On Analytic Signals with Nonnegative Instantaneous Frequency

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Abstract

In this paper, we characterize all analytic signals with band-limited amplitudes and polynomial phases. We show that a signal with band-limited amplitude and polynomial phase is analytic *if and only if* it has nonnegative constant instantaneous frequency, i.e., the derivative of the phase is a nonnegative constant, and the constant is greater than or equal to the minimum bandwidth of the amplitude.

1 Introduction

Both concepts of analytic signals and instantaneous frequencies play important roles in many areas including communication systems, physics, and joint time-frequency analysis in signal processing. They have been extensively studied, see for example [1-13]. A signal is analytic if its Fourier spectrum vanishes at negative frequencies. This implies that a nonzero analytic signal must be complex-valued. The instantaneous frequency (IF) of a complex-valued signal is commonly defined as the derivative of the phase of the signal. The IF is used as a generalization of the conventional frequency from the global sense to the local sense. With these two definitions, it is not unusual to expect that the IF of an analytic signal are nonnegative. Trivial examples of analytic signals with nonnegative IF are single tone signals $\exp(j\omega_0 t)$ for nonnegative constants ω_0. It is, however, not always true for an analytic signal to have nonnegative IF.

In this paper, we consider the class of signals with band-limited amplitudes and polynomial phases. This class is quite broad in our real applications, such as in communication systems. We characterize all analytic signals in the class as follows. Let $f(t)$ be a signal with polynomial phase and band-limited amplitude of minimal bandwidth B. It is shown that the signal $f(t)$ is analytic *if and only if* it has a constant nonnegative IF ω_0 with $\omega_0 \geq B$ for all t except possibly isolated points of t. Although this result looks simple, it turns out that the proof is not trivial.

2 Main Results

Before going to the main results, let us recall a characterization for a general analytic signal. In what follows, the letter z always stands for a complex value with its real part x and its imaginary part y, i.e., $z = x + jy$.

Let Ω be a region of the complex plane. A

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function $f(z)$ of the complex variable z is said *holomorphic*[1] in Ω if its first derivative $f'(z)$ exists for all the complex values $z \in \Omega$, see for example [14]. Let $f(t)$ be a signal (or function) defined for the real variable t. Its *complex extension* $f(z)$ is obtained from the signal $f(t)$ by replacing the real variable t with the complex variable z.

The function $f(z)$ with $z = x + jy$ belongs to the *Hardy class* H_2^+ if it is holomorphic in the half-plane $y > 0$ and satisfies the following inequality

$$M = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+jy)|^2 dx < \infty. \tag{2.1}$$

The following result gives a characterization of an analytic signal, see for example [15], pg. 72-74.

Proposition 1 *A finite energy signal $f(t)$ is analytic if and only if its complex extension $f(z)$ belongs to the Hardy class H_2^+.*

This Proposition is used in the proof of the following main result.

Theorem 1 *Let a finite energy signal*

$$f(t) = A(t)\exp(jP(t))$$

where $A(t)$ is real-valued and band-limited with its minimal bandwidth B, and $P(t)$ is a polynomial of t with real coefficients. Then, signal $f(t)$ is analytic if and only if

$$P(t) = \phi_0 + \omega_0 t, \tag{2.2}$$

where ϕ_0 and ω_0 are two real constants with $\omega_0 \geq B$.

Proof: The sufficient part is straightforward. We only need to prove the necessary part.

[1]It is also called *analytic* in the mathematics literature although the word "analytic" has a different meaning in the signal processing literature.

Since $A(t)$ is band-limited with minimal bandwidth B and has finite energy, we have

$$A(t) = \int_{-B}^{B} a(\omega) e^{jt\omega} d\omega, \tag{2.3}$$

where $a(\omega) \in L^2[-B, B]$. We first prove that, if $P(t) = bt^n$ for $n \geq 2$ and a real nonzero constant b, then $f(t)$ is not analytic. To do so, we want to apply Prop. 1, i.e., we want to prove that such an $f(t)$ is not in the Hardy class H_2^+. The complex extension of $f(t)$ is

$$f(x+jy) = A(x+jy)\exp[jb(x+jy)^n] =$$

$$A(x+jy)\exp\left[b\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} x^k y^{n-k} (j)^{n-k+1}\right].$$

We claim that $n < 4$ if $f(t)$ is analytic.

When $n \geq 4$, there is a term in the exponential:

$$bx^{n-3}y^3(j)^4 = bx^{n-3}y^3.$$

Since $A(t)$ has the form (2.3), $|A(x+jy)|^2$ has the order at least

$$\exp\left(-2B\sqrt{x^2+y^2}\right). \tag{2.4}$$

In the meantime, the exponential term has the order at least

$$\exp(bx^{n-3}y^3). \tag{2.5}$$

When $b > 0$, the order in (2.5), $bx^{n-3}y^3$, is greater than the one in (2.4), $2B\sqrt{x^2+y^2}$, when y is large enough, where x is the variable. Thus, the function $f(x+jy)$ of the variable x does not belong to $L^2(-\infty, \infty)$, i.e., $f(z) \notin H_2^+$, or $f(t)$ is not analytic.

When $b < 0$ and $n - 3$ is an odd number, we consider

$$\int_{-\infty}^{\infty} |f(x+jy)|^2 dx,$$

where the variable x runs from $-\infty$ to ∞. Therefore, when x runs negative values, the order in (2.5), $bx^{n-3}y^3$, is greater than the one in (2.4), $2B\sqrt{x^2+y^2}$, when y is large enough. Thus, $f(x+jy) \notin L^2(-\infty,\infty)$ and $f(t)$ is not analytic.

When $b<0$ and $n-3$ is an even number, we have $n \geq 5$. In this case, we consider the term of the exponential:

$$\exp(bx^{n-5}y^5(j)^6) = \exp(-bx^{n-5}y^5).$$

Since $b<0$, clearly either $f(x+jy) \notin L^2(-\infty,\infty)$ when $n>5$, or

$$M = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+jy)|^2 dx$$

$$= \sup_{y>0} \int_{-\infty}^{\infty} |A(x+jy)|^2 dx \exp(-by^5) = \infty,$$

when $n=5$. Thus, $f(t)$ is not analytic by Prop. 1.

Thus, we have proved the claim.

When $n=2$,

$$\exp(jb(x+jy)^2) = \exp(jbx^2 - 2bxy - jy^2).$$

In this case,

$$|f(x+jy)|^2 = |A(t)|^2 e^{-2bxy}.$$

No mater what the sign of b is, similar to before, $f(x+jy) \notin L^2(-\infty,\infty)$, or $f(t)$ is not analytic.

When $n=3$,

$$\exp(jb(x+jy)^3) = \exp(jbx^3 - 3bx^2y - 3jbxy^2 + by^3).$$

When $b>0$, similar to before,

$$M = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+jy)|^2 dx$$

$$= \sup_{y>0} \int_{-\infty}^{\infty} |A(x+jy)|^2 \exp(-3bx^2y) dx \exp(by^3)$$
$$= \infty.$$

Thus, $f(t)$ is not analytic by Prop. 1. When $b<0$, similar to before, $\exp(-3bx^2y)$ is unbounded in terms of x. Thus, $f(x+jy) \notin L^2(-\infty,\infty)$ and $f(t)$ is not analytic.

When the polynomial $P(t)$ has mixed terms of t^n, we only need to consider the highest order term. It is because in the expansions of $(x+jy)^n$ either the order x or the order of y resulted from the highest order term is higher than any one resulted from the lower order terms in $P(x+jy)$. Therefore, the highest order term dominates the signal growth order. Using the above result to the highest order term, we have proved that $P(t) = \phi_0 + \omega_0 t$. Since $A(t)$ has the minimal bandwidth B, for $f(t)$ to be analytic it is clear that $\omega_0 \geq B$. □

From Theorem 1, we immediately have the following corollary.

Corollary 1 *A finite energy signal with polynomial phase and band-limited amplitude of minimal bandwidth B is analytic if and only if it has a nonnegative constant instantaneous frequency with the constant greater than or equal to B at all t but possibly isolated points.*

Proof: By Theorem 1, we only need to prove that the IF of $f(t)$ is ω_0 at all t but possibly isolated points. Since the amplitude $A(t)$ is band-limited, it is an entire function [1]. Therefore, $A(t)$ has only zeros at possibly isolated points t_n for $n \in Z$ with $|t_n - t_m| > s$ for a positive constant s and $n \neq m$. In other words, $A(t)$ has the same sign in each time interval (t_n, t_{n+1}), which proves Corollary 1. □

Consider a signal with the following form

$$f(t) = \int_a^b F(\omega) e^{j\omega t} d\omega e^{j(\omega_0 t + \phi_0)}, \qquad (2.6)$$

where $0 \leq a \leq b < \infty$, ϕ_0 is a constant, $\omega_0 \geq 0$, and $F(\omega)$ is complex conjugate symmetrical around the center c:

$$F(c-\omega) = F^*(c+\omega), \quad \text{for } \omega \in [a,c],$$

where $c = (a+b)/2$. Clearly, $f(t)$ is analytic. Also,

$$f(t) = A(t)\exp(j(\bar{\omega}_0 t + \phi_0)),$$

where $\bar{\omega}_0 = \omega_0 + c$, and

$$A(t) = \int_{-d}^{d} \tilde{F}(\omega) \exp(j\omega t) d\omega,$$

where $d = (b-a)/2$, and $\tilde{F}(\omega) = F(\omega + c)$. One can see that $A(t)$ is real-valued and band-limited. Similar to the proof of Corollary 1, the IF of the above $f(t)$ is $\tilde{\omega}_0 = c + \omega_0 \geq 0$. Thus, the above family of signals are analytic and have nonnegative IF. Furthermore, by Corollary 1, this family is the only family of signals with band-limited amplitudes and polynomial phases that are analytic and have nonnegative instantaneous frequencies.

In the above we discussed the class of signals with band-limited amplitude and polynomial phase. One might want to ask what happens when the phase is not a polynomial. We have the following result.

Theorem 2 *Let $A(t)$ be a non-zero real band-limited signal, λ be a real constant, and*

$$f(t) = A(t) e^{j e^{\lambda t}}.$$

Then, $f(t)$ is not analytic.

The proof is similar to before by using Prop. 1.

3 Conclusions

In this paper, we have characterized all signals with band-limited amplitudes and polynomial phases that are analytic. It turns out that such signals are analytic if and only if they have nonnegative constant instantaneous frequencies where the constants are greater than or equal to the minimal bandwidths of the amplitudes.

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