DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

UNIVERSITY OF DELAWARE

ELEG 320 *ELECTROMAGNETIC FIELD THEORY I*

HOMEWORK SOLUTIONS #4 – DUE October 7, 2010

1. $\nabla \times E = -\frac{\partial B}{\partial t} \Rightarrow \nabla \times E = -j\omega\mu H$

$\nabla \times \vec{H} = \frac{dD}{dt} \Rightarrow \nabla \times \tilde{H} = j\omega\varepsilon \tilde{E}$

$\nabla \times (\nabla \times E) = -j\omega\mu (\nabla \times H)$

We know,

$\nabla \times (\nabla \times E) = \nabla^2 E - \nabla(\nabla \cdot E)$

$\therefore \quad \nabla^2 \tilde{E} = -j\omega\mu(-j\omega\varepsilon)\tilde{E}$

$\nabla^2 \tilde{E} = -\omega^2 \mu\varepsilon \tilde{E}$

or, $\nabla^2 \tilde{E} = -k^2 \tilde{E}$

Problem 7.4 The electric field of a plane wave propagating in a nonmagnetic material is given by

$$\mathbf{E} = [\hat{\mathbf{y}}\, 3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}}\, 4\cos(\pi \times 10^7 t - 0.2\pi x)] \quad \text{(V/m)}.$$

Determine (a) the wavelength, (b) ε_r, and (c) **H**.

Solution:

(a) Since $k = 0.2\pi$,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi} = 10 \text{ m}.$$

(b)

$$u_p = \frac{\omega}{k} = \frac{\pi \times 10^7}{0.2\pi} = 5 \times 10^7 \text{ m/s}.$$

But

$$u_p = \frac{c}{\sqrt{\varepsilon_r}}.$$

Hence,

$$\varepsilon_r = \left(\frac{c}{u_p}\right)^2 = \left(\frac{3 \times 10^8}{5 \times 10^7}\right)^2 = 36.$$

(c)

$$\begin{aligned}\mathbf{H} &= \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\eta}\hat{\mathbf{x}} \times \left[\hat{\mathbf{y}} 3\sin(\pi \times 10^7 t - 0.2\pi x) + \hat{\mathbf{z}} 4\cos(\pi \times 10^7 t - 0.2\pi x)\right] \\ &= \hat{\mathbf{z}}\frac{3}{\eta}\sin(\pi \times 10^7 t - 0.2\pi x) - \hat{\mathbf{y}}\frac{4}{\eta}\cos(\pi \times 10^7 t - 0.2\pi x) \quad \text{(A/m)},\end{aligned}$$

with

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{6} = 20\pi = 62.83 \quad (\Omega).$$

Problem 7.15 Dry soil is characterized by $\varepsilon_r = 2.5$, $\mu_r = 1$, and $\sigma = 10^{-4}$ (S/m). At each of the following frequencies, determine if dry soil may be considered a good conductor, a quasi-conductor, or a low-loss dielectric, and then calculate α, β, λ, μ_p, and η_c:

(a) 60 Hz,
(b) 1 kHz,
(c) 1 MHz,
(d) 1 GHz.

Solution: $\varepsilon_r = 2.5$, $\mu_r = 1$, $\sigma = 10^{-4}$ S/m.

$f \rightarrow$	60 Hz	1 kHz	1 MHz	1 GHz
$\dfrac{\varepsilon''}{\varepsilon'} = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{\sigma}{2\pi f \varepsilon_r \varepsilon_0}$	1.2×10^4	720	0.72	7.2×10^{-4}
Type of medium	Good conductor	Good conductor	Quasi-conductor	Low-loss dielectric
α (Np/m)	1.54×10^{-4}	6.28×10^{-4}	1.13×10^{-2}	1.19×10^{-2}
β (rad/m)	1.54×10^{-4}	6.28×10^{-4}	3.49×10^{-2}	33.14
λ (m)	4.08×10^4	10^4	180	0.19
u_p (m/s)	2.45×10^6	10^7	1.8×10^8	1.9×10^8
η_c (Ω)	$1.54(1+j)$	$6.28(1+j)$	$204.28 + j65.89$	238.27

Problem 7.22 In a nonmagnetic, lossy, dielectric medium, a 300-MHz plane wave is characterized by the magnetic field phasor

$$\tilde{\mathbf{H}} = (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y} \quad \text{(A/m)}.$$

Obtain time-domain expressions for the electric and magnetic field vectors.

Solution:

$$\widetilde{\mathbf{E}} = -\eta_c \hat{\mathbf{k}} \times \widetilde{\mathbf{H}}.$$

To find η_c, we need ε' and ε''. From the given expression for $\widetilde{\mathbf{H}}$,

$$\alpha = 2 \quad \text{(Np/m)},$$
$$\beta = 9 \quad \text{(rad/m)}.$$

Also, we are given than $f = 300$ MHz $= 3 \times 10^8$ Hz. From (7.65a),

$$\alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon',$$
$$4 - 81 = -(2\pi \times 3 \times 10^8)^2 \times 4\pi \times 10^{-7} \times \varepsilon_r' \times \frac{10^{-9}}{36\pi},$$

whose solution gives

$$\varepsilon_r' = 1.95.$$

Similarly, from (7.65b),

$$2\alpha\beta = \omega^2 \mu \varepsilon'',$$
$$2 \times 2 \times 9 = (2\pi \times 3 \times 10^8)^2 \times 4\pi \times 10^{-7} \times \varepsilon_r'' \times \frac{10^{-9}}{36\pi},$$

which gives

$$\varepsilon_r'' = 0.91.$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\,\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$
$$= \frac{\eta_0}{\sqrt{\varepsilon'_r}}\left(1 - j\,\frac{0.91}{1.95}\right)^{-1/2} = \frac{377}{\sqrt{1.95}}(0.93 + j0.21) = 256.9\,e^{j12.6^\circ}.$$

Hence,

$$\begin{aligned}
\widetilde{\mathbf{E}} &= -256.9\,e^{j12.6^\circ}\,\hat{\mathbf{y}} \times (\hat{\mathbf{x}} - j4\hat{\mathbf{z}})e^{-2y}e^{-j9y} \\
&= (\hat{\mathbf{x}}\,j4 + \hat{\mathbf{z}})\,256.9\,e^{-2y}e^{-j9y}e^{j12.6^\circ} \\
&= (\hat{\mathbf{x}}4e^{j\pi/2} + \hat{\mathbf{z}})\,256.9\,e^{-2y}e^{-j9y}e^{j12.6^\circ}, \\
\mathbf{E} &= \mathfrak{Re}\{\widetilde{\mathbf{E}}e^{j\omega t}\} \\
&= \hat{\mathbf{x}}\,1.03\times 10^3 e^{-2y}\cos(\omega t - 9y + 102.6^\circ) \\
&\quad + \hat{\mathbf{z}}\,256.9\,e^{-2y}\cos(\omega t - 9y + 12.6^\circ) \quad \text{(V/m)}, \\
\mathbf{H} &= \mathfrak{Re}\{\widetilde{\mathbf{H}}e^{j\omega t}\} \\
&= \mathfrak{Re}\{(\hat{\mathbf{x}} + j4\hat{\mathbf{z}})e^{-2y}e^{-j9y}e^{j\omega t}\} \\
&= \hat{\mathbf{x}}\,e^{-2y}\cos(\omega t - 9y) + \hat{\mathbf{z}}\,4e^{-2y}\sin(\omega t - 9y) \quad \text{(A/m)}.
\end{aligned}$$

Problem 7.31 Consider the imaginary rectangular box shown in Fig. 7-19 (P7.31).

(a) Determine the net power flux $P(t)$ entering the box due to a plane wave in air given by

$$\mathbf{E} = \hat{\mathbf{x}}E_0\cos(\omega t - ky) \quad \text{(V/m)}.$$

(b) Determine the net time-average power entering the box.

Solution:

(a)

$$\mathbf{E} = \hat{\mathbf{x}}E_0\cos(\omega t - ky),$$
$$\mathbf{H} = -\hat{\mathbf{z}}\,\frac{E_0}{\eta_0}\cos(\omega t - ky).$$
$$\mathbf{S}(t) = \mathbf{E}\times\mathbf{H} = \hat{\mathbf{y}}\,\frac{E_0^2}{\eta_0}\cos^2(\omega t - ky),$$
$$P(t) = S(t)\,A|_{y=0} - S(t)\,A|_{y=b} = \frac{E_0^2}{\eta_0}\,ac[\cos^2\omega t - \cos^2(\omega t - kb)].$$

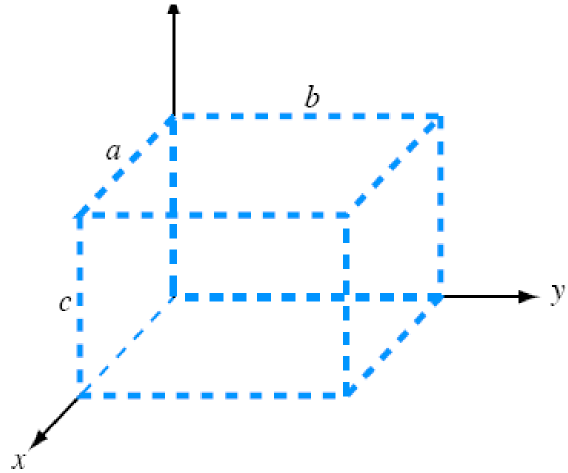

Figure P7.31: Imaginary rectangular box of Problems 7.31 and 7.32.

(b)

$$P_{\text{av}} = \frac{1}{T}\int_0^T P(t)\,dt.$$

where $T = 2\pi/\omega$.

$$P_{\text{av}} = \frac{E_0^2 ac}{\eta_0}\left\{\frac{\omega}{2\pi}\int_0^{2\pi/\omega}[\cos^2\omega t - \cos^2(\omega t - kb)]\,dt\right\} = 0.$$

Net average energy entering the box is zero, which is as expected since the box is in a lossless medium (air).

Problem 7.32 Repeat Problem 7.31 for a wave traveling in a lossy medium in which

$$\mathbf{E} = \hat{\mathbf{x}}\,100e^{-20y}\cos(2\pi\times 10^9 t - 40y) \quad \text{(V/m)},$$
$$\mathbf{H} = -\hat{\mathbf{z}}\,0.64e^{-20y}\cos(2\pi\times 10^9 t - 40y - 36.85^\circ) \quad \text{(A/m)}.$$

The box has dimensions $A = 1$ cm, $b = 2$ cm, and $c = 0.5$ cm.

Solution:

(a)

$$\begin{aligned}
\mathbf{S}(t) &= \mathbf{E}\times\mathbf{H} \\
&= \hat{\mathbf{x}}\,100e^{-20y}\cos(2\pi\times 10^9 t - 40y) \\
&\quad \times(-\hat{\mathbf{z}}\,0.64)e^{-20y}\cos(2\pi\times 10^9 t - 40y - 36.85^\circ) \\
&= \hat{\mathbf{y}}\,64e^{-40y}\cos(2\pi\times 10^9 t - 40y)\cos(2\pi\times 10^9 t - 40y - 36.85^\circ).
\end{aligned}$$

Using the identity $\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)]$,

$$\begin{aligned}
S(t) &= \frac{64}{2}e^{-40y}[\cos(4\pi\times10^9t-80y-36.85°)+\cos 36.85°],\\
P(t) &= S(t)\,A|_{y=0} - S(t)\,A|_{y=b}\\
&= 32ac\{[\cos(4\pi\times10^9t-36.85°)+\cos 36.85°]\\
&\quad - e^{-40b}[\cos(4\pi\times10^9t-80y-36.85°)+\cos 36.85°]\}.
\end{aligned}$$

(b)

$$P_{\text{av}} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)\,dt.$$

The average of $\cos(\omega t+\theta)$ over a period T is equal to zero, regardless of the value of θ. Hence,

$$P_{\text{av}} = 32ac(1-e^{-40b})\cos 36.85°.$$

With $a = 1$ cm, $b = 2$ cm, and $c = 0.5$ cm,

$$P_{\text{av}} = 7.05\times10^{-4} \quad \text{(W)}.$$

This is the average power absorbed by the lossy material in the box.

6. $\frac{d^2f}{dx^2} = -\beta_x^2 f$, $f = f_1 = A_1e^{-j\beta_x x} + B_1e^{+j\beta_x x}$

Using $f = f_1 = A_1e^{-j\beta_x x}$, then

$(-j\beta_x)^2A_1e^{-j\beta_x x} = -\beta_x^2A_1e^{-j\beta_x x} = -\beta_x A_1e^{-j\beta_x x}$

The same can be shown by letting $f = f_1 = B_1 e^{+j\beta_x x}$

Now let us try the cosinusoidal solutions.

Let $f = f_2 = C_1 \cos(\beta_x x)$

Substituting this into the differential equation leads to

$$-\beta_x^2 C_1 \cos(\beta_x) = -\beta_x^2 C_1 \cos(\beta_x x) \qquad \text{Q.E.D.}$$

The same can be shown by letting $f = f_2 = D_1 \sin(\beta_x x)$