Some 2×2 Unitary Space–Time Codes From Sphere Packing Theory With Optimal Diversity Product of Code Size 6

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***Abstract*—In this correspondence, we propose some new designs of 2×2 unitary space–time codes of sizes 6, 32, 48, 64 with best-known diversity products (or product distances) by partially using sphere packing theory. In particular, we present an optimal 2×2 unitary space–time code of size 6 in the sense that it reaches the maximal possible diversity product for 2×2 unitary space–time codes of size 6. The construction and the optimality of the code of size 6 provide the precise value of the maximal diversity product of a 2×2 unitary space–time code of size 6.**

***Index Terms*—Differential space–time modulation, optimal diversity product, packing theory, unitary space–time codes.**

I. Introduction

Unitary space–time codes have been recently proposed in [6], [5] for differential space–time modulation schemes and in [1]–[4] for possibly other space–time modulation schemes. Unitary space–time codes in differential encoding are useful not only when the channel information is not known at the receiver and noncoherent decoding is used but also when the channel information is known at the receiver and coherent decoding as a recursive trellis coding is used jointly with an error correction coding as a turbo type coding [19] where a super performance is achieved. There have been several unitary space–time code constructions in the literature: for example, group and optimal group constructions [6], [7], [5], [9]; orthogonal designs [8]; parametric codes [11]; Cayley transforms [10]; Lie groups [13], [16]; and Hamiltonian constellations or spherical codes using packing theory [9], [16]. It is known that the performance of a space–time code depends on its diversity product and having a good diversity product has become an important criterion in the design of a space–time code. In [11], some upper bounds on the diversity products of (unitary) space–time codes for a given size are presented. It is easy to reach the diversity product upper bound for 2×2 matrices of sizes below 4 and 2×2 unitary matrices of sizes 4 and 5 reaching the upper bound are also presented in [11] using the parametric forms of unitary matrices. In fact, 2×2 unitary matrices of sizes below 6 reaching the upper bound can be also constructed by using the Hamiltonian constellations from the packing theory, i.e., the optimal sphere packing points. However, in [11] it is shown that the upper bound is not reachable when the 2×2 unitary code size is above 5 and a tight upper bound on the diversity products remains open. The optimal or best-known sphere packing points of sizes above 5 do not provide optimal 2×2 unitary space–time codes with optimal diversity products.

Manuscript received March 26, 2003; revised June 15, 2004. This work was supported in part by the Air Force Office of Scientific Research under Grant F49620-02-1-0157 and the National Science Foundation under Grants CCR-0097240 and CCR-0325180. The material in this correspondence was presented in part at the IEEE International Symposium on Information Theory, Yokohama, Japan, June /July 2002.

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Communicated by G. Caire, Associate Editor for Communications.
Digital Object Identifier 10.1109/TIT.2004.838105

In this correspondence, we propose some 2×2 unitary space–time codes by partially using the optimal sphere packing points [20], [22]. We obtain a determinant relationship for difference matrices between Hamiltonian and general 2×2 unitary constellations. We present some best-known designs for size $L = 6, 32, 48, 64$, and also show that the code with size 6 reaches the optimal diversity product.

This correspondence is organized as follows. In Section II, we present new best-known diversity product designs for size $L = 6, 32, 48, 64$. In Section III, we show the optimality of the new code of size 6 presented in Section II. Since the proof is highly technical, we leave the most technical parts of the proofs in [23], which is downloadable via our website.

II. Some 2×2 Unitary Codes With Best Known Diversity Products

In this section, we present some new 2×2 unitary codes for sizes $L = 6, 32, 48, 64$ with best-known diversity products.

A. Diversity Product

Let $\mathcal{G} = \{V_1, V_2, \ldots, V_L\}$ be a 2×2 unitary space–time code of size L with ${V_l}^H V_l = I$ where ^{H} stands for the transpose and complex conjugate. Define

$$\xi(\mathcal{G}) \triangleq \min_{V_l, V_{l'} \in \mathcal{G}, l \neq l'} |\det(V_l - V_{l'})| \tag{1}$$

and

$$d_L \triangleq \max_{\mathcal{G}} \xi(\mathcal{G}) = \max_{\mathcal{G}} \min_{V_l, V_{l'} \in \mathcal{G}, l \neq l'} |\det(V_l - V_{l'})|. \tag{2}$$

Following the convention in the literature, the diversity product for a 2×2 code $\mathcal{G}$ is defined as follows:

$$\eta(\mathcal{G}) \triangleq \frac{1}{2}\sqrt{\xi(\mathcal{G})} \tag{3}$$

and the optimal diversity product for L-point constellation is defined as

$$\eta(L) \triangleq \max_{\mathcal{G}} \eta(\mathcal{G}) = \frac{1}{2}\sqrt{d_L}. \tag{4}$$

We are interested in designing a code $\mathcal{G}$ with large or optimal diversity product.

B. 2×2 *Unitary Matrices*

The content presented here can be found in many literatures, for example, [21], [16]. For the notational convenience for our later study, we briefly introduce some concepts on 2×2 unitary matrices below. Let $\boldsymbol{U}(2)$ be the set of all 2×2 unitary matrices, i.e.

$$\boldsymbol{U}(2) \triangleq \{A \mid A \text{ is a } 2 \times 2 \text{ matrix with } A^H A = I\}.$$

Between $\boldsymbol{U}(2)$ and the unit ball $\boldsymbol{S}^3 \subseteq \mathbb{R}^4$, there exists a close relationship as follows.

For any 2×2 matrix A with $A^H A = I$, we have $|\det(A)| = 1$ and thus there is a unique angle $\theta \in [0, 2\pi)$ such that $\det(A) = e^{j\theta}$. For any fixed angle $\theta \in [0, 2\pi)$, let

$$\boldsymbol{SU}(2, \theta) \triangleq \{A \in \boldsymbol{U}(2) \mid \det(A) = e^{j\theta}\}. \tag{5}$$

Thus, we have

$$\boldsymbol{U}(2) = \bigcup_{\theta \in [0, 2\pi)} \boldsymbol{SU}(2, \theta).$$

We are particularly interested in the case of $\theta = 0$ and denote the set $\boldsymbol{SU}(2,0)$ by $\boldsymbol{SU}(2)$ for short, i.e., $\boldsymbol{SU}(2) = \boldsymbol{SU}(2,0)$. We now investigate the structure of $\boldsymbol{SU}(2)$. From some theory of unitary matrices (for example, see [21]), $\boldsymbol{SU}(2)$ can be isometrically embedded onto the four-dimensional (4-D) Euclidean real unit sphere. Let $\boldsymbol{S}^3$ be the unit sphere of 4-D real Euclidean space $\mathbb{R}^4$, i.e.

$$\boldsymbol{S}^3 = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \|\boldsymbol{x}\| = 1\}$$

where $\|\cdot\|$ denotes the conventional l^2 norm. Because $\det(A) = 1$ and the unitariness for any element A in $\boldsymbol{SU}(2)$, it is not hard to see that there are two complex numbers $a = a_1 + ja_2$ and $b = b_1 + jb_2$ such that

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{6}$$

where * denotes the conjugate, and a_1, a_2, b_1, b_2 are real numbers governed by the condition $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$, i.e., $|a|^2 + |b|^2 = 1$, [21]. From this expression, the following embedding from $\boldsymbol{SU}(2)$ onto $\boldsymbol{S}^3$ can be obtained, also see for example [16]. Let i be the mapping $i : A \mapsto i(A)$ from $\boldsymbol{SU}(2)$ into $\boldsymbol{S}^3$ defined by

$$i(A) \triangleq (a_1, a_2, b_1, b_2) = (\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)) \tag{7}$$

where a_1, a_2, b_1, b_2 are the real numbers defined in (6) and Re and Im stand for the real and imaginary parts of a complex number, respectively. Clearly, the mapping i is one-to-one and onto. Furthermore, the following relationship holds:

$$\det(A - B) = \|i(A) - i(B)\|^2. \tag{8}$$

This equation also implies that all determinants of difference matrices of two distinct 2×2 unitary matrices in $\boldsymbol{SU}(2)$ are positive. From (8), one can see that the problem to find an optimal 2×2 space–time code in $\boldsymbol{SU}(2)$, i.e., it is restricted to have determinant 1, becomes to find optimal packing points on the sphere $\boldsymbol{S}^3$, which is called *Hamiltonian constellations* in [9]. Thus, as indicated in [9], if we denote D_L as the maximal minimum-distance of L-point packing on $\boldsymbol{S}^3$, then

$$d_L \geq D_L^2$$

i.e., the square of the maximal minimum-distance of L-point packing on $\boldsymbol{S}^3$ is a lower bound for d_L. However, as we shall see later, the above Hamiltonian constellation may not be enough to have good codes and we need to consider the entire 2×2 unitary matrix space $\boldsymbol{U}(2)$. To do so, we need a determinant formula.

C. A Useful Determinant Formula

Let us consider a relationship between $\boldsymbol{SU}(2)$ and $\boldsymbol{U}(2)$ or equivalently between $\boldsymbol{SU}(2)$ and $\boldsymbol{SU}(2,\theta)$ for any $\theta \in [0, 2\pi)$.

For a fixed θ, we define a mapping J_θ from $\boldsymbol{SU}(2,\theta)$ to $\boldsymbol{SU}(2)$ as follows:

$$J_\theta(A) \triangleq e^{-j\theta/2} A, \qquad \text{for } A \in \boldsymbol{SU}(2,\theta). \tag{9}$$

Since $\det(J_\theta(A)) = e^{-j\theta} \det(A) = e^{-j\theta} e^{j\theta} = 1$, this mapping is well defined. Furthermore, it is not hard to see that it is one-to-one and onto. With this notation, one can see that any 2×2 unitary matrix A can be represented by

$$A = e^{j\theta/2} J_\theta(A), \qquad \text{for some } \theta \in [0, 2\pi).$$

An important property from this mapping is that it also provides a determinant formula for a difference matrix of two matrices selected from different sets $\boldsymbol{SU}(2,\theta_1)$ and $\boldsymbol{SU}(2,\theta_2)$, which is stated in the following proposition.

Proposition 1: For any $A_0 \in \boldsymbol{SU}(2)$ and $A \in \boldsymbol{SU}(2,\theta)$, we have

$$|\det(A - A_0)| = |\det(A_0 - J_\theta(A)) - 4\sin^2(\theta/4)|.$$

Proof: Assume

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ -b_0^* & a_0^* \end{pmatrix} \quad \text{and} \quad J_\theta(A) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where $|a_0|^2 + |b_0|^2 = 1$ and $|a|^2 + |b|^2 = 1$. Then

$$\begin{aligned} \det(A_0 - J_\theta(A)) &= \det\left(\begin{pmatrix} a_0 & b_0 \\ -b_0^* & a_0^* \end{pmatrix} - \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\right) \\ &= 2 - (a_0 a^* + b_0 b^* + a_0^* a + b_0^* b). \end{aligned}$$

By the definition of J_θ in (5), we have $A = e^{j\theta/2} J_\theta(A)$. Therefore,

$$\begin{aligned} &|\det(A_0 - A)| \\ &= \left|\det\left(\begin{pmatrix} a_0 & b_0 \\ -b_0^* & a_0^* \end{pmatrix} - \begin{pmatrix} e^{j\theta/2}a & e^{j\theta/2}b \\ -e^{j\theta/2}b^* & e^{j\theta/2}a^* \end{pmatrix}\right)\right| \\ &= |1 + e^{j\theta} - e^{j\theta/2}(a_0 a^* + b_0 b^* + a_0^* a + b_0^* b)| \\ &= |1 + e^{j\theta} - e^{j\theta/2}(2 - \det(A_0 - J_\theta(A)))| \\ &= |e^{-j\theta/2} + e^{j\theta/2} - (2 - \det(A_0 - J_\theta(A)))| \end{aligned}$$

which is the same as the one in the proposition. QED

From this proposition, we immediately have the following corollary.

Corollary 1: For any $A_1 \in \boldsymbol{SU}(2,\theta_1)$ and $A_2 \in \boldsymbol{SU}(2,\theta_2)$, we have

$$|\det(A_1 - A_2)| = |\det(J_{\theta_1}(A_1) - J_{\theta_2}(A_2)) - 4\sin^2((\theta_1 - \theta_2)/4)|.$$

From the above proposition and corollary, one can see that the determinant absolute value of the difference matrix of two 2×2 unitary matrices depends on the distance between their embeddings and their angle difference. This motivates us to design a 2×2 unitary space–time code using two steps: one is to select good packing points on the sphere $\boldsymbol{S}^3$ and the other is to select good angles θ.

D. Some New Codes With Best Known Diversity Products

With the help of the above determinant formulas, we can construct some 2×2 unitary codes with best-known diversity products.

1) Size $L = 6$: Let $d = -5/2 + \sqrt{22}$. Select a four-point packing on $\boldsymbol{S}^3$ as follows:

$$\begin{aligned} \boldsymbol{a}_1 &= (-a, -b, b, -b), \ \boldsymbol{a}_2 = (-a, b, b, b) \\ \boldsymbol{a}_3 &= (-a, -b, -b, b), \ \boldsymbol{a}_4 = (-a, b, -b, -b) \end{aligned}$$

where $a = \sqrt{1 - 3d/8}$ and $b = \sqrt{(1 - a^2)/3}$. By mapping these points back to $\boldsymbol{SU}(2)$, we have the following four unitary matrices:

$$\begin{aligned} A_1 &= \begin{pmatrix} -a - bj & b - bj \\ -b - bj & -a + bj \end{pmatrix} \\ A_2 &= \begin{pmatrix} -a + bj & b + bj \\ -b + bj & -a - bj \end{pmatrix} \\ A_3 &= \begin{pmatrix} -a - bj & -b + bj \\ b + bj & -a + bj \end{pmatrix} \\ A_4 &= \begin{pmatrix} -a + bj & -b - bj \\ b - bj & -a - bj \end{pmatrix}. \end{aligned}$$

For other two unitary matrices, we use angle θ. Let

$$\theta_1 = 2\arccos(d/2 - a) \quad \text{and} \quad \theta_2 = 2\pi - \theta_1$$

and

$$A_5 = e^{j\theta_1/2} I \in \boldsymbol{SU}(2,\theta_1), \ A_6 = -e^{j\theta_2/2} I \in \boldsymbol{SU}(2,\theta_2).$$

It is easy to check that the diversity product of the code $\{A_1, A_2, \ldots, A_6\}$ is

$$\frac{1}{2}\sqrt{-5/2+\sqrt{22}} \approx 0.7400.$$

In next section, we shall prove that

$$\eta(6) = \frac{1}{2}\sqrt{-5/2+\sqrt{22}}$$

i.e., this code reaches the optimal diversity product of any 2×2 unitary space–time codes of size $L = 6$.

2) Sizes $L = 32, 48, 64$: To construct $32, 48$, or 64 unitary matrices with large diversity products, at first, we first construct four diamonds in $\boldsymbol{S}^3$ as follows.

Let t is a parameter, and

$$a = \sqrt{1-\frac{3}{8}t^2}, \quad r = \sqrt{1-a^2}$$
$$b = -\frac{\sqrt{6}}{12}t, \quad r_1 = \frac{\sqrt{3}}{3}t, \quad \beta = \frac{2\pi}{3}.$$

The four-point coordinates of the first diamond are

$$\boldsymbol{a}_1 = (a, r, 0, 0), \quad \boldsymbol{a}_2 = (a, b, r_1, 0)$$
$$\boldsymbol{a}_3 = (a, b, r_1\cos(\beta), r_1\sin(\beta)),$$
$$\boldsymbol{a}_4 = (a, b, r_1\cos(2\beta), r_1\sin(2\beta)).$$

The ones of the second diamond are

$$\boldsymbol{a}_5 = (a, -r, 0, 0), \quad \boldsymbol{a}_6 = (a, -b, -r_1, 0)$$
$$\boldsymbol{a}_7 = (a, -b, -r_1\cos(\beta), -r_1\sin(\beta))$$
$$\boldsymbol{a}_8 = (a, -b, -r_1\cos(2\beta), -r_1\sin(2\beta)).$$

The ones of the third diamond are

$$\boldsymbol{a}_9 = (-a, r, 0, 0), \quad \boldsymbol{a}_{10} = (-a, b, r_1, 0)$$
$$\boldsymbol{a}_{11} = (-a, b, r_1\cos(\beta), r_1\sin(\beta)),$$
$$\boldsymbol{a}_{12} = (-a, b, r_1\cos(2\beta), r_1\sin(2\beta)).$$

The ones of the fourth diamond are

$$\boldsymbol{a}_{13} = (-a, -r, 0, 0), \quad \boldsymbol{a}_{14} = (-a, -b, -r_1, 0)$$
$$\boldsymbol{a}_{15} = (-a, -b, -r_1\cos(\beta), -r_1\sin(\beta))$$
$$\boldsymbol{a}_{16} = (-a, -b, -r_1\cos(2\beta), -r_1\sin(2\beta)).$$

Mapping these points back to $\boldsymbol{SU}(2)$ using the map i^{-1} given in (7), we obtain 16 matrices, denoted by Q_j, i.e., $Q_j \triangleq i^{-1}(\boldsymbol{a}_j)$ for $j = 1, 2, \ldots, 16$. These matrices can be used to generate best-known diversity product unitary codes with $L = 32, 48$, and 64 as follows.

For $L = 32$, let $t = \sqrt{2}$, $\gamma = \arccos(3/4)$ and define

$$U_i = Q_i, \quad i = 1,2,3,4$$
$$U_i = Q_{8+i}, \quad i = 5,6,7,8$$
$$U_i = e^{\boldsymbol{j}(\pi/4+\gamma/2)}Q_{i-8}, \quad i = 9,10,11,12$$
$$U_i = e^{\boldsymbol{j}(\pi/4+\gamma/2)}Q_{i}, \quad i = 13,14,15,16$$
$$U_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-12}, \quad i = 17,18,19,20$$
$$U_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-12}, \quad i = 21,22,23,24$$
$$U_i = e^{\boldsymbol{j}(3\pi/4+\gamma/2)}Q_{i-20}, \quad i = 25,26,27,28$$
$$U_i = e^{\boldsymbol{j}(3\pi/4+\gamma/2)}Q_{i-20}, \quad i = 29,30,31,32$$

and put $\mathcal{G}_{32} = \{U_1, \ldots, U_{32}\}$, then the minimum determinant $\xi(\mathcal{G}_{32}) = \frac{\sqrt{7}-1}{2}$, and the diversity product $\eta(\mathcal{G}_{32})$ is

$$\frac{1}{2}\sqrt{\frac{\sqrt{7}-1}{2}} \cong 0.4536$$

which is best known for size $L = 32$.

For $L = 48$, let $t = \sqrt{2}$ and

$$V_i = Q_i, \quad i = 1,2,3,4;$$
$$V_i = Q_{8+i}, \quad i = 5,6,7,8$$
$$V_i = e^{\boldsymbol{j}(\pi/6)}Q_{i-4}, \quad i = 9,10,11,12$$
$$V_i = e^{\boldsymbol{j}(\pi/6)}Q_{i-4}, \quad i = 13,14,15,16$$
$$V_i = e^{\boldsymbol{j}(\pi/3)}Q_{i-16}, \quad i = 17,18,19,20$$
$$V_i = e^{\boldsymbol{j}(\pi/6)}Q_{i-8}, \quad i = 21,22,23,24$$
$$V_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-20}, \quad i = 25,26,27,28$$
$$V_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-20}, \quad i = 29,30,31,32$$
$$V_i = e^{\boldsymbol{j}(2\pi/3)}Q_{i-32}, \quad i = 33,34,35,36$$
$$V_i = e^{\boldsymbol{j}(2\pi/3)}Q_{i-24}, \quad i = 37,38,39,40$$
$$V_i = e^{\boldsymbol{j}(5\pi/6)}Q_{i-36}, \quad i = 41,42,43,44$$
$$V_i = e^{\boldsymbol{j}(5\pi/6)}Q_{i-36}, \quad i = 45,46,47,48$$

and define $\mathcal{G}_{48} = \{V_1, \ldots, V_{48}\}$, then the minimum determinant $\xi(\mathcal{G}_{48}) = \sqrt{3}-1$, and the diversity product $\eta(\mathcal{G}_{48})$ is

$$\frac{1}{2}\sqrt{\sqrt{3}-1} \cong 0.4278$$

which is best known for size $L = 48$.

For $L = 64$, let $t = \sqrt{1.3880}$, and define

$$W_i = Q_i, \quad i = 1,2,3,4$$
$$W_i = Q_{8+i}, \quad i = 5,6,7,8$$
$$W_i = e^{\boldsymbol{j}(\pi/8)}Q_{i-4}, \quad i = 9,10,11,12$$
$$W_i = e^{\boldsymbol{j}(\pi/8)}Q_{i-4}, \quad i = 13,14,15,16$$
$$W_i = e^{\boldsymbol{j}(\pi/4)}Q_{i-16}, \quad i = 17,18,19,20$$
$$W_i = e^{\boldsymbol{j}(\pi/4)}Q_{i-8}, \quad i = 21,22,23,24$$
$$W_i = e^{\boldsymbol{j}(3\pi/8)}Q_{i-20}, \quad i = 25,26,27,28$$
$$W_i = e^{\boldsymbol{j}(3\pi/8)}Q_{i-20}, \quad i = 29,30,31,32$$
$$W_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-32}, \quad i = 33,34,35,36$$
$$W_i = e^{\boldsymbol{j}(\pi/2)}Q_{i-24}, \quad i = 37,38,39,40$$
$$W_i = e^{\boldsymbol{j}(5\pi/8)}Q_{i-36}, \quad i = 41,42,43,44$$
$$W_i = e^{\boldsymbol{j}(5\pi/8)}Q_{i-36}, \quad i = 45,46,47,48$$
$$W_i = e^{\boldsymbol{j}(3\pi/4)}Q_{i-48}, \quad i = 49,50,51,52$$
$$W_i = e^{\boldsymbol{j}(3\pi/4)}Q_{i-40}, \quad i = 53,54,55,56;$$
$$W_i = e^{\boldsymbol{j}(7\pi/8)}Q_{i-52}, \quad i = 57,58,59,60$$
$$W_i = e^{\boldsymbol{j}(7\pi/8)}Q_{i-52}, \quad i = 61,62,63,64$$

and define $\mathcal{G}_{64} = \{W_1, \ldots, W_{64}\}$, then the minimum determinant $\xi(\mathcal{G}_{64}) = 0.5406$, and the diversity product $\eta(\mathcal{G}_{64})$ is

$$\frac{1}{2}\sqrt{0.5406} \cong 0.3676$$

which is best known for size $L = 64$.

TABLE I
DIVERSITY PRODUCT AND SUM COMPARISONS

	Hamiltonian Codes [9]		Parametric Codes [11]		New Codes	
Size	Diversity product	Diversity sum	Diversity product	Diversity sum	Diversity product	Diversity sum
6	0.7071	0.7071	0.7071	0.7746 (opt.)	0.7400 (opt.)	0.7400
32	0.4496	0.4496	0.4461	0.5621	0.4536	0.5217
48	0.3938	0.3938	0.3875		0.4278	0.5000
64	0.3609	0.3609	0.3535	0.4852	0.3676	0.3827

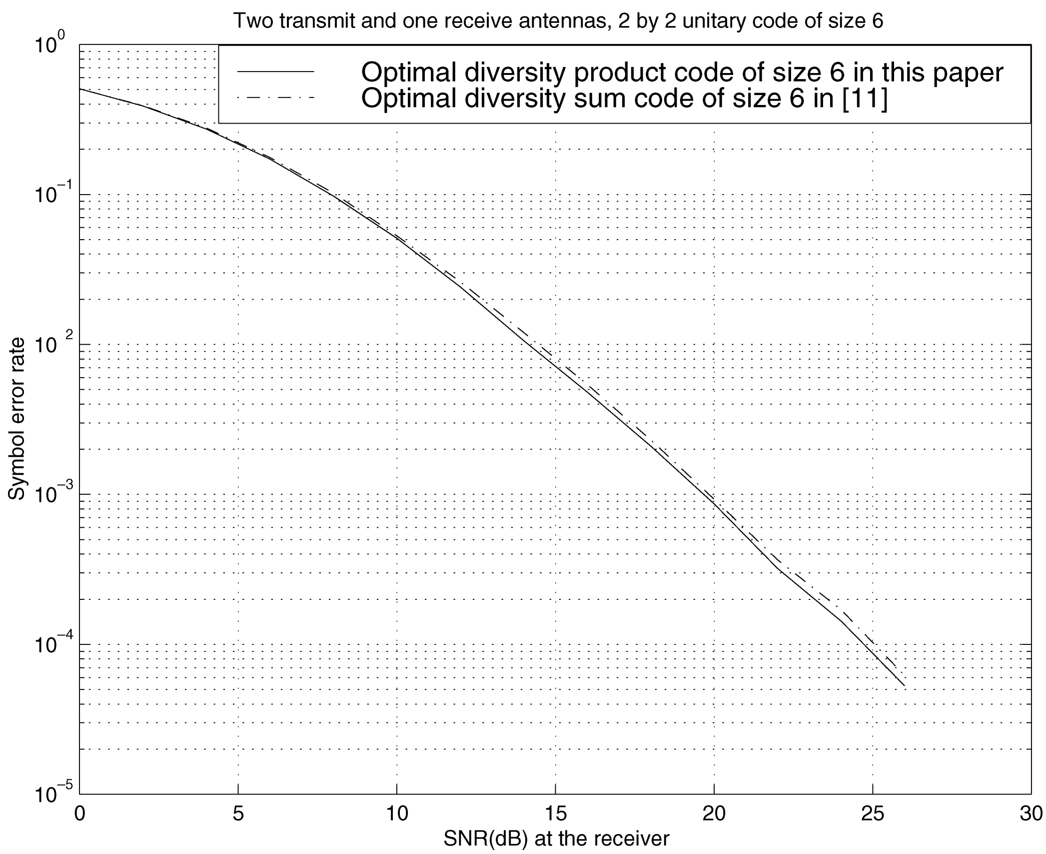

Fig. 1. SER comparison.

Table I summarizes the above results and compares with some existing codes, where diversity sum means the minimum Euclidean distance between codeword matrices [11]. From Table I, one can see that the optimal diversity sum 0.7746 of the 2×2 unitary code of size 6 presented in [11] is slightly better than the one 0.7400 of the 2×2 unitary code of size 6 with optimal diversity product presented in this paper. Fig. 1 shows the symbol error rates (SER) of these two codes of size 6 over a quasi-static fading channel and one can see that the one with the optimal diversity product performs slightly better than the one with the optimal diversity sum at high signal-to-noise ratio (SNR), which also confirms the argument between diversity product and diversity sum in [11].

III. OPTIMALITY OF 2×2 UNITARY SPACE–TIME CODES OF SIZE $L = 6$

The main goal of this section is to prove the optimality of the code of size 6 presented in Section II-D-1).

Theorem 1: The maximal diversity product of a 2×2 unitary space–time code of size 6 is $\frac{1}{2}\sqrt{-5/2+\sqrt{22}}$, i.e.,

$$\eta(6) = \frac{1}{2}\sqrt{-5/2+\sqrt{22}}.$$

This theorem implies that the code presented in Section II-D-1) has already reached the maximal diversity product.

To prove this theorem, we need some preparations.

First, we introduce the concept of *dual*. For any unitary matrix $A = e^{j\theta/2}J_\theta(A)$, its dual is defined as $e^{j(2\pi-\theta)/2}(-J_\theta(A))$ and denoted by $\tilde{A}$. If $\theta = 0$, then $\tilde{A} = e^{j2\pi/2}(-J_\theta(A)) = A$, i.e., the dual of $A \in \boldsymbol{SU}(2)$ is itself. With the definition of a dual matrix we have the following corollary.

Lemma 1: For any unitary matrices A_1 and A_2 with their duals $\tilde{A}_1$ and $\tilde{A}_2$, respectively, we have

i) $|\det(A_1 - B)| = |\det(\tilde{A}_1 - B)|$, for any unitary matrix $B \in \boldsymbol{SU}(2)$,
ii) $|\det(A_1 - A_2)| = |\det(\tilde{A}_1 - \tilde{A}_2)|$.

This lemma is a direct result of Proposition 1, we omit its proof. In what follows, for the notational convenience, we use $\hat{A}$ to denote $J_\theta(A)$ for a matrix $A \in \boldsymbol{SU}(2,\theta)$ by dropping the subscript θ without causing any confusion. Since there exists an embedding i from $\boldsymbol{SU}(2)$ onto $\boldsymbol{S}^3$, we do not distinguish a matrix in $\boldsymbol{SU}(2)$ and a vector on $\boldsymbol{S}^3$ and use the same notation A to express a matrix in $\boldsymbol{SU}(2)$ and a point on $\boldsymbol{S}^3$. If A is treated as a point on $\boldsymbol{S}^3$, it means its embedding, i.e., $i(A)$ in (7).

Lemma 2: Let $A_i \in \boldsymbol{SU}(2,\theta_i)$, $i = 1,2,\ldots,L$, and $\{A_1, A_2, \ldots, A_L\}$ be an optimal unitary space–time code of size L with the maximal diversity product $d_L \geq 2$. Then,

$$|\det(A_i - A_j)| = \det(\hat{A}_i - \hat{A}_j) - 4\sin^2((\theta_i - \theta_j)/4) \geq d_L, \quad \text{if } |\theta_i - \theta_j| \leq \pi$$

$$|\det(A_i - A_j)| = 4\sin^2((\theta_i - \theta_j)/4) - \det(\hat{A}_i - \hat{A}_j) \geq d_L, \quad \text{if } |\theta_i - \theta_j| \geq \pi$$

where $\hat{A}_l = J_\theta(A_l)$ is the projection of A_l from $\boldsymbol{SU}(2,\theta_l)$ to $\boldsymbol{SU}(2)$ as defined in Section II-B.

Its proof is in [23]. Lemma 2 basically provides an expression of the absolute value of a difference matrix determinant from the one of their projections to $\boldsymbol{SU}(2)$ and their angles for an optimal constellation.

Lemma 3: Let $\{A_1, \ldots, A_L\}$ be an unitary space–time code with the optimal diversity product $d_L > 2$ and $A_j \in \boldsymbol{SU}(2,\theta_j)$, $j = 1,2,\ldots,L$, with ≥ 6. If $0 = \theta_1 \leq \cdots \leq \theta_L < 2\pi$, then $\theta_{i+1} - \theta_i < \pi$ for $i = 1,2,\ldots,L-1$, i.e., the difference of two adjacent angles is less than π.

Its proof is in [23].

Lemma 4: Let $\{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_L\}$ be L points on the sphere $\boldsymbol{S}^3$. Assume that $\|\boldsymbol{P}_i - \boldsymbol{P}_j\|^2 \geq d$ for a constant $d \geq 2$ and $1 \leq i < j \leq L$. Let $\boldsymbol{P}_0$ be any a point on this sphere. Then

- if $L = 4$, there exist s and t, $1 \leq s,t \leq L$, such that
$$\|\boldsymbol{P}_0 - \boldsymbol{P}_s\|^2 \geq 2 - 2\sqrt{1 - 3d/8}$$
$$\text{and} \quad \|\boldsymbol{P}_0 - \boldsymbol{P}_t\|^2 \leq 2 + 2\sqrt{1 - 3d/8}$$
- if $L = 3$, there exist s and t, $1 \leq s,t \leq L$, such that
$$\|\boldsymbol{P}_0 - \boldsymbol{P}_s\|^2 \geq 2 - 2\sqrt{1 - d/3}$$
$$\text{and} \quad \|\boldsymbol{P}_0 - \boldsymbol{P}_t\|^2 \leq 2 + 2\sqrt{1 - d/3}$$
- if $L = 2$, there exist s and t, $1 \leq s,t \leq L$, such that
$$\|\boldsymbol{P}_0 - \boldsymbol{P}_s\|^2 \geq 2 - 2\sqrt{1 - d/4}$$
$$\text{and} \quad \|\boldsymbol{P}_0 - \boldsymbol{P}_t\|^2 \leq 2 + 2\sqrt{1 - d/4}.$$

Its proof is in [23].

Lemma 5: For any L points $\{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_L\}$ on the unit sphere $\boldsymbol{S}^n$ in the $n+1$-dimensional real Euclidean space $\mathbb{R}^{n+1}$, we have

$$\sum_{1 \leq i < j \leq L} \|\boldsymbol{P}_i - \boldsymbol{P}_j\|^2 \leq L^2.$$

Its proof can be found in, for example, [11].

Lemma 6: Let $A_i = e^{j\theta_i/2}\hat{A}_i \in \boldsymbol{SU}(2,\theta_i)$, $i = 1,2,3$, be three unitary matrices. Assume $\theta_1 \leq \theta_2 \leq \theta_3$. If $\theta_3 - \theta_1 \leq \pi$, and for $i \neq j$, $|\det(A_i - A_j)| \geq d_6 \geq \sqrt{22} - 5/2$, then,

$$\theta_3 - \theta_1 \leq 5\pi/6.$$

Its proof is in [23].

Lemma 7: Let $2 < d \leq 2.5$ and $-1 < a,b \leq 1 - d/2$. If $\arccos(d/2 + a) + \arccos(d/2 + b) \geq \pi/2$, then

$$2\sin(\arccos(a + d/2)/2) - \frac{b - \cos(\arccos(d/2 + b) + \arccos(d/2 + a)) - d/2}{\cos(\arccos(-a)/2)} \geq d, \tag{10}$$

where $0 \leq \arccos(x) \leq \pi$.

Its proof is in [23].

Proposition 2: Let $\{A_1, A_2, \ldots, A_6\}$ be an optimal constellation with $A_j = e^{j\theta_j/2}\hat{A}_j$ of the maximal diversity product d_6. Assume that $0 = \theta_1 \leq \cdots \leq \theta_5 \leq \pi \leq \theta_6$ and $\theta_6 - \theta_5 \leq \pi$, $\theta_6 - \theta_4 \geq \pi$. Then, $d_6 \leq -5/2 + \sqrt{22}$.

Its proof is in [23].

Proposition 3: Let $\{A_1, A_2, \ldots, A_6\}$ be an optimal constellation with $A_j = e^{j\theta_j/2}\hat{A}_j$. Assume that $0 = \theta_1 \leq \cdots \leq \theta_5 \leq \pi \leq \theta_6$ and $\theta_6 - \theta_4 \leq \pi$, $\theta_6 - \theta_3 \geq \pi$. Then, $d_6 < -5/2 + \sqrt{22}$.

Its proof is in [23].

Proposition 4: Let $\{A_1, A_2, \ldots, A_6\}$ be an optimal constellation with $A_j = e^{j\theta_j/2}\hat{A}_j$. Assume that $0 = \theta_1 \leq \cdots \leq \theta_4 \leq \pi \leq \theta_5 \leq \theta_6$ and $\theta_6 - \theta_4 \leq \pi$, $\theta_6 - \theta_3 \geq \pi$. Then, $d_6 < -5/2 + \sqrt{22}$.

Its proof is in [23].

Proposition 5: Let $\{A_1, A_2, \ldots, A_6\}$ be an optimal constellation with $A_j = e^{j\theta_j/2}\hat{A}_j$. Assume that $0 = \theta_1 \leq \cdots \leq \theta_4 \leq \pi \leq \theta_5 \leq \theta_6$ and $\theta_6 - \theta_3 \leq \pi$, $\theta_6 - \theta_2 \geq \pi$. Then, $d_6 < -5/2 + \sqrt{22}$.

Its proof is in [23].

Now we begin to prove Theorem 1.

Proof of Theorem 1: Assume signal constellation

$$\mathcal{G} = \{A_1, A_2, \ldots, A_6\}$$

is an optimal constellation with the maximal diversity product d_6 and $A_i \in \boldsymbol{SU}(2,\theta_i)$, $i = 1,2,\ldots,6$. By the construction in Section II-D 1), we have $d_6 \geq -5/2 + \sqrt{22}$. We next need to show that $d_6 \leq -5/2 + \sqrt{22}$. To do so, let us consider the different cases of the number of the zero angles of A_i: $p \triangleq \sharp\{i \mid \theta_i = 0\}$. Without loss of generality, we can assume that $1 \leq p \leq 6$. In this proof and the proofs in [23], we always use $0 \leq \arccos(x) \leq \pi$.

i) $p = 6$.

$p = 6$ means that all $A_i \in \boldsymbol{SU}(2)$, i.e., all six matrices A_i are on the sphere $\boldsymbol{S}^3$. In other words, there exist 6-point packing such that the minimal distance is greater than $\sqrt{2}$, which contradicts with the packing result on $\boldsymbol{S}^3$ (according to the result [20], the packing angle on $\boldsymbol{S}^3$ is $\pi/2$, that is, the maximal minimum distance is $\sqrt{2}$).

ii) $p = 5$.

Assume $\theta_1 = \cdots = \theta_5 = 0$ and $\theta_6 > 0$. Thus, $A_i = \hat{A}_i, i = 1,2,\ldots,5$. By Lemma 3, we have $\theta_6 - \theta_5 \leq \pi$, i.e., $\theta_6 \leq \pi$. By Lemma 2

$$\det(\hat{A}_i - \hat{A}_6) \geq d_6 + 4\sin^2(\theta_6/4) > 2, \qquad i = 1,2,\ldots,5.$$

For $1 \leq i \neq j \leq 5$, from the condition, $\det(\hat{A}_i - \hat{A}_j) > 2$. Therefore, there exists six points $\{\hat{A}_1, \ldots, \hat{A}_6\}$ on the sphere $\boldsymbol{S}^3$ such that the minimum distance is greater than $\sqrt{2}$, which contradicts with the packing result as in (i).

iii) $p = 4$.

Assume $\theta_1 = \cdots = \theta_4 = 0$ and $0 < \theta_5 \le \theta_6 < 2\pi$. By Lemma 3, we have $\theta_5 \le \pi$ and $\theta_6 - \theta_5 \le \pi$. If $\theta_6 \le \pi$, then, as shown as ii), $\{\hat{A}_1, \ldots, \hat{A}_4, \hat{A}_5, \hat{A}_6\}$ consists of a 6-point packing on $\boldsymbol{S}^3$ with minimum distance greater than $\sqrt{2}$, which results in a contradiction. Therefore, we can assume that $\theta_5 \le \pi \le \theta_6$ and $\theta_6 - \theta_5 \le \pi$.

We next investigate the packing position of $\{\hat{A}_1, \ldots, \hat{A}_4, \hat{A}_5, -\hat{A}_6\}$ on $\boldsymbol{S}^3$. Denote $\hat{A}_i = (a_i, b_i, c_i, e_i)$, $i = 1, \ldots, 6$. By a unitary transformation, we can assume $\hat{A}_5 = I$, i.e., $a_5 = 1, b_5 = c_5 = e_5 = 0$. We then convert this problem to a packing problem on the 3-D unit sphere $\boldsymbol{S}^2$ as follows. If $a_i \ne 1, -1$, define

$$\boldsymbol{b}_i = \frac{1}{r_i}(b_i, c_i, e_i), \qquad i = 1, 2, 3, 4, 6 \tag{11}$$

where $r_i = \sqrt{1 - a_i^2}$. Then $\boldsymbol{b}_i \in \boldsymbol{S}^2$ and clearly

$$\det(\hat{A}_i - \hat{A}_j) = 2(1 - a_i a_j) + r_i r_j (\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 - 2) \tag{12}$$

$$\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 = 2 - \frac{2(1 - a_i a_j) - \det(\hat{A}_i - \hat{A}_j)}{r_i r_j}. \tag{13}$$

Two remarks about this conversion are as follows. The mapping $\boldsymbol{S}^3 \ni (a, b, c, e) \to \boldsymbol{b} = (b/r, c/r, e/r) \in \boldsymbol{S}^2$ is *not* one-to-one. It is because, for different two points (a, b, c, e) and $(-a, b, c, e)$, the images are the same. However, when we restrict $a \ge 0$ or $a \le 0$, the mapping becomes one-to-one and onto. Another remark is that an image point $\boldsymbol{b}$ does *not* depend on a, when we restrict a to $a \le 0$ or $a \ge 0$. To explain this, we use the polar coordination. For any point $(a, b, c, e) \in \boldsymbol{S}^3$, there exist three angles ϕ_1, ϕ_2, ϕ_3, such that $a = \sin(\phi_1)$ and $b = \cos(\phi_1)\sin(\phi_2)$, $c = \cos(\phi_1)\cos(\phi_2)\sin(\phi_3)$, $e = \cos(\phi_1)\cos(\phi_2)\cos(\phi_3)$. Hence $\boldsymbol{b} = (b/r, c/r, e/r) = (\sin(\phi_2), \cos(\phi_2)\sin(\phi_3), \cos(\phi_2)\cos(\phi_3))$, which is independent of ϕ_1, i.e., a. Therefore, when we restrict a to $a \le 0$ or $a \ge 0$, the distance $\|\boldsymbol{b}_i - \boldsymbol{b}_j\|$ is independent of a_i and a_j.

For $1 \le i \le 4$ and $i = 6$, because $\theta_5 \le \pi$ and $\theta_6 - \theta_5 \le \pi$, by Lemma 2, we have

$$\begin{aligned} 2 < d_6 \le |\det(A_5 - A_i)| &= \det(\hat{A}_5 - \hat{A}_i) - 4\sin^2(\theta_5/4) \\ &= 2 - 2a_i - 4\sin^2(\theta_5/4). \end{aligned}$$

Therefore, $a_i < 0$ for $i = 1, 2, 3, 4, 6$.

Since $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, without loss of generality, we may assume that $-1 \le a_1 \le a_2 \le a_3 \le a_4 < 0$. If $a_1 = -1$, then $b_1 = c_1 = e_1 = 0$ and it is not hard to see that $\det(\hat{A}_4 - \hat{A}_1) = 2 + 2a_4$. It implies $d_6 \le 2 + 2a_4$, i.e., $a_4 > 0$, which contradicts with the result $a_4 < 0$ we derived before. Therefore, $a_1 > -1$.

For $1 \le i \ne j \le 4$, from (12) and the fact that $\theta_i = \theta_j = 0$, we have

$$\begin{aligned} d_6 &\le |\det(A_i - A_j)| = \det(\hat{A}_i - \hat{A}_j) - 4\sin^2((\theta_i - \theta_j)/4) \\ &= 2 - 2a_i a_j + r_i r_j (\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 - 2). \end{aligned}$$

Because $a_i a_j > 0$ and $r_i r_j > 0$, we have $\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 - 2 > 0$. Furthermore, it is easy to see that for a fixed a_i, the right hand side of the above inequality is increasing for a_j. Therefore,

$$\begin{aligned} d_6 &\le 2 - 2a_i a_j + r_i r_j (\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 - 2) \\ &\le 2 - 2a_4^2 + r_4^2 (\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 - 2) \end{aligned}$$

which implies that

$$a_4 \ge -\sqrt{1 - d_6/\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2}.$$

Because $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_4\}$ are on $\boldsymbol{S}^2$, by the packing theory on $\boldsymbol{S}^2$, there is at least one pair $\{\boldsymbol{b}_i, \boldsymbol{b}_j\}$ such that $\|\boldsymbol{b}_i - \boldsymbol{b}_j\|^2 \le 8/3$. Hence,

$$a_4 \ge -\sqrt{1 - 3d_6/8}. \tag{14}$$

Since $a_5 = 1, -1 \le a_6 < 0$, and $0 \le \theta_6 - \theta_5 \le \pi$, from Lemma 2 we have

$$\begin{aligned} d_6 \le |\det(A_6 - A_5)| &= \det(\hat{A}_6 - \hat{A}_5) - 4\sin^2((\theta_6 - \theta_5)/4) \\ &= 2 - 2a_6 - 4\sin^2((\theta_6 - \theta_5)/4) \\ &\le 2 - 2(-1) - 4\sin^2((\theta_6 - \theta_5)/4). \end{aligned}$$

Therefore,

$$\cos((\theta_6 - \theta_5)/2) \ge d_6/2 - 1. \tag{15}$$

Using the fact that

$$d_6 \le |\det(A_5 - A_4)| = \det(\hat{A}_5 - \hat{A}_4) - 4\sin^2(\theta_5/4)$$

and noting that $\det(\hat{A}_5 - \hat{A}_4) = 2 - 2a_4$, we have

$$\begin{aligned} d_6 &\le 2 - 2a_4 - 4\sin^2(\theta_5/4) \le 2 + 2\sqrt{1 - 3d_6/8} - 4\sin^2(\theta_5/4) \\ &= 2\cos(\theta_5/2) + 2\sqrt{1 - 3d_6/8} \end{aligned} \tag{16}$$

where the second inequality is from (14). Inequality (16) implies

$$\cos(\theta_5/2) \ge d_6/2 - \sqrt{1 - 3d_6/8}. \tag{17}$$

We now replace A_5, A_6 by their duals $\tilde{A}_5, \tilde{A}_6$. From the definition, we have

$$\tilde{A}_6 = e^{j(2\pi - \theta_6)/2}(-\hat{A}_6), \qquad \tilde{A}_5 = e^{j(2\pi - \theta_5)/2}(-\hat{A}_5).$$

Furthermore, $\{A_1, \ldots, A_4, \tilde{A}_6, \tilde{A}_5\}$ is also an optimal signal constellation by Lemma 1. We make a normalization by multiplying $-\hat{A}_6^H$ from left to the constellation to get a new constellation

$$\begin{aligned} \mathcal{G}_1 &\triangleq \{-\hat{A}_6^H A_1, -\hat{A}_6^H A_2, -\hat{A}_6^H A_3, -\hat{A}_6^H A_4, -\hat{A}_6^H \tilde{A}_6, -\hat{A}_6^H \tilde{A}_5\} \\ &= \{-\hat{A}_6^H A_1, -\hat{A}_6^H A_2, -\hat{A}_6^H A_3, \\ &\quad -\hat{A}_6^H A_4, e^{j(2\pi - \theta_6)/2} I, -\hat{A}_6^H \tilde{A}_5\}. \end{aligned}$$

Since $-\hat{A}_6^H$ is a unitary matrix, $\mathcal{G}_1$ is also an optimal constellation. Furthermore, $\mathcal{G}_1$ and $\{A_1, \ldots, A_6\}$ have the same angle relationships. Therefore, inequality (17) corresponding to this new constellation $\mathcal{G}_1$ also holds

$$\cos((2\pi - \theta_6)/2) \ge d_6/2 - \sqrt{1 - 3d_6/8}. \tag{18}$$

From (17) and (18), we have

$$\begin{aligned} \theta_5 &\le 2\arccos(d_6/2 - \sqrt{1 - 3d_6/8}) \\ \theta_6 &\ge 2\pi - 2\arccos(d_6/2 - \sqrt{1 - 3d_6/8}). \end{aligned}$$

Hence,

$$\theta_6 - \theta_5 \ge 2\pi - 4\arccos(d_6/2 - \sqrt{1 - 3d_6/8}).$$

From (15), we know that $\theta_6 - \theta_5 \le 2\arccos(d_6/2 - 1)$. Therefore,

$$2\pi - 4\arccos(d_6/2 - \sqrt{1 - 3d_6/8}) \le 2\arccos(d_6/2 - 1).$$

Hence,

$$d_6 \le -4(d_6/2 - \sqrt{1-3d_6/8})^2 + 4$$

which implies the desired result $d_6 \le -5/2 + \sqrt{22}$.

iv) $p \le 3$.

Assume $0 = \theta_1 \le \theta_2 \le \theta_3 \le \cdots \le \theta_6$. Using the same argument as in the beginning of Case (iii) when $p = 4$ and Lemma 3, we can also assume that $\pi \le \theta_6$ and $\theta_2 \le \pi$, $\theta_6 - \theta_5 \le \pi$. We divide the proof into several cases according to the relationships among the angles θ_j.

Case I $\theta_6 \ge \pi$ **and** $\theta_5 \le \pi$**.**

We divide this case into four subcases.

Case I.1 $\theta_6 \ge \pi, \theta_5 \le \pi$ **and** $\theta_6 - \theta_4 \ge \pi$**.**

This subcase is Proposition 2.

Case I.2 $\theta_6 \ge \pi, \theta_5 \le \pi$ **and** $\theta_6 - \theta_4 \le \pi, \theta_6 - \theta_3 \ge \pi$**.**

This subcase is Proposition 3.

Case I.3 $\theta_6 \ge \pi, \theta_5 \le \pi$ **and** $\theta_6 - \theta_3 \le \pi, \theta_6 - \theta_2 \ge \pi$**.**

By taking the rotation of angle $-\theta_5$ to $\mathcal{G}$, we obtain a new constellation

$$\mathcal{G}' = \{A'_1, A'_2, \ldots, A'_6\}$$

where $A'_j = e^{-j\theta_5/2}A_j$. For $j = 1, 2, 3, 4$

$$A'_j = e^{-j\theta_5/2}e^{j\theta_j/2}\hat{A}_j = e^{(2\pi-(\theta_5-\theta_j))/2}(-\hat{A}_5).$$

For $j = 5$, we have $A'_5 = e^{-j\theta_5/2}A_5 = \hat{A}_5$. For $j = 6$, we have

$$A'_6 = e^{-j\theta_5/2}e^{j\theta_6/2}\hat{A}_6 = e^{(\theta_6-\theta_5)/2}\hat{A}_6.$$

Therefore, the relationship between $\mathcal{G}'$ and $\mathcal{G}$ is

$$\{\hat{A}'_1, \hat{A}'_2, \hat{A}'_3, \hat{A}'_4, \hat{A}'_5, \hat{A}'_6\} = \{-\hat{A}_1, -\hat{A}_2, -\hat{A}_3, -\hat{A}_4, \hat{A}_5, \hat{A}_6\},$$

and

$$\begin{aligned}\{\theta'_1, \theta'_2, \theta'_3, \theta'_4, \theta'_5, \theta'_6\} = \{&2\pi - \theta_5, 2\pi - (\theta_5 - \theta_2),\\ &2\pi - (\theta_5 - \theta_3), 2\pi - (\theta_5 - \theta_4),\\ &0, \theta_6 - \theta_5\}.\end{aligned}$$

Clearly, the diversity product of $\mathcal{G}'$ is still d_6.

We now consider the dual of $\mathcal{G}'$, denoted by $\tilde{\mathcal{G}}'$, which has the same diversity product as $\mathcal{G}'$ by Corollary 1: $\tilde{\mathcal{G}}' = \{\tilde{A}'_1, \tilde{A}'_2, \ldots, \tilde{A}'_6\}$, where $\tilde{A}'_j = e^{\theta''_j/2}\hat{\tilde{A}}'_j, 1 \le j \le 6$. By the definition of a dual, the relationship between $\tilde{\mathcal{G}}'$ and $\mathcal{G}'$ or $\mathcal{G}$ is

$$\begin{aligned}\{\hat{\tilde{A}}'_1, \hat{\tilde{A}}'_2, \hat{\tilde{A}}'_3, \hat{\tilde{A}}'_4, \hat{\tilde{A}}'_5, \hat{\tilde{A}}'_6\} &= \{-\hat{A}'_1, -\hat{A}'_2, -\hat{A}'_3, -\hat{A}'_4, \hat{A}'_5, -\hat{A}'_6\}\\ &= \{\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4, \hat{A}_5, -\hat{A}_6\}\end{aligned}$$

and the corresponding angles are

$$\begin{aligned}&\{\theta''_1, \theta''_2, \theta''_3, \theta''_4, \theta''_5, \theta''_6\}\\ &\quad = \{2\pi - \theta'_1, 2\pi - \theta'_2, 2\pi - \theta'_3, 2\pi - \theta'_4, 0, 2\pi - \theta'_6\}\\ &\quad = \{\theta_5, \theta_5 - \theta_2, \theta_5 - \theta_3, \theta_5 - \theta_4, 0, 2\pi - (\theta_6 - \theta_5)\}.\end{aligned}$$

It is easy to see that $\theta''_6 \ge \pi$ and $\theta''_j \le \pi$ for $j = 1, 2, 3, 4, 5$. Furthermore,

$$\theta''_6 - \theta''_1 \le \pi, \quad \theta''_6 - \theta''_2 \le \pi, \theta''_6 - \theta''_j \ge \pi, \quad j = 3, 4, 5.$$

Thus, if we rearrange $\tilde{\mathcal{G}}'$ into

$$\mathcal{G}'' = \{\tilde{A}'_5, \tilde{A}'_4, \tilde{A}'_3, \tilde{A}'_2, \tilde{A}'_1, \tilde{A}'_6\}$$

then the conditions on $\mathcal{G}''$ are exactly the same as the ones in Case I.2. By Proposition 3, we have proved this theorem in this subcase.

Case I.4 $\theta_6 \ge \pi, \theta_5 \le \pi$ **and** $\theta_6 - \theta_2 \le \pi$

Make a rotation angle $-\theta_2$ to the constellation $\mathcal{G}$ as done in Case I.3 and we find that the new constellation has the same conditions as in Case I.1. Therefore, by Proposition 2, we have proved this theorem in this subcase.

Case II $\theta_6, \theta_5 \ge \pi$ **and** $\theta_4 \le \pi$**.**

We divide this proof into four subcases.

Case II.1 $\theta_6, \theta_5 \ge \pi, \theta_4 \le \pi$ **and** $\theta_6 - \theta_4 \ge \pi$**.**

In this case, we make a rotation to the constellation as follows. Let $A'_j = e^{-j\theta_6/2}A_j$ for $j = 1, 2, \ldots, 6$. Then $\{A'_1, A'_2, \ldots, A'_6\}$ is also an optimal constellation. Since, for $j = 1, 2, 3, 4, 5$, $A'_j = e^{-j\theta_6/2}A_j = e^{j(2\pi-(\theta_6-\theta_j))/2} \cdot (-\hat{A}_j)$, we obtain $\hat{A}'_j = -\hat{A}_j$ and the angle θ'_j of A'_j is $2\pi - (\theta_6 - \theta_j)$. For $j = 6$, $\theta'_6 = 0$, i.e., A'_6 belongs to $\boldsymbol{SU}(2)$ and $A'_6 = \hat{A}_6$. Furthermore, we have that $\theta'_6, \theta'_1, \theta'_2, \theta'_3, \theta'_4$ are all less than or equal to π, and θ'_5 is greater than or equal to π. Therefore, $\{A'_1, A'_2, \ldots, A'_6\}$ satisfies the conditions in Case I. Thus we have proved this theorem in this subcase.

Case II.2 $\theta_6, \theta_5 \ge \pi, \theta_4 \le \pi$ **and** $\theta_6 - \theta_4 \le \pi, \theta_6 - \theta_3 \ge \pi$**.**

It is proved in Proposition 4.

Case II.3 $\theta_6, \theta_5 \ge \pi, \theta_4 \le \pi$ **and** $\theta_6 - \theta_3 \le \pi, \theta_6 - \theta_2 \ge \pi$**.**

It is proved in Proposition 5.

Case II.4 $\theta_6, \theta_5 \ge \pi, \theta_4 \le \pi$ **and** $\theta_6 - \theta_2 \le \pi$**.**

Let $A'_{j-1} = e^{-j\theta_2/2}A_j$ for $j = 2, 3, 4, 5, 6$, and $A'_6 = e^{(2\pi - j\theta_2)/2}A_1$. Note that $\hat{A}_1 = A_1$ since $\theta_1 = 0$. Then, $\{A'_1, A'_2, A'_3, A'_4, A'_5, A'_6\}$ satisfies the conditions of Case I.

Case III $\theta_6, \theta_5, \theta_4 \ge \pi$ **and** $\theta_3 \le \pi$**.**

Under this assumption, we consider the dual constellation: $\theta_1 = 0$ is fixed, and θ_2, θ_3 are changed to $2\pi - \theta_2, 2\pi - \theta_3$, which belong to $[\pi, 2\pi]$, and $\theta_4, \theta_5 . \theta_6$ are transferred to $2\pi - \theta_4, 2\pi - \theta_5, 2\pi - \theta_6$, which belong to $[0, \pi]$. Therefore, through this duality, we change this subcase into Case II.

Case IV $\theta_6, \theta_5, \theta_4, \theta_3 \ge \pi$ **and** $\theta_2 \le \pi$**.**

Also we consider its dual constellation and find that this case can be converted to Case I.

By summarizing all the above cases, this theorem is proved. $\boldsymbol{q}.e.d.$

IV. Conclusion

In this correspondence, we have partially used sphere packing theory to construct 2×2 unitary space–time codes. Although the optimal ones of sizes L below 6 can be constructed from the sphere packings on $\boldsymbol{S}^3$, i.e., Hamiltonian constellations [9], [16] that reach the upper bound $\frac{1}{2}\sqrt{2L/(L-1)}$ of the maximal diversity products derived in [11]. This upper bound can not be reached when the sizes are above 5 as shown in [11]. The critical boundary on the sizes is size $L = 6$. In this correspondence, we have constructed 2×2 unitary space–time code of size 6 that has been shown in this correspondence to have the optimal diversity product. The optimal diversity product

$$d_6 = \frac{1}{2}\sqrt{-5/2 + \sqrt{22}} \approx 0.74 < 0.7746 \approx \frac{1}{2}\sqrt{2L/(L-1)}$$

when $L = 6$. Some constructions of 2×2 unitary space–time codes of sizes $32, 48, 64$ of non-Hamiltonian constellations with best-known diversity products have been also presented by partially using sphere packing theory. To obtain these results, we have presented a determinant identity between the difference matrices of two matrices in a Hamiltonian constellation and two matrices in non-Hamiltonian constellations. Since some of the proofs of the lemmas and propositions in Section III are tedious and long, the longer version of this correspondence including all the proofs is downloadable through the website [23].

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