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Upper Bounds of Rates of Complex Orthogonal Space–Time Block Codes

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***Abstract*—In this correspondence, we derive some upper bounds of the rates of (generalized) complex orthogonal space–time block codes. We first present some new properties of complex orthogonal designs and then show that the rates of complex orthogonal space–time block codes for more than two transmit antennas are upper-bounded by $3/4$. We show that the rates of generalized complex orthogonal space–time block codes for more than two transmit antennas are upper-bounded by $4/5$, where the norms of column vectors may not be necessarily the same. We also present another upper bound under a certain condition.**

For a (generalized) complex orthogonal design, its variables are not restricted to any alphabet sets but are on the whole complex plane. In this correspondence, a (generalized) complex orthogonal design with variables over some alphabet sets on the complex plane is also considered. We obtain a condition on the alphabet sets such that a (generalized) complex orthogonal design with variables over these alphabet sets is also a conventional (generalized) complex orthogonal design and, therefore, the above upper bounds on its rate also hold. We show that commonly used quadrature amplitude modulation (QAM) constellations of sizes above 4 satisfy this condition.

***Index Terms*—Complex orthogonal designs, complex orthogonal space–time block codes, Hermitian compositions of quadratic forms, Hurwitz family, Hurwitz–Radon theory.**

I. Introduction

The first real/complex orthogonal space–time block code was proposed by Alamouti [1] for two transmit antennas. It was then generalized to real/complex orthogonal space–time block codes for more than two transmit antennas by Tarokh, Jafarkhani, and Calderbank [3]. There are two important properties of real/complex orthogonal space–time block codes: 1) they have fast maximum-likelihood (ML) decoding, namely, symbol-by-symbol decoding; 2) they have the full diversity. These two properties make real/complex orthogonal space–time block codes attractive in space–time code designs. By utilizing the Hurwitz–Radon theory [17]–[19], [23], [26], Tarokh, Jafarkhani, and Calderbank [3] provided a systematic method to construct *real* orthogonal space–time block codes of size $p \times n$ and rate 1 for k pulse-amplitude modulation (PAM) symbols, where n is the number of transmit antennas, p is the time delay (or block size), and $R = k/p$ is the code rate. They also provided a construction of rate $1/2$ *complex* orthogonal space–time block codes for phase-shift keying (PSK) and quadrature amplitude modulation (QAM) symbols using real orthogonal space–time block codes of rate 1. In order to maintain the fast ML decoding and the full diversity of a space–time block code, the orthonormality in the sense that the norms of all column vectors are the same can be relaxed to a general orthogonality where the norms of column vectors may not be necessarily the same [3]. A complex orthogonal space–time block code with the generalized orthonormality is called a generalized complex orthogonal space–time block code. In [2], [3], it has been shown that the rate $R \le 1$ for both real and complex orthogonal space–time block codes for any number of transmit antennas. While the maximal rate 1, i.e., $R = 1$, is reachable for real orthogonal space–time block codes as we previously mentioned from the Hurwitz–Radon's constructive theory, it has been recently shown in [8] that $k \le p - 1$ when $n > 2$, i.e., $R < 1$ and $R = 1$ is not reachable for (generalized) complex orthogonal space–time block codes no matter what the time delay p is unless the number of transmit antennas is two, i.e., the Alamouti's scheme. Notice that, if condition $p = n$ is required, i.e., *square* codes or *square* complex orthogonal designs, then $R < 1$ when $n > 2$ directly follows from the results on amicable designs [18], [21]–[23], [3], [5]–[7] that have small rates when $n \ge 8$. While both square and nonsquare *real* orthogonal designs (or compositions of quadratic forms) are well understood, not much is known for nonsquare *complex* orthogonal designs (or Hermitian compositions of quadratic forms [26]), [3], [26], [27].

In this correspondence, we derive some upper bounds on the rates R of (generalized) complex orthogonal space–time block codes (or complex orthogonal designs). We emphasize that the sizes of (generalized) complex orthogonal space–time block codes (or complex orthogonal designs) here are general and they may not be square, i.e., p may not be equal to n. We show that, when the number of transmit antennas is more than two, i.e., $n > 2$, the rates of complex orthogonal space–time block codes are upper-bounded by $3/4$, i.e.,

$$R \le \frac{3}{4}$$

and the rates of generalized complex orthogonal space–time block codes are upper-bounded by $4/5$, i.e.,

$$R \le \frac{4}{5}.$$

Note that rate–$3/4$ complex orthogonal space–time block codes for three and four transmit antennas have appeared in [3]–[6]. Therefore, the above upper bound tells us that these complex orthogonal space–time block codes have already reached the optimal rate. Also note that the above upper bound $3/4$ on the rates is not new for *square* complex orthogonal designs. In fact, it has been shown and reviewed from amicable designs in [18], [21]–[23], [3], [5]–[7]. However, this upper bound is *new* for nonsquare complex orthogonal designs. In the meantime, it is known that to generate orthogonal space–time codes, a square orthogonal design is not necessary [3].

In a conventional (generalized) complex orthogonal design, its variables may take any values in the complex plane. However, as we shall see later, to generate a space–time code, the variables only take values in some finite subsets, called alphabet sets, on the complex plane. The question then becomes whether it is helpful to produce more (generalized) complex orthogonal designs of high rates when their variables are restricted to some alphabet sets. This question has been partially

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studied lately in [13], [14], [16], and [15]. For square real orthogonal designs, when their variables are restricted to finite or infinite subsets of the real line (or field), they are called restricted orthogonal designs in [13] and pseudo-orthogonal designs in [14]. It is shown in [13], [14] that there does not exist new square real orthogonal designs even when their variables are restricted to subsets of the real line, if the number of the elements of the alphabet set is greater than two. For square complex orthogonal designs, it is known that (also as mentioned previously) the maximal rate of 4×4 complex orthogonal designs is $3/4$ when all the variables can take any values on the complex plane. However, examples of rate–1 complex orthogonal designs of size 4×4 have been shown in [16] when their variables take some alphabet sets on the complex plane, where in the examples all the alphabet sets are PAM and a rotation of PAM, i.e., all points in an alphabet set are collinear. In this correspondence, we also consider this problem. We obtain a condition on the alphabet sets such that a (generalized) complex orthogonal design with variables over these alphabet sets is also a conventional (generalized) complex orthogonal design and, therefore, the above upper bounds on its rate also hold. We show that commonly used QAM signal constellations of size above 4 do satisfy this condition and, therefore, a (generalized) complex orthogonal design with their variables over QAM constellations of size above 4 is also a conventional (generalized) complex orthogonal design. For convenience, in what follows, we adopt the name "*restricted* (generalized) complex orthogonal design" as used in [13] for real orthogonal designs, when their variables are restricted to some alphabet sets.

This correspondence is organized as follows. In Section II, we provide some preparations and new properties on (generalized) complex orthogonal designs. In Section III, we prove several upper bounds. In Section IV, we study restricted (generalized) complex orthogonal designs.

II. Some Preliminaries and New Properties on Complex Orthogonal Designs

In this section, we present some properties of a (generalized) complex orthogonal design used in a (generalized) complex orthogonal space–time block code. In what follows, $\mathbb{C}$ denotes the field of all complex numbers and $\mathbb{R}$ denotes the field of all real numbers. For convenience, symbol 0 means scalar 0 or all 0 matrices of possibly different sizes and I means the identity matrices of possibly different sizes unless specified otherwise. For two matrices A and B of the same number of rows, $(A\, B)$ denotes the concatenation matrix of A and B, i.e., $(A\, B)$ is a new matrix with the columns of A as its first part columns and the columns of B as its second part columns.

A *complex orthogonal design* $\mathcal{G}(x_1, x_2, \ldots, x_k)$ of size $p \times n$ is a $p \times n$ matrix satisfying the following conditions:

- the entries of $\mathcal{G}(x_1, x_2, \ldots, x_k)$ are complex linear combinations of $x_1, x_2, \ldots, x_k$ and their complex conjugates $x_1^*, x_2^*, \ldots, x_k^*$;
- the orthonormality
$$(\mathcal{G}(x_1, x_2, \ldots, x_k))^H \mathcal{G}(x_1, x_2, \ldots, x_k) = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I$$
holds for *any* complex values x_i, $i = 1, 2, \ldots, k$, where ^{H} stands for the complex conjugate transpose and I is the $n \times n$ identity matrix.

The orthonormality in the preceding definition can be generalized to the orthogonality as follows for preserving the full-diversity and the fast ML decoding [3].

A *generalized complex orthogonal design*
$$\mathcal{G}(x_1, x_2, \ldots, x_k)$$
of size $p \times n$ is a $p \times n$ matrix satisfying the following conditions:

- the entries of $\mathcal{G}(x_1, x_2, \ldots, x_k)$ are complex linear combinations of $x_1, x_2, \ldots, x_k$ and their complex conjugates $x_1^*, x_2^*, \ldots, x_k^*$;
- the orthogonality
$$(\mathcal{G}(x_1, x_2, \ldots, x_k))^H \mathcal{G}(x_1, x_2, \ldots, x_k) = (|x_1|^2 D_1 + |x_2|^2 D_2 + \cdots + |x_k|^2 D_k)$$
holds for *any* complex values x_i, $i = 1, 2, \ldots, k$, where D_i, $i = 1, 2, \ldots, k$, are $n \times n$ diagonal positive definite constant matrices, i.e., their diagonal elements are all positive constants.

Let $\mathcal{A}$ denote a signal constellation alphabet set and
$$\mathcal{C} = \{\mathcal{G}(x_1, x_2, \ldots, x_k): x_i \in \mathcal{A}\}.$$
Then, $\mathcal{C}$ is called a complex (or generalized) orthogonal space–time block code. For this block code, every p time slots carries k information symbols, $x_1, x_2, \ldots, x_k$. The *rate* of this complex orthogonal space–time (or generalized complex orthogonal space–time) block code is defined as k/p and denoted by R, i.e., $R = k/p$. Without any confusion in understanding, in what follows we use complex orthogonal space–time (or generalized orthogonal complex space–time) block code $\mathcal{C}$ and (generalized) complex orthogonal design $\mathcal{G}(x_1, x_2, \ldots, x_k)$ interchangeably.

For a real orthogonal design, x_i are real valued in the above definition and the coefficients in the linear combinations of x_i of components of $\mathcal{G}(x_1, x_2, \ldots, x_k)$ are all real. It is known that there exist real orthogonal designs with $R = 1$ for any number n of transmit antennas, see [19], [23], [26], [3]. We refer the reader to [1], [3] for the properties of the fast ML decoding and the full diversity of a complex orthogonal space–time (or generalized complex orthogonal space–time) block code, where the full diversity means that any difference matrix of two different complex orthogonal space–time (or generalized complex orthogonal space–time) block codewords (or code matrices) has full rank. The main goal of this correspondence is to show that: 1) if $\mathcal{G} = \mathcal{G}(x_1, x_2, \ldots, x_k)$ of size $p \times n$ is a complex orthogonal design and $n \geq 3$, then its rate $R = k/p \leq 3/4$; 2) if $\mathcal{G} = \mathcal{G}(x_1, x_2, \ldots, x_k)$ of size $p \times n$ is a generalized complex orthogonal design and $n \geq 3$, then its rate $R = k/p \leq 4/5$. To do so, we need some preparations.

Let $\mathcal{G} = \mathcal{G}(x_1, x_2, \ldots, x_k)$ be a matrix of size $p \times n$, where its entries are complex linear combinations of $x_1, x_2, \ldots, x_k$ and their complex conjugates $x_1^*, x_2^*, \ldots, x_k^*$. Then, $\mathcal{G}$ can be expressed in terms of its column vectors as follows:
$$\mathcal{G} = (A_1\boldsymbol{x} + B_1\boldsymbol{x}^* \;\; A_2\boldsymbol{x} + B_2\boldsymbol{x}^* \;\; \cdots \;\; A_n\boldsymbol{x} + B_n\boldsymbol{x}^*) \tag{1}$$
where A_i, B_i, $i = 1, \ldots, n$, are $p \times k$ constant complex matrices, $\boldsymbol{x} = (x_1, \ldots, x_k)^t$, and ^{t} stands for the transpose while * stands for the complex conjugate.

For the $n \times n$ diagonal matrices D_i given in the preceding definition of a generalized complex orthogonal design, we denote
$$D_i = \mathrm{diag}(d_1^i, d_2^i, \ldots, d_n^i).$$
For each j, $j = 1, \ldots, n$, all the (j, j)-entries d_j^i of matrices D_i, $i = 1, \ldots, k$, form a new $k \times k$ diagonal matrix E_j as follows:
$$E_j \triangleq \mathrm{diag}(d_j^1, d_j^2, \ldots, d_j^k). \tag{2}$$
Clearly, when all D_i are positive definite, all E_j are positive definite. Using these matrices, we can transfer the orthogonal condition on $\mathcal{G}$ into the conditions on the matrices A_i, B_j, $1 \leq i, j \leq n$.

The following Lemma 1 is from [8].

Lemma 1 [8]: Let A, B, and C be three $m \times m$ complex constant matrices. If for any $\boldsymbol{x} \in \mathbb{C}^m$
$$\boldsymbol{x}^H A\boldsymbol{x} + \boldsymbol{x}^H B\boldsymbol{x}^* + \boldsymbol{x}^t C\boldsymbol{x} = 0$$

then

$$A = B + B^t = C + C^t = 0.$$

This lemma is used to prove the following proposition.

Proposition 1: Matrix $\mathcal{G}$ in (1) is a generalized complex orthogonal design, i.e.,

$$\mathcal{G}^H \mathcal{G} = |x_1|^2 D_1 + |x_2|^2 D_2 + \cdots + |x_k|^2 D_k$$

for some $n \times n$ diagonal positive definite constant matrices D_i, $1 \le i \le k$, if and only if there exist diagonal positive definite matrices E_i, $i = 1, 2, \ldots, n$, such that their associated matrices A_i and B_i, $i = 1, \ldots, n$, in (1) satisfy the following conditions:

$$\begin{cases} A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i \\ A_i^H B_j + B_j^t A_i^* = 0, \; B_i^H A_j + A_j^t B_i^* = 0 \end{cases} \tag{3}$$

or equivalently

$$\begin{pmatrix} A_i & B_i \\ B_j^* & A_j^* \end{pmatrix}^H \begin{pmatrix} A_j & B_j \\ B_i^* & A_i^* \end{pmatrix} = \delta_{ij} \begin{pmatrix} E_i & 0 \\ 0 & E_j \end{pmatrix} \tag{4}$$

for all $i, j = 1, \ldots, n$, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

In particular, $\mathcal{G}$ is a complex orthogonal design if and only if (3) or (4) holds for $E_i = I$ for $1 \le i \le n$.

Proof: By the orthogonality of a generalized complex orthogonal design in terms of its column vectors, we have

$$(A_i \boldsymbol{x} + B_i \boldsymbol{x}^*)^H (A_j \boldsymbol{x} + B_j \boldsymbol{x}^*) = \boldsymbol{x}^H \delta_{ij} E_i \boldsymbol{x}$$

i.e.,

$$\boldsymbol{x}^H A_i^H A_j \boldsymbol{x} + \boldsymbol{x}^H A_i^H B_j \boldsymbol{x}^* + \boldsymbol{x}^t B_i^H A_j \boldsymbol{x} + \boldsymbol{x}^t B_i^H B_j \boldsymbol{x}^* = \boldsymbol{x}^H \delta_{ij} E_i \boldsymbol{x}$$

where E_j are from D_i as in (2) and, therefore, they are positive definite. Note that

$$\boldsymbol{x}^t B_i^H B_j \boldsymbol{x}^* = (\boldsymbol{x}^t B_i^H B_j \boldsymbol{x}^*)^t = \boldsymbol{x}^H B_j^t B_i^* \boldsymbol{x}$$

the above equation can be rewritten as

$$\boldsymbol{x}^H (A_i^H A_j + B_j^t B_i^* - \delta_{ij} E_i) \boldsymbol{x} + \boldsymbol{x}^H A_i^H B_j \boldsymbol{x}^* + \boldsymbol{x}^t B_i^H A_j \boldsymbol{x} = 0, \quad \text{for any } \boldsymbol{x} \in \mathbb{C}^k.$$

By Lemma 1, we obtain

$$\begin{aligned} A_i^H A_j + B_j^t B_i^* &= \delta_{ij} E_i \\ A_i^H B_j + (A_i^H B_j)^t &= 0 \end{aligned}$$

and

$$B_i^H A_j + (B_i^H A_j)^t = 0.$$

The sufficiency part is easy to verify. QED

As a remark, equation $A_i^H B_j + (A_i^H B_j)^t = 0$ holds is equivalent to matrix $A_i^H B_j$ is skew symmetry,[1] which are used interchangeably in what follows.

We next investigate some properties of a generalized complex orthogonal design $\mathcal{G}$ under a unitary transformation. Let U be a unitary matrix and $\mathcal{G}(\boldsymbol{x})$ be a generalized complex orthogonal design, then $\mathcal{G}(U\boldsymbol{x})$ may not be a generalized complex orthogonal design due to the fact that $U^H E_i U$ may not be diagonal, i.e., a unitary transform on variables x_i does not preserve a generalized complex orthogonal design. On the other hand, if $\mathcal{G}(\boldsymbol{x})$ is a complex orthogonal design, then $\mathcal{G}(U\boldsymbol{x})$ is also a complex orthogonal design due to $E_i = I$ and $U^H E_i U = I$, i.e., a unitary transform on variables x_i preserves a complex orthogonal design.

[1]A matrix $\boldsymbol{S} = (\boldsymbol{s}_{ij})$ is called skew symmetric if $\boldsymbol{s}_{ij} = -\boldsymbol{s}_{ji}$. For a $\boldsymbol{k} \times \boldsymbol{k}$ skew-symmetric matrix $\boldsymbol{S} = (\boldsymbol{s}_{ij})$, we always have $\boldsymbol{x}^t \boldsymbol{S} \boldsymbol{x} = \boldsymbol{0}$ for any $\boldsymbol{k} \times \boldsymbol{1}$ vector $\boldsymbol{x} \in \mathbb{C}^k$.

In order to implement unitary transformations on variables of a generalized complex orthogonal design to simplify its corresponding matrices, we introduce the following concept of Hurwitz families, which is preserved by a unitary transformation as we can see later.

Definition 1: A set of $p \times 2k$ matrices

$$\{(A_1 \;\; B_1), (A_2 \;\; B_2), \ldots, (A_n \;\; B_n)\}$$

is called a Hurwitz family if there exist n positive definite matrices E_i, $i = 1, 2, \ldots, n$, such that

$$A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i, \qquad 1 \le i, j \le n \tag{5}$$

and

$$\begin{aligned} A_i^H B_j + B_j^t A_i^* &= 0 \\ B_i^H A_j + A_j^t B_i^* &= 0, \qquad 1 \le i \neq j \le n. \end{aligned} \tag{6}$$

In the preceding definition of a Hurwitz family, the diagonality of the matrices E_i is *not* required. Clearly, by Proposition 1, the matrices

$$\{(A_1 \;\; B_1), (A_2 \;\; B_2), \ldots, (A_n \;\; B_n)\}$$

of a generalized complex orthogonal design $\mathcal{G}(\boldsymbol{x})$ form a Hurwitz family, and

$$\{(A_1 U \;\; B_1 U^*), (A_2 U \;\; B_2 U^*), \ldots, (A_n U \;\; B_n U^*)\}$$

of $\mathcal{G}(U\boldsymbol{x})$ for a unitary transform U also form a Hurwitz family.

Note that in (6), we have the restriction $i \neq j$ due to the fact that it cannot be deduced for $i = j$ when E_i is not the identity matrix when a unitary transform is applied to a generalized complex orthogonal design as we shall see after the proof of Lemma 4. Thus, the condition for a Hurwitz family is weaker than the one for a generalized complex orthogonal design. Also note that the above definition coincides with the one in [23] when $B_i = 0$, A_i are real and $E_i = I$, i.e., the real case.

For a Hurwitz family

$$\{(A_1 \;\; B_1), (A_2 \;\; B_2), \ldots, (A_n \;\; B_n)\}$$

by using some proper unitary transformations, we can diagonalize the first matrix $(A_1 \;\; B_1)$ as follows, which plays a key role in the proof of our main theorem in next section.

Lemma 2: Let

$$\mathcal{G} = (A_1 \boldsymbol{x} + B_1 \boldsymbol{x}^* \;\; A_2 \boldsymbol{x} + B_2 \boldsymbol{x}^* \;\; \ldots \;\; A_n \boldsymbol{x} + B_n \boldsymbol{x}^*)$$

be a generalized complex orthogonal design. Then, $\mathcal{G}$ can be reduced to a new generalized complex orthogonal design $\tilde{\mathcal{G}}$ with the same parameters p, k, n as in $\mathcal{G}$ as follows:

$$\tilde{\mathcal{G}} = (\tilde{A}_1 \boldsymbol{y} + \tilde{B}_1 \boldsymbol{y}^* \;\; \tilde{A}_2 \boldsymbol{y} + \tilde{B}_2 \boldsymbol{y}^* \;\; \ldots \;\; \tilde{A}_n \boldsymbol{y} + \tilde{B}_n \boldsymbol{y}^*)$$

with $\tilde{A}_1^H \tilde{A}_1 + \tilde{B}_1^t \tilde{B}_1^* = I$, that is, $E_1 = I$ in (3) for $\tilde{A}_1$ and $\tilde{B}_1$, where $\boldsymbol{y} = (y_1 \;\; y_2 \cdots y_k)^t$.

Proof: By Proposition 1, E_1 is diagonal positive definite. Let

$$U = \sqrt{E_1^{-1}}$$

and then U is also diagonal positive definite. Make the transformation $\boldsymbol{x} = U\boldsymbol{y}$ and let $\tilde{A}_i = A_i U$, $\tilde{B}_i = B_i U$, and $\tilde{E}_i = U E_i U$, then $\tilde{A}_i$ and $\tilde{B}_i$ satisfy (3) and $\tilde{E}_i$ are all diagonal positive definite. Furthermore, $\tilde{A}_1^H \tilde{A}_1 + \tilde{B}_1^t \tilde{B}_1^* = U E_1 U = I$. QED

The following lemma can be also thought of as an independent result in linear algebra on special singular value decomposition (SVD) forms of special matrices.

Lemma 3: Let A and B be two $p \times k$ matrices and satisfy conditions $A^H A + B^t B^* = I$, and $A^H B$ and $B^H A$ are skew symmetric. Then, there exist a unitary matrix V of size $p \times p$ and a unitary matrix U of size $2k \times 2k$ such that the $p \times 2k$ matrix $(A\, B)$ can be diagonalized as follows:

$$V(A\, B)U = \Sigma \triangleq \begin{pmatrix} D_\lambda & 0 & 0 & 0 \\ 0 & I_{k-s} & 0 & 0 \\ 0 & 0 & D_\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{p\times 2k} \tag{7}$$

where $k - s \geq 2k - p$, $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s)$, $D_\mu = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_s)$ and $\lambda_i^2 + \mu_i^2 = 1, 1 > \lambda_i \geq \sqrt{1/2} \geq \mu_j > 0$, $i, j = 1, 2, \ldots, s$, $k + s = \kappa$, and $\kappa = \mathrm{rank}(A\, B) \geq k$, and, furthermore, the $2k \times 2k$ unitary matrix U has the following form:

$$U = \begin{pmatrix} W_1 & W_2 \\ W_2^* & W_1^* \end{pmatrix}, \tag{8}$$

where W_i, $i = 1, 2$, are $k \times k$ matrices.

The proof of Lemma 3 is included in the longer version of this correspondence [12] (it corresponds to [12, Lemma 6]).

Note that the speciality of the above SVD of matrix $(A\, B)$ comes from the special form of U in (8) that may not hold for an SVD of a general matrix.

As a consequence of Lemma 3, if the rank of $(A\, B)$ in Lemma 3 is k, then $s = 0$ in (7) and, therefore, all the diagonal elements are 1, i.e., all singular values of $(A\, B)$ are 1. Another remark is that, when $p = k$, i.e., A and B are square, then the above proof can be simplified as follows. When $p = k$, the matrix $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$ is square. Then, the condition in this lemma implies

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}^H = I_{2k}.$$

In this case, if we take $U = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}^H$ that certainly has the form of (8), then (7) is proved.

We next want to make a transformation to the variables of a generalized complex orthogonal design.

Lemma 4: Let

$$\mathcal{G} = (A_1\boldsymbol{x} + B_1\boldsymbol{x}^* \;\; A_2\boldsymbol{x} + B_2\boldsymbol{x}^* \;\; \cdots \;\; A_n\boldsymbol{x} + B_n\boldsymbol{x}^*)$$

be a generalized complex orthogonal design and matrix $\begin{pmatrix} W_1 & W_2 \\ W_2^* & W_1^* \end{pmatrix}$ and matrix V be unitary. Make the transformation $\boldsymbol{x} = W_1\boldsymbol{y} + W_2\boldsymbol{y}^*$ and let $\tilde{A}_i = VA_iW_1 + VB_iW_2^*$ and $\tilde{B}_i = VA_iW_2 + VB_iW_1^*$, then

$$\tilde{A}_i^H\tilde{A}_j + \tilde{B}_j^t\tilde{B}_i^* = \delta_{ij}\tilde{E}_i, \qquad 1 \leq i, j \leq n$$

and

$$\begin{aligned} \tilde{A}_i^H\tilde{B}_j + \tilde{B}_j^t\tilde{A}_i^* &= 0 \\ \tilde{B}_i^H\tilde{A}_j + \tilde{A}_j^t\tilde{B}_i^* &= 0, \qquad 1 \leq i \neq j \leq n \end{aligned} \tag{9}$$

where $\tilde{E}_i = W_1^HE_iW_1 + W_2^tE_iW_2^*$ are positive definite. In other words, $\{(\tilde{A}_1 \;\; \tilde{B}_1), \ldots, (\tilde{A}_n \;\; \tilde{B}_n)\}$ form a Hurwitz family. In particular, if $\mathcal{G}$ is a complex orthogonal design, then its transformation $\tilde{\mathcal{G}} = (\tilde{A}_1\boldsymbol{x} + \tilde{B}_1\boldsymbol{x}^* \;\; \tilde{A}_2\boldsymbol{x} + \tilde{B}_2\boldsymbol{x}^* \;\; \cdots \;\; \tilde{A}_n\boldsymbol{x} + \tilde{B}_n\boldsymbol{x}^*)$ is also a complex orthogonal design.

Proof: It is enough to notice that

$$\begin{pmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{B}_j^* & \tilde{A}_j^* \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} A_i & B_i \\ B_j^* & A_j^* \end{pmatrix} \begin{pmatrix} W_1 & W_2 \\ W_2^* & W_1^* \end{pmatrix}. \quad \text{QED}$$

As a remark, if $E_i \neq I$, then $\tilde{A}_i^H\tilde{B}_i + \tilde{B}_i^t\tilde{A}_i^*$ may not be 0, which is the reason why condition $i \neq j$ in (6) in Definition 1 for a Hurwitz family is required. On the other hand, by reviewing Proposition 1, condition $\tilde{A}_i^H\tilde{B}_i + \tilde{B}_i^t\tilde{A}_i^* = 0$ is crucial for a generalized complex orthogonal design as in (3).

Proposition 2: Let

$$\mathcal{G} = (A_1\boldsymbol{x} + B_1\boldsymbol{x}^* \;\; A_2\boldsymbol{x} + B_2\boldsymbol{x}^* \cdots A_n\boldsymbol{x} + B_n\boldsymbol{x}^*)$$

be a generalized complex orthogonal design. Then, there exists a Hurwitz family

$$\{(\tilde{A}_1 \;\; \tilde{B}_1), (\tilde{A}_2 \;\; \tilde{B}_2), \ldots, (\tilde{A}_n \;\; \tilde{B}_n)\} \tag{10}$$

with the same parameters p, n, k as $\mathcal{G}$ and

$$\begin{aligned} &\tilde{A}_1^H\tilde{A}_1 + \tilde{B}_1^t\tilde{B}_1^* = I \\ &\tilde{A}_1^H\tilde{B}_1 + \tilde{B}_1^t\tilde{A}_1^* = 0, \qquad \tilde{B}_1^H\tilde{A}_1 + \tilde{A}_1^t\tilde{B}_1^* = 0 \end{aligned} \tag{11}$$

and, furthermore, $\tilde{A}_1$ and $\tilde{B}_1$ have the following forms:

$$\tilde{A}_1 = \begin{pmatrix} D_\lambda & 0 \\ 0 & I_{k-s} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{B}_1 = \begin{pmatrix} 0_{s\times s} & 0 \\ 0_{(k-s)\times s} & 0 \\ D_\mu & 0 \\ 0 & 0 \end{pmatrix} \tag{12}$$

where $k - s \geq 2k - p$, $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s)$, $D_\mu = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_s)$ and $\lambda_i^2 + \mu_i^2 = 1, 1 > \lambda_i \geq \sqrt{1/2} \geq \mu_j > 0$, $i, j = 1, 2, \ldots, s$, $k + s = \kappa$, and $\kappa \triangleq \mathrm{rank}((A \;\; B)) \geq k$. In particular, if $\mathcal{G}$ is a complex orthogonal design, then there exists a complex orthogonal design $\tilde{\mathcal{G}}$ with the same parameters p, n, k as $\mathcal{G}$ such that its corresponding matrices $\tilde{A}_1$ and $\tilde{B}_1$ have the forms in (12).

Proof: Proposition 2 is a direct consequence of Lemmas 2–4. QED.

In the proof of the main theorem in the next section, we need the following rank inequalities.

1) (*Sylvester's Inequality*) Let A be a $k \times p$ matrix and B be a $p \times n$ matrix. Then

$$\mathrm{rank}(A) + \mathrm{rank}(B) - p \leq \mathrm{rank}(AB).$$

2) Let A be a $k \times p$ matrix and B be a $p \times n$ matrix. Then

$$\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

3) Let A, B be two $n \times m$ matrices and E be an $m \times m$ positive definite matrix. If $A^HA + B^HB = E$, then

$$\mathrm{rank}(A) + \mathrm{rank}(B) \geq m.$$

The above rank inequalities 1) and 2) are fundamental and can be found in linear algebra books, e.g., see [25]. Rank inequality 3) can be obtained from

$$\begin{aligned} \mathrm{rank}(A) + \mathrm{rank}(B) &= \mathrm{rank}(A^HA) + \mathrm{rank}(B^HB) \\ &\geq \mathrm{rank}(A^HA + B^HB) = m. \end{aligned}$$

III. Upper Bounds of Rates for Three or More Antennas

In this section, we present several upper bounds of the rates for both complex orthogonal designs and generalized complex orthogonal designs.

Theorem 1: Let $\mathcal{G} = \mathcal{G}(x_1, x_2, \ldots, x_k)$ be a generalized complex orthogonal design of size $p \times n$. If $n \geq 3$, then, its rate is upper-bounded by $4/5$, i.e.,

$$R = \frac{k}{p} \leq \frac{4}{5}. \tag{13}$$

If $\mathcal{G}$ is a complex orthogonal design and $n \geq 3$, then its rate is upper-bounded by $3/4$, i.e.,

$$R = \frac{k}{p} \leq \frac{3}{4}. \tag{14}$$

Proof: We first want to prove the first part of this theorem. Let

$$\mathcal{G} = (A_1\boldsymbol{x} + B_1\boldsymbol{x}^* \;\; A_2\boldsymbol{x} + B_2\boldsymbol{x}^* \;\; \cdots \;\; A_n\boldsymbol{x} + B_n\boldsymbol{x}^*).$$

By Proposition 2, we can assume that A_1 and B_1 have the following forms:

$$A_1 = \begin{pmatrix} D_\lambda & 0 \\ 0 & I_{k-s} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{s\times s} & 0 \\ 0_{(k-s)\times s} & 0 \\ D_\mu & 0 \\ 0 & 0 \end{pmatrix}$$

where $D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_s)$, $D_\mu = \mathrm{diag}(\mu_1, \ldots, \mu_s)$, I_{k-s} is the identity matrix of size $k-s$, and $1 > \lambda_i \geq \mu_j > 0$, $i, j = 1, \ldots, s$, and $\{(A_1 \;\; B_1), (A_2 \;\; B_2), \ldots, (A_n \;\; B_n)\}$ is a Hurwitz family with $A_1^H A_1 + B_1^t B_1^* = I$.

Divide $p \times k$ matrices A_i and B_i into block matrices as follows:

$$A_i = \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \\ A_{i5} & A_{i6} \\ A_{i7} & A_{i8} \end{pmatrix}, \quad B_i = \begin{pmatrix} B_{i1} & B_{i2} \\ B_{i3} & B_{i4} \\ B_{i5} & B_{i6} \\ B_{i7} & B_{i8} \end{pmatrix}$$

where A_{i1} and B_{i1} are $s \times s$ matrices, A_{i3} and B_{i3} are $(k-s) \times s$ matrices, A_{i5} and B_{i5} are $s \times s$ matrices, A_{i7} and B_{i7} are $(p-k-s) \times s$ matrices, A_{i2} and B_{i2} are $s \times (k-s)$ matrices, A_{i4} and B_{i4} are $(k-s) \times (k-s)$ matrices, A_{i6} and B_{i6} are $s \times (k-s)$ matrices, and A_{i8} and B_{i8} are $(p-k-s) \times (k-s)$ matrices.

Since $A_1^H A_i + B_i^t B_1^* = 0$ for $i > 1$, we have

$$\begin{pmatrix} D_\lambda & 0 & 0 & 0 \\ 0 & I_{k-s} & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \\ A_{i5} & A_{i6} \\ A_{i7} & A_{i8} \end{pmatrix} + \begin{pmatrix} B_{i1}^t & B_{i3}^t & B_{i5}^t & B_{i7}^t \\ B_{i2}^t & B_{i4}^t & B_{i6}^t & B_{i8}^t \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ D_\mu & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This matrix equation implies

$$A_{i2} = 0, \; A_{i4} = 0, \qquad i = 2, \ldots, n.$$

From the skew symmetry of $B_1^H A_i$, we obtain $A_{i6} = 0$ for $i = 2, \ldots, n$.

Define $\{(\hat{A}_2 \;\; \hat{B}_2), (\hat{A}_3 \;\; \hat{B}_3), \ldots, (\hat{A}_n \;\; \hat{B}_n)\}$ as follows:

$$\hat{A}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ A_{i8} \end{pmatrix}, \quad \hat{B}_i = \begin{pmatrix} B_{i2} \\ B_{i4} \\ B_{i6} \\ B_{i8} \end{pmatrix}$$

i.e., $\hat{A}_i$ and $\hat{B}_i$ are the second columns of the block matrices in A_i and B_i, respectively.

Since $A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i$ for $i, j \geq 2$, we have

$$\begin{pmatrix} A_{i1}^H & A_{i3}^H & A_{i5}^H & A_{i7}^H \\ 0 & 0 & 0 & A_{i8}^H \end{pmatrix} \begin{pmatrix} A_{j1} & 0 \\ A_{j3} & 0 \\ A_{j5} & 0 \\ A_{j7} & A_{j8} \end{pmatrix} + \begin{pmatrix} B_{j1}^t & B_{j3}^t & B_{j5}^t & B_{j7}^t \\ B_{j2}^t & B_{j4}^t & B_{j6}^t & B_{j8}^t \end{pmatrix} \begin{pmatrix} B_{i1}^* & B_{i2}^* \\ B_{i3}^* & B_{i4}^* \\ B_{i5}^* & B_{i6}^* \\ B_{i7}^* & B_{i8}^* \end{pmatrix} = \delta_{ij} E_i,$$

where E_i are positive definite. By noting the second row and the second column in the above products, we obtain

$$\hat{A}_i^H \hat{A}_j + \hat{B}_j^t \hat{B}_i^* = \delta_{ij} \hat{E}_i, \qquad i, j \geq 2 \tag{15}$$

where $\hat{E}_i$ are the $(k-s) \times (k-s)$ matrix taken from the last $k-s$ rows and the last $k-s$ columns of E_i, and therefore, $\hat{E}_i$ are also positive definite. By similarly, showing that other conditions $\{(\hat{A}_2 \;\; \hat{B}_2), \ldots, (\hat{A}_n \;\; \hat{B}_n)\}$ are also a Hurwitz family of size $(k-s) \times p$ matrices.

By the rank inequality 3) at the end of Section II, (15) implies

$$\mathrm{rank}(A_{i8}) + \mathrm{rank}(\hat{B}_i) \geq k - s, \qquad i = 2, \ldots, n. \tag{16}$$

Since $n \geq 3$, there exists a pair i and j with $i \neq j \geq 2$. When $i \neq j \geq 2$, we have $\hat{A}_i^H \hat{A}_j + \hat{B}_j^t \hat{B}_i^* = 0$, that is, $A_{i8}^H A_{j8} + \hat{B}_j^t \hat{B}_i^* = 0$, which implies

$$\begin{aligned} \mathrm{rank}(\hat{B}_i) + \mathrm{rank}(\hat{B}_j) - p &\leq \mathrm{rank}\,(\hat{B}_j^t \hat{B}_i^*) \\ &= \mathrm{rank}\,(A_{i8}^H A_{j8}) \leq p - k - s \end{aligned} \tag{17}$$

where the first inequality is due to Sylvester's inequality and the row size of $\hat{B}_i$ and $\hat{B}_j$ is p, and the last inequality is because A_{i8} and A_{j8} are all of size $(p-k-s) \times (k-s)$ and the rank inequality 2) at the end of Section II. Hence, from (17) and (16), rank $(A_{j8}) \leq p - k - s$ and rank $(A_{i8}) \leq p - k - s$, we have

$$\begin{aligned} p - k - s &\geq \mathrm{rank}(\hat{B}_i) + \mathrm{rank}(\hat{B}_j) - p \\ &\geq k - s - \mathrm{rank}(A_{i8}) + k - s - \mathrm{rank}(A_{j8}) - p \\ &\geq k - s - (p - k - s) + k - s - (p - k - s) - p \\ &= 4k - 3p \end{aligned}$$

which implies

$$4p - 5k \geq s \geq 0.$$

Therefore, the first half of the theorem is proved.

We next want to show the second half of the theorem and assume that $\mathcal{G}$ is a complex orthogonal design. All the above derivations still hold for $\mathcal{G}$ and are adopted in the following proof. Since $\mathcal{G}$ is a complex orthogonal design, by Proposition 2

$$\{(A_2 \;\; B_2), (A_3 \;\; B_3), \ldots, (A_n \;\; B_n)\}$$

satisfies (3) Proposition 1 with $E_i = I$. Therefore, it is not hard to see that

$$\{(\hat{A}_2 \;\; \hat{B}_2), (\hat{A}_3 \;\; \hat{B}_3), \ldots, (\hat{A}_n \;\; \hat{B}_n)\}$$

also satisfies (3) in Proposition 1 with $\hat{E}_i = I$.

Consider matrix A_{28} in $\hat{A}_2$, which has size $(p-k-s) \times (k-s)$. There exist a $(p-k-s) \times (p-k-s)$ unitary matrix U and a $(k-s) \times (k-s)$ unitary matrix R such that

$$U A_{28} R = \begin{pmatrix} 0 & 0 \\ D_\beta & 0 \end{pmatrix}$$

where $D_\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_r)$, r is the rank of the matrix A_{28}, and $\beta_i > 0$, $i = 1, 2, \ldots, r$, are the positive square roots of the eigenvalues of $A_{28} A_{28}^H$. Clearly

$$r \leq p - k - s.$$

Using these unitary matrices U and R, we rewrite the matrix pairs $\{(\hat{A}_2 \;\; \hat{B}_2), \ldots, (\hat{A}_n \;\; \hat{B}_n)\}$ as follows.

Let

$$P = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$$

where I is the identity matrix of size $k+s$. Then P is a $p \times p$ unitary matrix and

$$\{(P\hat{A}_2 R \;\; P\hat{B}_2 R^*), \ldots, (P\hat{A}_n R \;\; P\hat{B}_n R^*)\}$$

also satisfies (3) in Proposition 1 with $E_i = I$. Furthermore, $\{P\hat{A}_2R,\ P\hat{B}_2R^*,\ P\hat{A}_3R,\ P\hat{B}_3R^*\}$ can be written as follows:

$$\left\{\begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ D_\beta & 0\end{pmatrix},\ \begin{pmatrix} \tilde{B}_{22} & \overline{B}_{22}\\ \tilde{B}_{24} & \overline{B}_{24}\\ \tilde{B}_{26} & \overline{B}_{26}\\ \tilde{B}_{281} & \overline{B}_{282}\\ \tilde{B}_{283} & \overline{B}_{284}\end{pmatrix},\ \begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ \tilde{A}_{381} & \overline{A}_{382}\\ \tilde{A}_{383} & \overline{A}_{384}\end{pmatrix},\ \begin{pmatrix} \tilde{B}_{32} & \overline{B}_{32}\\ \tilde{B}_{34} & \overline{B}_{34}\\ \tilde{B}_{36} & \overline{B}_{36}\\ \tilde{B}_{381} & \overline{B}_{382}\\ \tilde{B}_{383} & \overline{B}_{384}\end{pmatrix}\right\} \tag{19}$$

where the sizes of $\tilde{B}_{i2},\ \tilde{B}_{i4},\ \tilde{B}_{i6}$ are $s\times r,\ (k-s)\times r,\ s\times r$, respectively, the sizes of $\overline{B}_{i2},\ \overline{B}_{i4},\ \overline{B}_{i6}$ are $s\times(k-s-r),\ (k-s)\times(k-s-r),\ s\times(k-s-r)$, respectively, the sizes of $\tilde{B}_{281},\ \tilde{A}_{381},\ \tilde{B}_{381}$ are $(p-k-s-r)\times r$, the sizes of $\tilde{B}_{283},\ \tilde{A}_{383},\ \tilde{B}_{383}$ are $r\times r$, the sizes of $\overline{B}_{282},\ \overline{A}_{382},\ \overline{B}_{382}$ are $(p-k-s-r)\times(k-s-r)$, and the sizes of $\overline{B}_{284},\ \overline{A}_{384},\ \overline{B}_{384}$ are $r\times(k-s-r)$.

We next want to show that $\overline{B}_{284}=0$ and $\overline{B}_{384}=0$. From (3) in Proposition 1, the matrix $(P\hat{A}_2R)^H(P\hat{B}_2R^*)$ is skew symmetry, i.e.,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & D_\beta\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \tilde{B}_{22} & \overline{B}_{22}\\ \tilde{B}_{24} & \overline{B}_{24}\\ \tilde{B}_{26} & \overline{B}_{26}\\ \tilde{B}_{281} & \overline{B}_{282}\\ \tilde{B}_{283} & \overline{B}_{284}\end{pmatrix}$$

is skew symmetry, which implies $D_\beta\overline{B}_{284}=0$, therefore, $\overline{B}_{284}=0$ because D_β is invertible. Similarly, the matrix $(P\hat{A}_2R)^H(P\hat{B}_3R^*)$ is also skew symmetry, which implies $\overline{B}_{384}=0$.

Again by (3) in Proposition 1, we have

$$(P\hat{A}_2R)^H(P\hat{A}_3R)+(P\hat{B}_3R^*)^t(P\hat{B}_2R^*)^*=0$$

i.e.,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & D_\beta\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ \tilde{A}_{381} & \overline{A}_{382}\\ \tilde{A}_{383} & \overline{A}_{384}\end{pmatrix} + \begin{pmatrix} \tilde{B}^t_{32} & \tilde{B}^t_{34} & \tilde{B}^t_{36} & \tilde{B}^t_{381} & \tilde{B}^t_{383}\\ \overline{B}^{\,t}_{32} & \overline{B}^{\,t}_{34} & \overline{B}^{\,t}_{36} & \overline{B}^{\,t}_{382} & 0\end{pmatrix}\begin{pmatrix} \tilde{B}^*_{22} & \overline{B}^*_{22}\\ \tilde{B}^*_{24} & \overline{B}^*_{24}\\ \tilde{B}^*_{26} & \overline{B}^*_{26}\\ \tilde{B}^*_{281} & \overline{B}^*_{282}\\ \tilde{B}^*_{283} & 0\end{pmatrix}=0.$$

From the second row and the second column, the above equation implies

$$\overline{B}^{\,t}_3\overline{B}^{\,*}_2=0 \tag{20}$$

where $\overline{B}_i=(\overline{B}^{\,t}_{i2}\ \ \overline{B}^{\,t}_{i4}\ \ \overline{B}^{\,t}_{i6}\ \ \overline{B}^{\,t}_{i82})^t$ for $i=2,3$. By Sylvester's inequality, and noting that the size of matrices $\overline{B}_2$ and $\overline{B}_3$ is $(p-r)\times(k-s-r)$, (20) implies

$$\mathrm{rank}(\overline{B}_2)+\mathrm{rank}(\overline{B}_3)\le p-r. \tag{21}$$

We next want to determine the ranks of $\overline{B}_2$ and $\overline{B}_3$.

Because

$$(P\hat{A}_2R)^H(P\hat{A}_2R)+(P\hat{B}_2R^*)^t(P\hat{B}_2R^*)^*=I$$

we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & D_\beta\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ D_\beta & 0\end{pmatrix} + \begin{pmatrix} \tilde{B}^t_{22} & \tilde{B}^t_{24} & \tilde{B}^t_{26} & \tilde{B}^t_{281} & \tilde{B}^t_{283}\\ \overline{B}^{t}_{22} & \overline{B}^{t}_{24} & \overline{B}^{t}_{26} & \overline{B}^{t}_{282} & 0\end{pmatrix}\begin{pmatrix} \tilde{B}^*_{22} & \overline{B}^*_{22}\\ \tilde{B}^*_{24} & \overline{B}^*_{24}\\ \tilde{B}^*_{26} & \overline{B}^*_{26}\\ \tilde{B}^*_{281} & \overline{B}^*_{282}\\ \tilde{B}^*_{283} & 0\end{pmatrix}=I.$$

Therefore, by noting the second row and the second column, we have $\overline{B}^{\,t}_2\overline{B}^{\,*}_2=I$, hence,

$$\mathrm{rank}(\overline{B}_2)=k-s-r. \tag{22}$$

For the rank of $\overline{B}_3$, we first use the fact that

$$(P\hat{A}_3R)^H(P\hat{A}_3R)+(P\hat{B}_3R^*)^t(P\hat{B}_3R^*)^*=I$$

and we then use the forms of $P\hat{A}_3R$ and $P\hat{B}_3R^*$ in (19) and expand the summation. We then conclude that $\overline{A}^{\,H}_{382}\overline{A}_{382}+\overline{A}^{\,H}_{384}\overline{A}_{384}+\overline{B}^{\,t}_3\overline{B}^{\,*}_3=I$. Finally, from the rank inequality 3) at the end of Section II, we have

$$\mathrm{rank}(\overline{B}_3)\ge k-s-r-r_1 \tag{23}$$

where r_1 is the rank of matrix $(\overline{A}^{\,t}_{382}\ \ \overline{A}^{\,t}_{384})^t$ that has $p-k-s$ rows from (19). Thus, we also have

$$r_1\le p-k-s. \tag{24}$$

Combining (18) and (21)–(24), we have

$$2k-p\le 2s+r+r_1\le 2s+(p-k-s)+(p-k-s)=2p-2k$$

i.e.,

$$\frac{k}{p}\le\frac{3}{4}.$$

This proves Theorem 1. QED

From the above proof, one can see that the difference between the above upper bounds that we obtained on the rates of complex orthogonal designs and the generalized orthogonal designs depends on whether the property

$$\hat{A}^H_i\hat{B}_i+\hat{B}^t_i\hat{A}^*_i=0 \tag{25}$$

holds. It holds for the complex orthogonal designs due to the orthonormality but may not hold for generalized complex orthogonal designs.

As another application of Proposition 2, we have another upper bound for the rates of a complex orthogonal space–time block code, which sharpens the result in Theorem 1 if an additional condition is satisfied. Let $\mathcal{G}=(A_1\boldsymbol{x}+B_1\boldsymbol{x}^*\ \ \ldots\ \ A_n\boldsymbol{x}+B_n\boldsymbol{x}^*)$ be a generalized complex orthogonal design. Define

$$\rho\triangleq\max_{i=1,2,\ldots,n}\mathrm{rank}((A_i\ \ B_i)). \tag{26}$$

Theorem 2: Let $\mathcal{G}=(A_1\boldsymbol{x}+B_1\boldsymbol{x}^*\ A_2\boldsymbol{x}+B_2\boldsymbol{x}^*\cdots A_n\boldsymbol{x}+B_n\boldsymbol{x}^*)$ be a generalized complex orthogonal design. If $\rho=p$ and $n\ge 2$, then the rate of $\mathcal{G}$ is upper-bounded by $n/(2n-2)$, i.e.,

$$R=\frac{k}{p}\le\frac{n}{2n-2}.$$

Proof: If $2k \leq p$, then, the theorem is proved. So, in what follows, we assume $2k > p$. Without loss of generality, we assume rank $((A_1 \;\; B_1)) = p$. By Proposition 2, we may assume A_1 and B_1 have the following forms:

$$A_1 = \begin{pmatrix} D_\lambda & 0 \\ 0 & I_{k-s} \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0_{s\times s} & 0 \\ 0_{(k-s)\times s} & 0 \\ D_\mu & 0 \end{pmatrix}$$

where $s = p - k$, $D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_s)$, $D_\mu = \mathrm{diag}(\mu_1, \ldots, \mu_s)$, I_{k-s} is the identity matrix of size $k - s$, and $1 > \lambda_i > \mu_j > 0$, $i, j = 1, \ldots, s$, and $\{(A_1\; B_1), (A_2\; B_2), \ldots, (A_n\; B_n)\}$ is a Hurwitz family.

Divide matrices A_i and B_i into block matrices as follows:

$$A_i = \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \\ A_{i5} & A_{i6} \end{pmatrix}, \quad B_i = \begin{pmatrix} B_{i1} & B_{i2} \\ B_{i3} & B_{i4} \\ B_{i5} & B_{i6} \end{pmatrix}$$

where the sizes of A_{i1} and B_{i1} are $s \times s$, the sizes of A_{i3} and B_{i3} are $(k-s) \times s$, and the sizes of the remaining submatrices can be determined accordingly. By the properties of a Hurwitz family in Definition 1, we have $A_1^H A_i + B_i^t B_1^* = 0$ for $i > 1$, that is,

$$\begin{pmatrix} D_\lambda & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \\ A_{i5} & A_{i6} \end{pmatrix} + \begin{pmatrix} B_{i1}^t & B_{i3}^t & B_{i5}^t \\ B_{i2}^t & B_{i4}^t & B_{i6}^t \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ D_\mu & 0 \end{pmatrix} = 0$$

which implies $A_{i2} = A_{i4} = 0$ for $i > 1$. Similarly, $A_{i6} = 0$ can been obtained from the skew symmetry of $A_i^H B_1$.

Define

$$\hat{A}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{B}_i = \begin{pmatrix} B_{i2} \\ B_{i4} \\ B_{i6} \end{pmatrix}.$$

Then, by the same method as that used in the proof of Theorem 1, $\{(\hat{A}_2 \;\; \hat{B}_2), \ldots, (\hat{A}_n \;\; \hat{B}_n)\}$ is also a Hurwitz family.

By applying the properties of a Hurwitz family, it is not hard to verify that the $(n-1)(k-s)$ column vectors of the following matrix:

$$\begin{pmatrix} B_{22} & B_{32} & \cdots & B_{n2} \\ B_{24} & B_{34} & \cdots & B_{n4} \\ B_{26} & B_{36} & \cdots & B_{n6} \end{pmatrix}$$

are linearly independent in $\mathbb{C}^p$. Therefore, by noticing $s = p - k$, we have

$$(n-1)(k-s) = (n-1)(k-(p-k)) \leq p$$

which establishes the theorem. QED

It is not hard to check that the rate–$3/4$ complex orthogonal space–time block codes in [3], [4]–[6] for three and four transmit antennas do not satisfy the condition in Theorem 2 and, therefore, the upper bound in Theorem 2 does not apply to them.

IV. Restricted (Generalized) Complex Orthogonal Designs

In the previous sections, we consider the conventional (generalized) complex orthogonal designs in the sense that all the variables in the designs may take any values on the complex plane. In this section, we consider restricted (generalized) complex orthogonal designs where the variables only take values from subsets of the complex plane. To do so, we first introduce some necessary notations and concepts.

Let $\mathcal{A}$ be a subset (finite or infinite) of the complex plane, which is called the alphabet set. The *difference* set of $\mathcal{A}$, denoted $\Delta\mathcal{A}$, is defined by

$$\Delta\mathcal{A} \triangleq \{z_1 - z_2 \mid \text{for any } z_1, z_2 \in \mathcal{A}\}. \tag{27}$$

Note that for any alphabet set $\mathcal{A}$, we have $0 \in \Delta\mathcal{A}$.

An alphabet set $\mathcal{A}$ is called *admissible* if it contains at least three distinct points such that they are *not* collinear, i.e., they do not lie on a straight line on the complex plane, or more precisely, there exist $z_j = p_j + \boldsymbol{j} q_j \in \mathcal{A}$, $j = 1, 2, 3$, such that

$$\det\begin{pmatrix} p_1 - p_2 & q_1 - q_2 \\ p_1 - p_3 & q_1 - q_3 \end{pmatrix} \neq 0 \tag{28}$$

where $\boldsymbol{j} = \sqrt{-1}$. We next want to see the admissibility condition (28) on the difference set $\Delta\mathcal{A}$. It is clear that condition (28) is equivalent to any one of the following:

$$\det\begin{pmatrix} p_2 - p_1 & q_2 - q_1 \\ p_2 - p_3 & q_2 - q_3 \end{pmatrix} \neq 0 \tag{29}$$

$$\det\begin{pmatrix} p_3 - p_1 & q_3 - q_1 \\ p_3 - p_2 & q_3 - q_3 \end{pmatrix} \neq 0. \tag{30}$$

Let $x_1 = z_1 - z_2$, $x_2 = z_2 - z_3$, $x_3 = z_3 - z_1$. Then $x_j \in \Delta\mathcal{A}$, furthermore, condition (28) can be rewritten as

$$\det\begin{pmatrix} \mathrm{Re}(x_1) & \mathrm{Im}(x_1) \\ -\mathrm{Re}(x_3) & -\mathrm{Im}(x_3) \end{pmatrix} \neq 0 \tag{31}$$

where $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are the real and image parts of x, respectively. By some simple calculations, we may find that condition (31) or (28) is equivalent to

$$\det\begin{pmatrix} x_1 & x_1^* \\ x_3 & x_3^* \end{pmatrix} \neq 0. \tag{32}$$

Similarly, (29) and (30) are equivalent to, respectively,

$$\det\begin{pmatrix} x_1 & x_1^* \\ x_2 & x_2^* \end{pmatrix} \neq 0, \quad \det\begin{pmatrix} x_3 & x_3^* \\ x_2 & x_2^* \end{pmatrix} \neq 0. \tag{33}$$

In summary, an alphabet set $\mathcal{A}$ is admissible if and only if there exist at least three points $\{x_1, x_2, x_3\}$ in $\Delta\mathcal{A}$ such that condition (32) or any one of the two in (33) holds. Note that a constellation set M-PSK $(M > 2)$ or M-QAM $(M > 2)$ is admissible.

We next give the definition of a *restricted* generalized complex orthogonal design. Let

$$\mathcal{G} = (A_1 \boldsymbol{z} + B_1 \boldsymbol{z}^* \;\; \cdots \;\; A_n \boldsymbol{z} + B_n \boldsymbol{z}^*)$$

be a $p \times n$ matrix, where $\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t \in \mathbb{C}^k$, and A_i, B_i, $i = 1, \ldots, n$, are $p \times k$ complex constant matrices.

Definition 2: Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be k complex point sets. $\{\mathcal{G}; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$ is called a restricted orthogonal space–time design if for any

$$\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t \in (\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k)^t$$

the following orthogonality holds:

$$\begin{aligned} \mathcal{G}^H \mathcal{G} &= (A_1 \boldsymbol{z} + B_1 \boldsymbol{z}^* \cdots A_n \boldsymbol{z} + B_n \boldsymbol{z}^*)^H \\ &\quad \cdot (A_1 \boldsymbol{z} + B_1 \boldsymbol{z}^* \cdots A_n \boldsymbol{z} + B_n \boldsymbol{z}^*) \\ &= |\boldsymbol{z}_1|^2 D_1 + |\boldsymbol{z}_2|^2 D_2 + \cdots + |\boldsymbol{z}_k|^2 D_k \end{aligned} \tag{34}$$

where D_i, $i = 1, 2, \ldots, k$, are some $n \times n$ diagonal positive definite constant matrices.

For a restricted generalized complex orthogonal design, we have the following theorem.

Theorem 3: Let $\{\mathcal{G}; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\}$ be a restricted generalized complex orthogonal design. If for each i with $1 \le i \le k$, the following conditions hold for alphabet set $\mathcal{A}_i$:

i) $\mathcal{A}_i$ is admissible, i.e., $\mathcal{A}_i$ does not contain only collinear points;

ii) there exists $z = p + q\boldsymbol{j} \in \mathcal{A}_i$ with $pq \neq 0$ such that $z \neq z^*$ and $z^* \in \mathcal{A}_i$;

iii) there exist $z_j = p_j + q_j\boldsymbol{j} \in \mathcal{A}_i$, $j = 1, 2, 3$, such that

$$\det \begin{pmatrix} p_1^2 - p_2^2 & q_1^2 - q_2^2 \\ p_1^2 - p_3^2 & q_1^2 - q_3^2 \end{pmatrix} \neq 0 \tag{35}$$

then $\mathcal{G}$ is also a generalized complex orthogonal design.

Proof: By Proposition 1, it is enough to prove that under the conditions of this theorem, the following matrix equations hold:

$$A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i, \qquad i, j = 1, 2, \ldots, n \tag{36}$$
$$A_i^H B_j + B_j^t A_i^* = 0, \qquad i, j = 1, 2, \ldots, n \tag{37}$$

where E_i are positive definite diagonal matrices, which can be proved by using Lemma 5 stated below similar to the proof of Proposition 1 by using Lemma 1. We omit the details. QED

Lemma 5: Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be k complex alphabet sets. Let A, B, and C be three $k \times k$ complex matrices such that, for any $\boldsymbol{z} = (z_1, z_2, \ldots, z_k)^t \in (\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k)^t$, the following holds:

$$\boldsymbol{z}^H A\boldsymbol{z} + \boldsymbol{z}^H B\boldsymbol{z}^* + \boldsymbol{z}^t C\boldsymbol{z} = 0. \tag{38}$$

i) If for any $1 \le i \le k$, $\mathcal{A}_i$ satisfies condition i) in Theorem 3, i.e., none of the alphabet sets contains only collinear points, then matrices A, $B + B^t$, and $C + C^t$ are all diagonal.

ii) If for any $1 \le i \le k$, $\mathcal{A}_i$ satisfies conditions i)–iii) in Theorem 3, then

$$A = B + B^t = C + C^t = 0. \tag{39}$$

iii) If for any $1 \le i \le k$, $\mathcal{A}_i$ satisfies condition i) in Theorem 3 with three nonzero points and $0 \in \mathcal{A}_i$, then, (39) holds.

Lemma 5 sharpens Lemma 1. Its proof is included in the longer version of this paper [12] (it corresponds to [12, Lemma 8]).

The result in Theorem 3 can be thought of a generalization of the results in [13], [14] in the sense that the size of an orthogonal design is not necessarily square, i.e., p does not have to be equal to n, and it is in the complex field instead of the real field, and the orthonormality is generalized to the orthogonality. One can see that PSK constellations do not satisfy condition iii) of Theorem 3. However, it is not difficult to check that the commonly used QAM signal constellations of sizes above 4 located on a square lattice satisfy conditions i)–iii) in the above theorem. Therefore, we have the following corollary.

Corollary 1: A restricted generalized complex orthogonal design with its variables restricted to QAM constellations of sizes above 4 on square lattices is also a generalized complex orthogonal design and, therefore, the upper bounds on its rate in Section III hold.

The fact that PSK constellations do not satisfy condition iii) in Theorem 3 shows that the admissibility (28) does not imply condition iii) in Theorem 3. On the other hand, by considering points on a straight line (it is neither the x-axis nor the y-axis), condition iii) in Theorem 3 may hold. This shows that condition iii) in Theorem 3 does not imply the admissibility (28) in general.

V. Conclusion

In this correspondence, we have shown that the rates of complex orthogonal space–time block codes for three or more transmit antennas are upper-bounded by $3/4$ and the rates of generalized complex orthogonal space–time codes for three or more transmit antennas are upper-bounded by $4/5$. We have presented another sharper upper bound for the rates under a certain condition. Notice that the maximal rate of real orthogonal space–time codes is 1 for any number of transmit antennas, which is achievable using the Hurwitz–Radon constructive proof. For complex orthogonal space–time block codes or generalized complex orthogonal space–time block codes, the maximal rate 1 is reached only for two transmit antennas. For generalized complex orthogonal space–time block codes, rate $7/11$ and $3/5$ generalized complex orthogonal designs for $n = 5$ and $n = 6$ have been constructed in [9], which are $9/55$ and $1/5$ away from the upper bound $4/5$ we derived in this correspondence for generalized complex orthogonal space–time block codes, respectively. For complex orthogonal space–time codes, rate–$2/3$ complex orthogonal design for $n = 5$ has been constructed in [11], which is $1/12$ away from the upper bound $3/4$. For a general n, we conjecture that the upper bound $3/4$ of the rate of complex orthogonal designs can be sharpened as

$$R \le \frac{\lceil \frac{n}{2} \rceil + 1}{2\lceil \frac{n}{2} \rceil}$$

which can be achieved for $n = 1, 2, 3, 4, 5$.

Note that the upper bound of the rates $R \le 3/4$ when $n > 2$ was proved in [10] for a special family of complex orthogonal space–time block codes from the complex orthogonal designs $\mathcal{G}$, where the entries of $\mathcal{G}$ do not consist of any linear processing of $\boldsymbol{x}_i$ and $\boldsymbol{x}_i^*$, $i = 1, 2, \ldots, k$, and can only be 0 or single variables $\pm\boldsymbol{x}_i$ or $\pm\boldsymbol{x}_i^*$, $i = 1, 2, \ldots, k$, and these variables do not repeat in any column of $\mathcal{G}$. The method used in [10] was based more on a combinatorial argument that is different from what was used in this work.

In the last part of this correspondence, we have considered the restricted generalized complex orthogonal designs by restricting the variables to subsets of the complex plane. We have obtained a condition on the alphabet sets such that a restricted generalized complex orthogonal design is a generalized complex orthogonal design. The commonly used QAM constellations of size above 4 on square lattices do satisfy the condition. Thus, the upper bounds on the rates presented in this correspondence also apply to restricted generalized complex orthogonal designs for commonly used QAM signal constellations of sizes above 4. This result can be thought of as a generalization of the results in [13], [14] from square real orthogonal designs to (not necessary square) generalized complex orthogonal designs.

Due to the lengthy proofs of some of the main lemmas in this correspondence, Lemmas 3 and 5, these proofs have been omitted in the text but can be found online through our website http://www.ee.udel.edu/∼xxia/Pub.html. We would like to mention here that Lemma 3 presents a new SVD factorization of a special structure for a particular family of matrices, which does not exist in the mathematics literature.

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