An Efficient Frequency-Determination Algorithm from Multiple Undersampled Waveforms

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***Abstract*—Frequency estimation/determination has applications in various areas, where the sampling rate is usually above the Nyquist rate. In some applications, it is preferred that the range of the frequencies is as large as possible for a given sampling rate and in some applications, the sampling rate is below the Nyquist rate. In both cases, frequency estimation from undersampled waveforms is needed. In this letter, we present an efficient algorithm to determine multiple frequencies from multiple undersampled waveforms with sampling rates below the Nyquist rates.**

***Index Terms*—Chinese remainder theorem, undersampling.**

I. INTRODUCTION

FREQUENCY estimation/determination has applications in almost all engineering fields. In frequency estimation, it is known that frequencies can be uniquely determined if the sampling rate is above the Nyquist rate. The simplest frequency-estimation method is the discrete Fourier transform (DFT) from a finite data. However, when the sampling rate is below the Nyquist rate, it is impossible to uniquely determine the frequencies from a sampled waveform. Recently, frequency determination from multiple undersampled waveforms has been studied in [1]–[3], [7], where multiple sequences sampled from the same analog waveform using multiple sampling rates were used. If there is only a single frequency in a complex-valued waveform, the frequency determination from multiple undersampled waveforms can be achieved by using the Chinese Remainder Theorem (CRT), as we will see later. When there is a single frequency in a real-valued waveform, an algorithm for the frequency determination using multiple undersamplings was obtained in [3], which corresponds to two symmetric frequencies in a complex-valued waveform. When there are multiple frequencies in a complex-valued waveform, a range for the detectable multiple frequencies was given in [1] in terms of the multiple sampling rates, where no other conditions of the frequencies are needed. This range was maximized in [2] by imposing a condition on the distance between the multiple frequencies. A different approach was studied in [7], where the two sampling rates are the same but the analog waveform is slightly delayed in the second sampling.

The multiple frequency-determination method proposed in [1] is the looking-up table method, which is expensive when the number of frequencies and the number of sampling rates are large. Although an efficient multiple frequency-determination method was proposed in [2] with some restrictions on the frequencies, there is no general efficient multiple frequency-determination algorithm from multiple undersampled waveforms. In this letter, we present such a general and efficient algorithm, which can be thought of as a generalization of the CRT from a single frequency-determination to multiple undersampled waveforms.

There are at least two cases where the frequency determination from undersampled waveforms is useful. The first case is when only undersampled data is available. The second case is when it is desired to have as large detectable frequencies as possible given certain sampling rates. One such application is the synthetic aperture radar (SAR) imaging of moving targets using a linear antenna array such as [4], [5], in particular using a linear antenna array and multiple wavelength-transmission signals [5]. To detect the accurate locations of a moving target, it was proved in [5] that it is essential to use multiple antennas and multiple wavelength-transmission signals unless the velocity of a moving target is as slow as walking people. The number of different wavelengths in a transmission signal corresponds to the number of sampling rates of an analog signal. The multiple frequencies of interest correspond to the velocities/locations of the multiple targets. Given the number of wavelengths and the pattern of an antenna array (corresponding to given sampling rates), it is desired that the detectable velocities/locations of the moving targets (corresponding to multiple frequencies to determine) are as large as possible (such as the velocities of moving vehicles). Another such application is to increase the dynamic range of the detectable parameters for polynomial phase signals using multiple lag diversities in high-order ambiguity functions [6]. Notice that this paragraph only describes some application examples and the rest of this letter is self-contained (and the approach is not hard to follow).

This letter is organized as follows. In Section II, we describe and formulate the problem. In Section III, we present an efficient algorithm.

II. PROBLEM FORMULATION

Without loss of generality, we assume that the multiple frequencies in a waveform $x(t)$ are $f_1 = N_1$ Hz, $f_2 = N_2$ Hz, $\ldots, f_\rho = N_\rho$ Hz, and $N_1, N_2, \ldots, N_\rho$ are all nonnegative integers

$$x(t) = \sum_{l=1}^{\rho} A_l e^{2\pi j f_l t} \tag{1}$$

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where $A_l, 1 \le l \le \rho$ are ρ nonzero complex-valued coefficients, and ρ is assumed known throughout this letter. The simplest method to determine these frequencies f_l is to first sample $x(t)$ at a sampling rate $f_s = m$ Hz

$$x_m[n] = x\left(\frac{n}{m}\right) = \sum_{l=1}^{\rho} A_l e^{2\pi j N_l n/m}, \quad n \in \mathbf{Z} \tag{2}$$

then take the m-point DFT of $x_m[n]$, $n = 0, 1, \ldots, m-1$, which gives the nonzero values A_l at frequencies f_l, respectively. For explanation convenience, in the above, the additive noise is not mentioned, otherwise more sophisticated spectra-estimation methods are needed when the SNR is low, which is beyond the scope of this letter. Also notice that the assumption of integer-valued frequencies $f_l = N_l$ is for the study convenience similar to the conventional DFT frequency estimation.

The above frequency determination, however, works only when $m \ge \max\{N_1, N_2, \ldots, N_\rho\}$ (i.e., the sampling rate $f_s = m$ must be above the Nyquist rate). Otherwise, the detected frequencies have the modulo m ambiguities. In the rest of this letter, we are only interested in the latter case (i.e., the sampling rate is below the Nyquist rate). In this case, although it is not possible to uniquely determine all the frequencies N_l, $1 \le l \le \rho$, from a single sampled waveform $x_m[n]$ in (2), it may be possible to do so from multiple sampled waveforms $x_{m_r}[n]$ with multiple sampling rates m_r, $1 \le r \le \gamma$:

$$x_{m_r}[n] = x\left(\frac{n}{m_r}\right) = \sum_{l=1}^{\rho} A_l e^{2\pi j N_l n/m_r}, \quad n \in \mathbf{Z} \tag{3}$$

which is the problem of interest in this letter and can be restated as follows:

Problem A: Determine $N_1, N_2, \ldots, N_\rho$ from the multiple sampled data in (3)

$$x_{m_r}[n], \quad 0 \le n \le m_r - 1, \quad 1 \le r \le \gamma$$

where $\max\{m_1, m_2, \ldots, m_\gamma\}$ may be smaller than $\max\{N_1, N_2, \ldots, N_\rho\}$.

This problem can be reformulated after taking the multiple DFT's of $x_{m_r}[n], 0 \le n \le m_r - 1$, ($\gamma$ many DFT's) as follows. By taking the m_r-point DFT (DFT_{m_r}) of $x_{m_r}[n]$ in (3), we obtain

$$\mathrm{DFT}_{m_r}(x_{m_r}[n]) = \sum_{l=1}^{\rho} A_l \delta(k - k_{l,r}), \quad 0 \le k \le m_r - 1 \tag{4}$$

where

$$k_{l,r} = N_l \bmod m_r, \quad 1 \le l \le \rho, \quad 1 \le r \le \gamma. \tag{5}$$

For each r, $1 \le r \le \gamma$ from (4), the residue set

$$S_r(N_1, N_2, \ldots, N_\rho) \triangleq \{k_{l,r} : l = 1, 2, \ldots, \rho\} \tag{6}$$

can be determined. Note that in the above formulation, the amplitude information of A_l is not used, which is because the amplitudes may be distorted by the additive noise in practical applications. Problem A then becomes the following equivalent Problem B.

Problem B: Determine $N_1, N_2, \ldots, N_\rho$ from the γ residue sets $S_r(N_1, N_2, \ldots, N_\rho)$, $1 \le r \le \gamma$ obtained in (4)–(6).

The simplest case for Problem B is when there is only a single frequency in the waveform (i.e., $\rho = 1$). In this case, the problem is to determine N_1 from its γ residues $k_{1,r} = N_1 \bmod m_r$, $r = 1, 2, \ldots, \gamma$, which can be solved by using the CRT (see for example [8]) if and only if

$$0 \le N_1 < \mathrm{lcm}(m_1, m_2, \ldots, m_\gamma) \tag{7}$$

where lcm stands for the least common multiple. Under the condition that each pair m_i and m_j for $i \ne j$ are coprime, the solution is given by the following formulas. Let $m = \mathrm{lcm}(m_1, m_2, \ldots, m_\gamma)$ and $M_r = m/m_r$ and n_r be the number with $1 \le n_r \le m_r - 1$, such that $n_r M_r = 1 \bmod m_r$, then

$$N_1 = \sum_{r=1}^{\gamma} k_{1,r} n_r M_r. \tag{8}$$

Notice that the above determination formula requires the coprimeness of each pair m_i, and m_j for $i \ne j$, while the uniqueness does not as we can see, for example, from [1].

When there are multiple frequencies in the waveform (i.e., $\rho > 1$), the problem becomes more complicated. The complication comes from the fact that the known residue sets $S_r(N_1, N_2, \ldots, N_\rho)$, $1 \le r \le \gamma$ do not specify the order of the residues $r_{l,r}$ with respect to l but only γ sets of numbers. In other words, it is not known from these sets which frequency an element in set $S_r(N_1, N_2, \ldots, N_\rho)$ comes from (or modulo from), although m_r is known. Otherwise, the frequency could be determined using the CRT the same as the single frequency case, as described in (7) and (8).

There are two issues with Problem B. The first issue is the range of the detectable frequencies $N_1, N_2, \ldots, N_\rho$ when $\rho > 1$. It is given in (7) when $\rho = 1$. The second issue is the efficient determination algorithm when all the frequencies are in the range from the first issue. A result on the first issue has been recently obtained in [1], which is stated in the following.

Theorem 1: Assume that a complex valued waveform $x(t)$ contains ρ different frequencies $f_l = N_l \ge 0$ for $1 \le l \le \rho$. Let m_r, $1 \le r \le \gamma$ be γ sampling rates in the undersampled versions $x_{m_r}[n]$ of $x(t)$ in (3). Let

$$\gamma = \eta\rho + \theta, \quad 0 \le \theta < \rho \tag{9}$$

where η is a nonnegative integer. Then the ρ frequencies $f_l = N_l \ge 0$ for $1 \le l \le \rho$ can be uniquely determined by using the m_r-point DFT of $x_{m_r}[n]$ for $1 \le r \le \gamma$ if

$$\max\{N_1, N_2, \ldots, N_\rho\} < \max\{m, m_1, m_2, \ldots, m_\gamma\} \tag{10}$$

where where η is defined in (9).

As an example, let us consider the case of two frequencies N_1 and N_2. We choose four sampling rates $m_1 = 19$ Hz, $m_2 = 21$ Hz, $m_3 = 22$ Hz, and $m_4 = 23$ Hz. In this case, $\rho = 2$, $\gamma = 4$, and therefore $\eta = 2$ in (9). Clearly, $m = m_1 m_2 = 399$ in (11), as shown at the bottom of the following page. By Theorem 1, all two different frequencies N_1 and N_2 in the range $[0, 398]$ can be uniquely determined from the undersampled waveforms with sampling rates 19 Hz, 21 Hz, 22 Hz, and 23 Hz by using 19, 21, 22, and 23 point DFT's, respectively. We can see that the sampling rates required are about 20 times less than the Nyquist sampling rate in this simple example.

Regarding the second issue, the determination method suggested in [1] is the looking-up table method as follows. Define the product set

$$\begin{aligned} S(N_1, N_2, \ldots, N_\rho) & \\ &\triangleq S_1(N_1, N_2, \ldots, N_\rho) \\ &\cdot S_2(N_1, N_2, \ldots, N_\rho) \\ &\times \cdots \times S_\gamma(N_1, N_2, \ldots, N_\rho). \end{aligned} \tag{12}$$

List a table including all possible such product sets $S(N_1, N_2, \ldots, N_\rho)$ for all possible frequencies $N_1, N_2, \ldots, N_\rho$ in the range given in (9)–(11). Then the solution can be matched by looking up this table as long as the true γ frequencies in the range given in (9)–(11). Clearly, this matching process is expensive when $m_1, \ldots, m_\gamma$, ρ, and γ are large. Next, we want to propose an efficient determination algorithm.

III. Efficient Algorithm

In this section, we want to present an efficient algorithm to determine the multiple frequencies $N_1, N_2, \ldots, N_\rho$ from their residue product set $S(N_1, N_2, \ldots, N_\rho)$ defined in (12) by assuming the range (9)–(11) of $N_1, N_2, \ldots, N_\rho$. In the following, we always assume that m in (11) is greater than $\max\{m_1, m_2, \ldots, m_\gamma\}$, otherwise there are no modulo ambiguities in the DFT's in (4). This also implies $\eta > 1$ in (9).

A. Multiple Frequency-Determination Algorithm

Step 1: Arbitrarily take a vector $(k_1, k_2, \ldots, k_\gamma) \in S(N_1, N_2, \ldots, N_\rho)$, defined in (12).

Step 2: For each r with $1 \le r \le \gamma$, define a set

$$\mathcal{N}_r = \{k_r + n m_r : k_r \le k_r + n m_r < m \text{ and integers } n\} \tag{13}$$

where m is defined in (11). Note that all the numbers in set $\mathcal{N}_r$ have the same residue modulo m_r, which is also called the equivalent class (or coset) of k_r.

Step 3: By (9), there are η times more residues than the frequencies themselves (i.e., $\gamma \ge \rho\eta$). Therefore, there exist integers $r_1, r_2, \ldots, r_\eta$ with $1 \le r_1 < r_2 < \cdots < r_\eta \le \gamma$, such that the residues $k_{r_1}, k_{r_2}, \ldots, k_{r_\eta}$ are from a common frequency (i.e., $\mathcal{U} \triangleq \mathcal{N}_{r_1} \cap \mathcal{N}_{r_2} \cap \cdots \cap \mathcal{N}_{r_\eta} \neq \emptyset$). By the following Lemma 1, $\mathcal{U}$ has only one element $\bar{N}$ (i.e., $\mathcal{U} = \{\bar{N}\}$).

Check whether $\bar{N}$ is a valid frequency by checking whether its residue vector $(\bar{k}_1, \ldots, \bar{k}_\gamma) \bmod (m_1, \ldots, m_\gamma)$ belongs to the set $S(N_1, N_2, \ldots, N_\rho)$. If not, find another set of $1 \le r_1 < r_2 < \cdots < r_\eta \le \gamma$, such that $\mathcal{N}_{r_1} \cap \mathcal{N}_{r_2} \cap \cdots \cap \mathcal{N}_{r_\eta} \neq \emptyset$. Repeat this process until $\bar{N}$ is a valid frequency denoted by $N_\rho \in \mathcal{U}$.

Step 4: For $r = 1, 2, \ldots, \gamma$, remove $k_{\rho,r} = N_\rho \bmod m_r$ from the set $S_r(N_1, N_2, \ldots, N_\rho)$: shown in (14), at the bottom of the page, where $|S|$ denotes the cardinality of set S, as shown in (15), at the bottom of the page.

Step 5: Go to Step 1 by replacing ρ with $\rho - 1$ and replacing $S(N_1, N_2, \ldots, N_\rho)$ with $S(N_1, N_2, \ldots, N_{\rho-1})$. Repeat this process until N_1 is determined.

Lemma 1: The intersection set $\mathcal{U}$ in Step 3 has only one element.

Proof:: Let $N \in \mathcal{U}$. By (13) and (11), we have

$$N < m \le \operatorname{lcm}\{m_{r_1}, m_{r_2}, \ldots, m_{r_\eta}\}.$$

Therefore, N can be uniquely determined from its residues $N = k_{r_e} \bmod m_{r_e}$ for $e = 1, 2, \ldots, \eta$ (see for example, Lemma 1 in [1]). This proves Lemma 1. □

The advantages of this algorithm over the looking-up table method proposed in [1] is that this algorithm only deals with the current detected residues and therefore, it does not need the table, which may be huge, or matching process.

IV. Conclusion

In this letter, we presented an efficient algorithm to determine multiple frequencies from multiple undersampled waveforms, where the multiple sampling rates are below the Nyquist rate. The proposed algorithm can be thought of as a generalization of the CRT. It is believed that its applications may not be limited to SAR imaging of moving targets mentioned in the introduction.

$$m \triangleq \begin{cases} \min_{1 \le r_1 < r_2 < \cdots < r_\eta \le \gamma} \operatorname{lcm}\{m_{r_1}, m_{r_2}, \ldots, m_{r_\eta}\}, & \text{if } \eta > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

$$S_r(N_1, N_2, \ldots, N_{\rho-1}) = \begin{cases} S_r(N_1, N_2, \ldots, N_\rho) - \{k_{\rho,r}\}, & \text{if } |S_r(N_1, N_2, \ldots, N_\rho)| = \rho, \\ S_r(N_1, N_2, \ldots, N_\rho) & \text{otherwise} \end{cases} \tag{14}$$

$$\begin{aligned} S(N_1, N_2, \ldots, N_{\rho-1}) = S_1(N_1, N_2, \ldots, N_{\rho-1}) \times S_2(N_1, N_2, \ldots, N_{\rho-1}) \\ \times \cdots \times S_\gamma(N_1, N_2, \ldots, N_{\rho-1}). \end{aligned} \tag{15}$$

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REFERENCES

[1] X.-G. Xia, "On estimation of multiple frequencies in undersampled complex valued waveforms," *IEEE Trans. Signal Processing*, vol. 47, pp. 3417–3419, Dec. 1999.

[2] G. C. Zhou and X.-G. Xia, "Multiple frequency detection in undersampled complex-valued waveforms with close multiple frequencies," *Electron. Lett.*, vol. 33, pp. 1294–1295, July 1997.

[3] P. E. Pace, R. E. Leino, and D. Styer, "Use of the symmetrical number system in resolving single-frequency undersampled aliases," *IEEE Trans. Signal Processing*, vol. 45, pp. 1153–1160, May 1997.

[4] B. Friedlander and B. Porat, "VSAR: A high resolution radar system for ocean imaging," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 34, pp. 755–776, May 1998.

[5] G. Wang, X.-G. Xia, and V. C. Chen, "Multi-frequency VSAR imaging of moving targets," in *Proc. SPIE: Radar Processing, Technology, and Applications IV*, vol. 3810, Denver, CO, July 1999.

[6] X.-G. Xia, "Dynamic range determination of the detectable parameters for polynomial phase signals using multiple lag diversities in high-order ambiguity functions," in *Proc. IEEE-SP Internal. Symp. Time-Frequency and Time-Scale Analysis*, Pittsburgh, PA, Oct. 6–9, 1998.

[7] M. D. Zoltowski and C. P. Mathews, "Real-time frequency and 2-D angle estimation with sub-Nyquist spatio-temporal sampling," *IEEE Trans. Signal Processing*, vol. 42, pp. 2781–2794, Oct. 1994.

[8] J. H. McClellan and C. M. Rader, *Number Theory in Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1979.