Processing Society. He is the General Co-Chair of ICASSP 2005 in Philadelphia, PA.