Electromagnetic Theorems

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ELEG 648—Electromagnetic Theorems

Outline

Outline

Duality—The Main Idea

Electric Sources	Magnetic Sources
$\nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J}$	$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H} + \mathbf{M}$
$-\nabla \times \mathbf{E} = \hat{z}\mathbf{H}$	$\nabla \times \mathbf{H} = \hat{y}\mathbf{E}$
$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$	$-\mathbf{E} = \frac{1}{\epsilon}\nabla \times \mathbf{F}$
$\mathbf{E} = -j\omega\left[\mathbf{A} + \frac{1}{k^2}\nabla\left(\nabla \cdot \mathbf{A}\right)\right]$	$\mathbf{H} = -j\omega\left[\mathbf{F} + \frac{1}{k^2}\nabla\left(\nabla \cdot \mathbf{F}\right)\right]$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint \mathbf{J}(\mathbf{r})\frac{e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r}-\mathbf{r}'\rvert}\,d\mathbf{r}'$	$\mathbf{F} = \frac{\epsilon}{4\pi}\iiint \mathbf{M}(\mathbf{r})\frac{e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{\lvert\mathbf{r}-\mathbf{r}'\rvert}\,d\mathbf{r}'$

- These equations are almost the same.
- By systematically replacing one quantity with another, we can get to the right column from the left.

Duality—The Replacements

Electric Sources	Magnetic Sources
$\mathbf{E}$	$\mathbf{H}$
$\mathbf{H}$	$-\mathbf{E}$
$\mathbf{J}$	$\mathbf{M}$
$\mathbf{A}$	$\mathbf{F}$
$\hat{y}$	$\hat{z}$
$\hat{z}$	$\hat{y}$
k	k
η	$\frac{1}{\eta}$

- Why do we care?
- There is no such thing as magnetic current, right?

The First Reason

- Where there is no "magnetic current" the solution provided by the "electric vector potential" is a legitimate free space solution to the Maxwell Equations.
- We will see why this may need to be done later...

Consider:

$$\begin{aligned} \mathbf{E} &= \mathbf{E_A} + \mathbf{E_F} \\ \mathbf{H} &= \mathbf{H_A} + \mathbf{H_F} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \iiint \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}' \\ \mathbf{F}(\mathbf{r}) &= \frac{\epsilon}{4\pi} \iiint \mathbf{M}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}' \end{aligned}$$

The First Reason

As we have seen,

$$\begin{aligned}
\mathbf{E_A} &= -j\omega\mathbf{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{A} \\
\mathbf{E_F} &= -\frac{1}{\epsilon}\nabla\times\mathbf{F} \\
\mathbf{H_A} &= \frac{1}{\mu}\nabla\times\mathbf{H} \\
\mathbf{H_F} &= -j\omega\mathbf{F} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{F}
\end{aligned}$$

This is the most general solution to the Maxwell Equations in free space.

The Second Reason

Consider a loop of

- Constant current I, and
- Radius a.

We will compute the radiation of this loop for small $a \ll \lambda$. The loop can be parameterized as

$$\mathbf{r}'(\phi') = a\cos\phi'\mathbf{u}_x + a\sin\phi'\mathbf{u}_y$$

Clearly, the current will produce an azimuthally symmetric **A**, and **A** will be azimuthally directed. We can therefore seek

$$A_\phi = A_y(\phi = 0)$$

and set

$$\mathbf{r}(r,\theta) = r\sin\theta\mathbf{u}_x + r\cos\theta\mathbf{u}_z.$$

Small Current Loops

Given this,

$$\mathbf{R} = \left(r\sin\theta - a\cos\phi'\right)\mathbf{u}_x + \left(-a\sin\phi'\right)\mathbf{u}_y + \left(r\cos\theta\right)\mathbf{u}_z$$

Therefore

$$\begin{aligned} R^2 &= \left(r\sin\theta - a\cos\phi'\right)^2 + \left(a\sin\phi'\right)^2 + \left(r\cos\theta\right)^2 \\ &= r^2\sin^2\theta - 2ar\sin\theta\cos\phi' + a^2\cos^2\phi' + a^2\sin^2\phi' \\ &\quad + r^2\cos^2\theta \\ &= r^2 + a^2 - 2ar\sin\theta\cos\phi' \end{aligned}$$

Now since

$$A_y = \mu \mathbf{u}_y \cdot \int \mathbf{J}(\mathbf{r}') \frac{e^{-jkR}}{4\pi R}\, \mathrm{d}\mathbf{r}'$$

we find the

Magnetic Vector Potential of a Current Loop

$$A_\phi = \frac{\mu I a}{4\pi} \int_0^{2\pi} \frac{\exp\left(-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}\right)}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}} \cos\phi'\, \mathrm{d}\phi'$$

Let

$$f(R(a)) = \frac{\exp\left(-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}\right)}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}}$$

Small Current Loops

We are interested in small loops, so we can take the limit as $a \to 0$. We expand $f(R(a))$ in a Maclaurin series:

$$f(R(a)) \approx f(R(0)) + f'(R(0))R'(0)a$$

Now

$$\begin{aligned} R(a) &= \left[r^2 + a^2 - 2ar\sin\theta\cos\phi'\right]^{\frac{1}{2}} \\ R(0) &= r \\ R'(a) &= \tfrac{1}{2}\left[r^2 + a^2 - 2ar\sin\theta\cos\phi'\right]^{-\frac{1}{2}}\left[2a - 2r\sin\theta\cos\phi'\right] \\ R'(0) &= -\sin\theta\cos\phi' \end{aligned}$$

Similarly,

$$
\begin{aligned}
f(R) &= \frac{e^{-jkR}}{R} \\
f(r) &= \frac{e^{-jkr}}{r} \\
f'(R) &= -\left(\frac{jk}{R} + \frac{1}{R^2}\right) e^{-jkR} \\
f'(r) &= -\left(\frac{jk}{r} + \frac{1}{r^2}\right) e^{-jkr}
\end{aligned}
$$

Therefore

$$
f(R(a)) \approx \frac{e^{-jkr}}{r} + \left(\frac{jk}{r} + \frac{1}{r^2}\right) e^{-jkr} \sin\theta \cos\phi' a
$$

Given this, we compute A_ϕ

$$A_\phi \rightarrow \mu \frac{Ia}{4\pi} \int_0^{2\pi} \left[\frac{e^{-jkr}}{r} + \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} a \sin\theta \cos\phi' \right] \cos\phi' \, \mathrm{d}\phi'$$

Small Current Loops

Given this, we compute A_ϕ

$$\begin{aligned}
A_\phi &\rightarrow \mu \frac{Ia}{4\pi} \int_0^{2\pi} \left[\frac{e^{-jkr}}{r} + \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} a \sin\theta \cos\phi' \right] \cos\phi' \, d\phi' \\
&= \mu \frac{I\pi a^2}{4\pi^2} \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} \sin\theta \int_0^{2\pi} \cos^2\phi' \, d\phi' \\
&= \mu \frac{I\pi a^2}{4\pi} \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} \sin\theta \\
&= \mu \frac{IS}{4\pi} \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} \sin\theta
\end{aligned}$$

where $S = \pi a^2$ is the area of the loop. The quantity $m = IS$ is known as the magnetic moment of the loop.

Small Current Loops

Once we have computed **A**, we can compute the fields. Here we compare them to an electric dipole of moment $p = I\ell$.

Electric Dipole	Magnetic Dipole
$E_r = \frac{p\cos\theta}{2\pi} e^{-jkr} \left(\frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^2} \right)$	$H_r = \frac{m\cos\theta}{2\pi} e^{-jkr} \left(\frac{jk}{r^2} + \frac{1}{r^2} \right)$
$E_\theta = \frac{p\sin\theta}{2\pi} e^{-jkr} \left(\frac{j\omega\mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^3} \right)$	$H_\theta = \frac{m\sin\theta}{2\pi} e^{-jkr} \left(-\frac{k^2}{r} + \frac{jk}{r^2} + \frac{1}{r^3} \right)$
$H_\phi = \frac{p\sin\theta}{4\pi} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} \right)$	$E_\phi = \eta \frac{m\sin\theta}{4\pi} e^{-jkr} \left(\frac{k^2}{r} - \frac{jk}{r^2} \right)$

These equations are duals using the

Magnetic Current—Dipole Duality

$$p = jk\eta m$$

In short, we can understand magnetic currents as small loops of electric current.

Outline

Why Bother With Uniqueness?

- A uniqueness theorem tells us what information we need to get an answer.
- Under some circumstances, problems do not have unique solutions. We want to know
  - Why, and
  - What it means.
- By deploying the uniqueness theorem intelligently, we might be able to come up with alternative formulations of problems that are more useful for our purposes.

The Uniqueness Theorem

Consider a region of space V,

- filled with linear matter
- occupied by sources $\mathbf{J}$ and $\mathbf{M}$, and
- bounded by a surface S with outward normal $\mathbf{u}_\mathrm{n}$.

Suppose that two sets of fields $(\mathbf{E}^a, \mathbf{H}^a)$ and $(\mathbf{E}^b, \mathbf{H}^b)$ solve this problem. Then:

$$
\begin{aligned}
-\nabla \times \mathbf{E}^a &= \hat{z}\mathbf{H}^a + \mathbf{M} \\
\nabla \times \mathbf{H}^a &= \hat{y}\mathbf{E}^a + \mathbf{J} \\
-\nabla \times \mathbf{E}^b &= \hat{z}\mathbf{H}^b + \mathbf{M} \\
\nabla \times \mathbf{H}^b &= \hat{y}\mathbf{E}^b + \mathbf{J}
\end{aligned}
$$

Define $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$.

The Uniqueness Theorem

The difference fields solve the source-free Maxwell Equations:

$$\begin{aligned} -\nabla \times \delta\mathbf{E} &= \hat{z}\mathbf{H} \\ \nabla \times \delta\mathbf{H} &= \hat{y}\mathbf{E} \end{aligned}$$

We now apply Poynting's Theorem to these equations to find the

Uniqueness Theorem

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S} + \iiint_V \left(\hat{z}|\delta\mathbf{H}|^2 + \hat{y}^*|\delta\mathbf{E}|^2\right) dv = 0$$

Huh??? How is this a uniqueness theorem???

An Explanation

If we can show that

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S} = 0$$

then we can conclude that

$$\iiint_V \left(\hat{z}|\delta\mathbf{H}|^2 + \hat{y}^*|\delta\mathbf{E}|^2\right) dv = 0$$

The real part of this equation is

$$\iiint_V \left[\text{Re}(\hat{z})|\delta\mathbf{H}^2| + \text{Re}(\hat{y})|\delta\mathbf{E}^2|\right] dv = 0$$

If we are in a lossy medium, $\text{Re}(\hat{z}) > 0$ and $\text{Re}(\hat{y}) > 0$, and we can conclude

$$\delta\mathbf{E} = \delta\mathbf{H} = 0.$$

More Explanation

So, for the moment, let's assume a lossy medium. What does the condition

$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \mathbf{u}_{\mathrm{n}} \mathrm{d}S = 0$$

mean?

More Explanation

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mean?
Suppose we specify the tangential electric field $\mathbf{u}_\mathrm{n} \times \mathbf{E}$ over S. Then

- $\mathbf{u}_\mathrm{n} \times \delta\mathbf{E} = 0$

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- $\mathbf{u}_\mathrm{n} \times \delta\mathbf{E} = 0$
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More Explanation

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mean?

Suppose we specify the tangential electric field $\mathbf{u}_\mathrm{n} \times \mathbf{E}$ over S. Then

- $\mathbf{u}_\mathrm{n} \times \delta\mathbf{E} = 0$
- $(\mathbf{u}_\mathrm{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* = 0$, and
- $(\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \mathbf{u}_\mathrm{n} = 0$

Uniqueness So Far

Thus, if we are in a volume V bounded by a surface S which contains some lossy matter, the solution is unique if we specify

- The tangential electric field over S,
- The tangential magnetic field over S, or
- The tangential electric field over part of S and the tangential magnetic field over the rest of S.

Infinite surfaces can be thought of as the limit of finite surfaces, so there is really no problem there if we specify the field vanishes (or is at least outward traveling) at ∞.

Uniqueness for Lossless Regions

Why do we need loss to prove uniqueness???

Consider an enclosed metal box. It resonates at certain frequencies so...

Uniqueness for Lossless Regions

Why do we need loss to prove uniqueness???

Consider an enclosed metal box. It resonates at certain frequencies so...

An Ugly Fact

A field can be sustained inside without any excitation!

This field can be multiplied by an arbitrary constant and added to any other solution inside the box!!

In general, we take the solution in lossless cases to be the limit of the lossy case. Nonetheless, this bizarre fact causes problem in computational electromagnetics...

Outline

The Field of a Dipole in Free Space

We have seen that a z-directed dipole in free space radiates

Dipole Fields

$$
\begin{aligned}
E_r &= \frac{I\ell\cos\theta}{2\pi} e^{-jkr}\left(\frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^2}\right) = f_r(r)\cos\theta \\
E_\theta &= \frac{I\ell\sin\theta}{2\pi} e^{-jkr}\left(\frac{j\omega\mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega\epsilon r^3}\right) = f_\theta(r)\sin\theta \\
H_\phi &= \frac{I\ell\sin\theta}{4\pi} e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2}\right)
\end{aligned}
$$

Let us consider a dipole over a PEC plane.

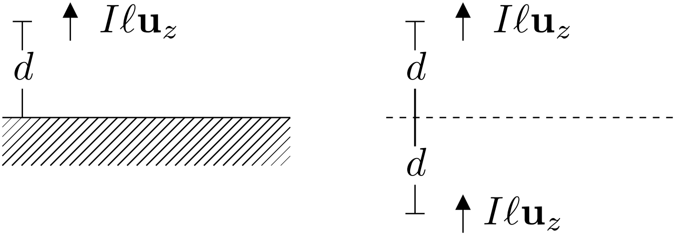


Image theory states that the picture on the left may be replaced by that on the right as far as the field above the plane is concerned. This requires that the resultant field:

1. Solve Maxwell's Equations above the plane, and
2. Satisfy the boundary condition on the PEC.

Proof of Image Theory: Maxwell's Equations

- Refer to the source above the plane and the fields it generates with the subscript "1"
- Refer to the source below the plane and the fields it generates with the subscript "2"
- Subscript free fields are total fields.

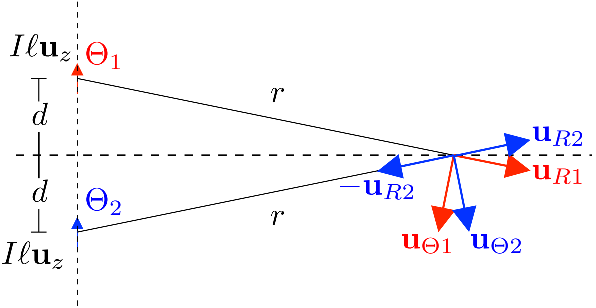
Then above the plane the fields satisfy

$$\begin{aligned} \nabla \times \mathbf{H}_1 &= \hat{y}\mathbf{E}_1 + \mathbf{J}_1 \\ \nabla \times \mathbf{H}_2 &= \hat{y}\mathbf{E}_2 \end{aligned}$$

Summing these equations gives

$$\nabla \times \mathbf{H} = \hat{y}\mathbf{E} + \mathbf{J}_1$$

The other equations are similar, so Maxwell's equations are satisfied.



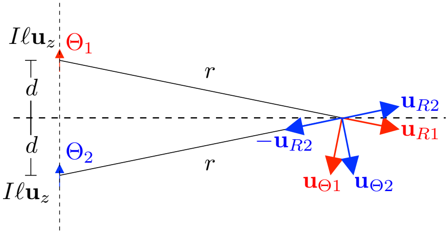
Now

$$\Theta_2 = \pi - \Theta_1$$

so

$$\begin{aligned} \sin\Theta_2 &= \quad \sin\pi\cos\Theta_1 - \cos\pi\sin\Theta_1 \quad &= \sin\Theta_1 \\ \cos\Theta_2 &= \quad \cos\pi\cos\Theta_1 + \sin\pi\sin\Theta_1 \quad &= -\cos\Theta_1 \end{aligned}$$

Proof of Image Theory: Boundary Conditions



$$\begin{aligned}
\mathbf{E}_1 &= f_r(r)\cos\Theta_1\mathbf{u}_{R1} + f_\theta(r)\sin\Theta_1\mathbf{u}_{\Theta 1} \\
&= a\mathbf{u}_{R1} + b\mathbf{u}_{\Theta 1} \\
\mathbf{E}_2 &= f_r(r)\cos\Theta_2\mathbf{u}_{R2} + f_\theta(r)\sin\Theta_2\mathbf{u}_{\Theta 2} \\
&= -a\mathbf{u}_{R2} + b\mathbf{u}_{\Theta 2}
\end{aligned}$$

Thus, the field is normal to the conductor. $\square$

Image Theory Summary

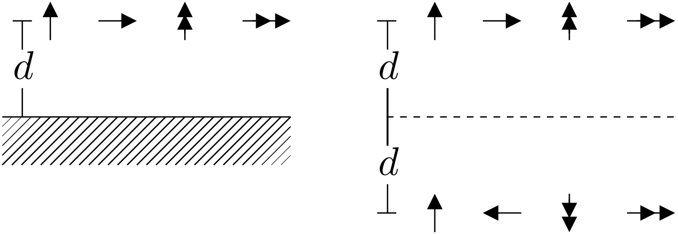
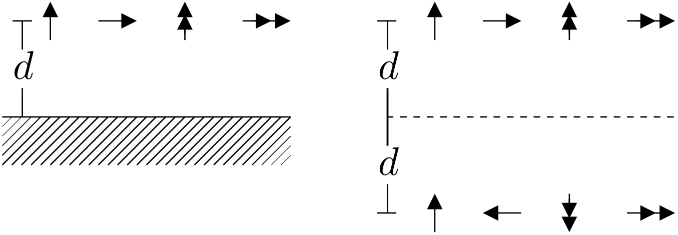


Image Theory Summary

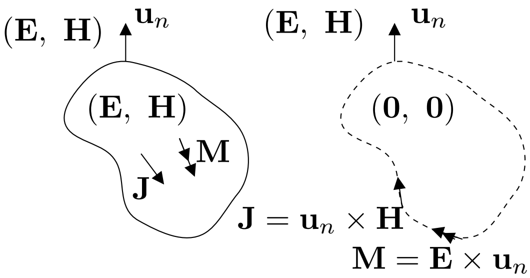


Of course, conductors and dielectrics are also imaged. Why?

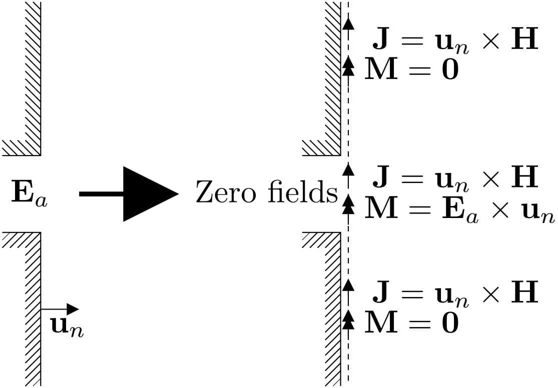
Outline

The Equivalence Principle

- Very often we are interested in finding alternative sources that produce a given field.
- The uniqueness theorem provides a way to do this.

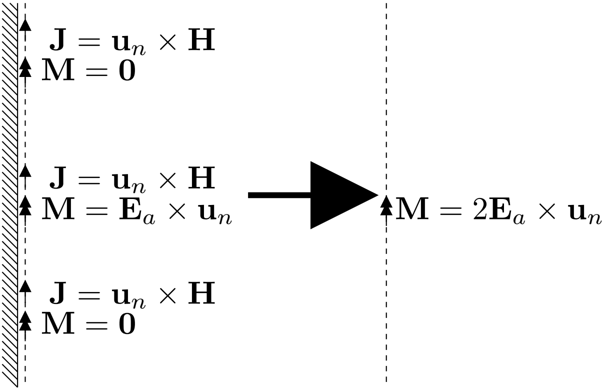


How do we know the fields outside the surface on the left are identical to fields on the right?



We can simplify radiation from an aperture with the equivalence principle.

Aperture Antennas



- We can alter the zero-field region to
- Apply image theory...

Thus, if we let

$$\mathbf{M} = \mathbf{E}_a \times \mathbf{u}_\mathrm{n}$$

we can write

The Fields

$$\begin{aligned}
\mathbf{F}(\mathbf{r}) &= \frac{\epsilon_0}{2\pi} \iint_a \mathbf{M}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dS \\
\mathbf{H}(\mathbf{r}) &= -\frac{1}{\epsilon_0} \nabla \times \mathbf{F}(\mathbf{r}) \\
\mathbf{E}(\mathbf{r}) &= -j\omega \mathbf{F}(\mathbf{r}) + \frac{1}{j\omega\mu_0\epsilon_0} \nabla(\nabla \cdot \mathbf{F}(\mathbf{r}))
\end{aligned}$$

The Induction Theorem

Consider a conductive (PEC) scatterer illuminated by a source. Define the

Incident field, $\mathbf{E}^i$ as the field that would exist in the absence of the scatterer.

Total field, $\mathbf{E}$ as the total field that exists in the presence of the scatter.

Scattered field, $\mathbf{E}^s$ as the field that must therefore be due to the scatterer.

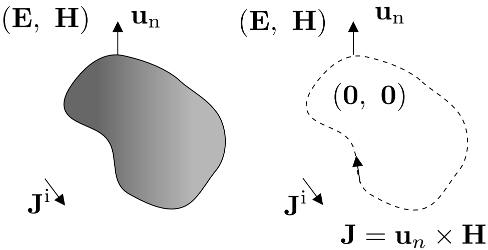
In short, the scattered field is defined to be

Scattered Field Definition

$$\mathbf{E}^s \triangleq \mathbf{E} - \mathbf{E}^i$$

The Induction Theorem

The equivalence principle can be applied to this problem resulting in the following picture.



Here, we clearly see that $\mathbf{E}^s$ is due to $\mathbf{J}$.

The Electric Field Integral Equation

Thus,

$$\mathbf{E}^{\mathrm{s}} = -j\omega \mathbf{A}(\mathbf{r}) - \nabla \phi(\mathbf{r})$$

where

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \iint_S \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}S \\ \phi(\mathbf{r}) &= \frac{-1}{4\pi j\omega\epsilon} \iint_S \nabla' \cdot \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}S \end{aligned}$$

Now, on S, we must have

Satsifaction of Boundary Conditions

$$\mathbf{E}^{\mathrm{i}} = -\mathbf{E}^{\mathrm{s}}\Big|_{\text{Tan to } s}$$

The Electric Field Integral Equation

Defining $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = |\mathbf{R}|$, and noting

$$\nabla\left(\frac{e^{-jkR}}{R}\right) = -e^{-jkR}\frac{1+jkR}{R^2}\frac{\mathbf{R}}{R}$$

we have the

Electric Field Integral Equation

$$\begin{aligned}\mathbf{E}^{\mathrm{i}} = {} & \frac{j\omega\mu}{4\pi}\iint_S \mathbf{J}(\mathbf{r}')\frac{e^{-jkR}}{R}\,\mathrm{d}S \\ & + \frac{1}{4\pi j\omega\epsilon}\iint_S \nabla'\cdot\mathbf{J}(\mathbf{r}')e^{-jkR}\frac{1+jkR}{R^2}\frac{\mathbf{R}}{R}\,\mathrm{d}S\bigg|_{\text{Tan to } s}\end{aligned}$$

This is the starting point for computational methods based on integral equations.

Outline

The Volume Equivalence Principle

- The surface equivalence principle is only helpful where homogeneous media are concerned.
- This is because finding the radiation of a point source (the Green's Function) in inhomogeneous media is nearly impossible.
- To cope we introduce a volume equivalence principle.

Consider (recalling that ϵ and μ are functions of position):

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} - \mathbf{M}^{\mathrm{i}} \\ \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} + \mathbf{J}^{\mathrm{i}} \end{aligned}$$

The Volume Equivalence Principle

These equations can be rewritten as

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega\mu_0\mathbf{H} - j\omega(\mu - \mu_0)\mathbf{H} - \mathbf{M}^{\mathrm{i}} \\ \nabla \times \mathbf{H} &= j\omega\epsilon_0\mathbf{E} + j\omega(\epsilon - \epsilon_0)\mathbf{E} + \mathbf{J}^{\mathrm{i}}\end{aligned}$$

or

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega\mu_0\mathbf{H} - \mathbf{M}^{\mathrm{eq}} - \mathbf{M}^{\mathrm{i}} \\ \nabla \times \mathbf{H} &= j\omega\epsilon_0\mathbf{E} + \mathbf{J}^{\mathrm{eq}} + \mathbf{J}^{\mathrm{i}}\end{aligned}$$

with

Equivalent Currents

$$\mathbf{M}^{\mathrm{eq}} \triangleq j\omega(\mu - \mu_0)\mathbf{H} \qquad \mathbf{J}^{\mathrm{eq}} \triangleq j\omega(\epsilon - \epsilon_0)\mathbf{E}$$

The point of this is that these currents, radiating in free space, create the same field as the scatterer.

Volume Integral Equations

Consider an inhomogeneous scatterer in free space. Inside the scatterer, we have

$$\mathbf{E} = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{s}}$$

- $\mathbf{E}^{\mathrm{i}}$ is known, by definition.
- $\mathbf{E}^{\mathrm{s}}$ can be computed if the equivalent current $\mathbf{J}^{\mathrm{eq}}$ is known.
- $\mathbf{E} = \frac{\mathbf{J}^{\mathrm{eq}}}{j\omega(\epsilon - \epsilon_0)}$

Thus, the above is an integral equation for the equivalent current.

Outline

The Main Idea

- Reciprocity theorems state that the response of a system is unchanged when source and measurement are exchanged.
- More generally, they deal with the reaction of on set of sources to the fields of another set.

To get a mathematical statement:

- Call the two sets of sources $\mathbf{J}^a, \mathbf{M}^a$ and $\mathbf{J}^b, \mathbf{M}^b$
- Let the fields of each set of sources operating along be $\mathbf{E}^a, \mathbf{H}^a$ and $\mathbf{E}^b, \mathbf{H}^b$

Reciprocity

$$\nabla \times \mathbf{H}^a = \hat{y}\mathbf{E}^a + \mathbf{J}^a \qquad \nabla \times \mathbf{H}^b = \hat{y}\mathbf{E}^a + \mathbf{J}^b$$

$$-\nabla \times \mathbf{E}^a = \hat{z}\mathbf{H}^a + \mathbf{M}^a \qquad -\nabla \times \mathbf{E}^b = \hat{z}\mathbf{H}^a + \mathbf{M}^b$$

Dot

- The first equation with $\mathbf{E}^b$, and
- The last equation with $\mathbf{H}^a$

and subtract. This gives

$$-\nabla \cdot \left(\mathbf{E}^b \times \mathbf{H}^a\right) = \hat{y}\mathbf{E}^a \cdot \mathbf{E}^b + \hat{z}\mathbf{H}^a \cdot \mathbf{H}^b + \mathbf{E}^b \cdot \mathbf{J}^a + \mathbf{H}^a \cdot \mathbf{M}^b$$

Switching a and b,

$$-\nabla \cdot \left(\mathbf{E}^a \times \mathbf{H}^b\right) = \hat{y}\mathbf{E}^b \cdot \mathbf{E}^a + \hat{z}\mathbf{H}^b \cdot \mathbf{H}^a + \mathbf{E}^a \cdot \mathbf{J}^b + \mathbf{H}^b \cdot \mathbf{M}^a$$

Reciprocity

Subtracting given the

General Reciprocity Theorem (Differential Form)

$$-\nabla\cdot\left(\mathbf{E}^a\times\mathbf{H}^b-\mathbf{E}^b\times\mathbf{H}^a\right)=\mathbf{E}^a\cdot\mathbf{J}^b-\mathbf{H}^a\cdot\mathbf{M}^b-\mathbf{E}^b\cdot\mathbf{J}^a+\mathbf{H}^b\cdot\mathbf{M}^a$$

Integrating over an arbitrary volume gives the

General Reciprocity Theorem (Integral Form)

$$\oiint\left(\mathbf{E}^a\times\mathbf{H}^b-\mathbf{E}^b\times\mathbf{H}^a\right)\cdot d\mathbf{S}=\iiint\left(\mathbf{E}^a\cdot\mathbf{J}^b-\mathbf{H}^a\cdot\mathbf{M}^b-\mathbf{E}^b\cdot\mathbf{J}^a+\mathbf{H}^b\cdot\mathbf{M}^a\right)dv$$

Outline

The Lorentz Reciprocity Theorem

Where there are no sources we have the

Lorentz Reciprocity Theorem

$$\nabla \cdot \left(\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a \right) = 0$$

which becomes

Lorentz Reciprocity Theorem

$$\oiint \left(\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a \right) \cdot d\mathbf{S} = 0$$

in integral form.

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in integral form.
There is no currently known use for this theorem.

Whole Space Reciprocity

Far from all sources and matter, we have seen that the field has the property

$$\mathbf{E} = \eta_0 \mathbf{H} \times \mathbf{u}_r.$$

Consider applying reciprocity to a large sphere S_a of radius a. Now,

$$\lim_{a\to\infty} \oiint_{S_a} \left(\mathbf{E}^{\mathrm{a}} \times \mathbf{H}^{\mathrm{b}} - \mathbf{E}^{\mathrm{b}} \times \mathbf{H}^{\mathrm{a}}\right) \cdot d\mathbf{S}$$

$$= \eta_0 \lim_{a\to\infty} \oiint_{S_a} \left[(\mathbf{H}^{\mathrm{a}} \times \mathbf{u}_r) \times \mathbf{H}^{\mathrm{b}} - \left(\mathbf{H}^{\mathrm{b}} \times \mathbf{u}_r\right) \times \mathbf{H}^{\mathrm{a}}\right] \cdot d\mathbf{S}$$

$$= \eta_0 \lim_{a\to\infty} \oiint_{S_a} \left[\mathbf{u}_r \left(\mathbf{H}^{\mathrm{a}} \cdot \mathbf{H}^{\mathrm{b}}\right) - \mathbf{H}^{\mathrm{a}} \left(\mathbf{u}_r \cdot \mathbf{H}^{\mathrm{b}}\right) - \mathbf{u}_r \left(\mathbf{H}^{\mathrm{a}} \cdot \mathbf{H}^{\mathrm{b}}\right) + \mathbf{H}^{\mathrm{b}} \left(\mathbf{u}_r \cdot \mathbf{H}^{\mathrm{a}}\right)\right] \cdot d\mathbf{S}$$

$$= 0$$

Whole Space Reciprocity

Thus, considering all of space, we have

The Reciprocity Theorem

$$\iiint \left(\mathbf{E}^{\mathrm{a}} \cdot \mathbf{J}^{\mathrm{b}} - \mathbf{H}^{\mathrm{a}} \cdot \mathbf{M}^{\mathrm{b}}\right) \mathrm{d}v = \iiint \left(\mathbf{E}^{\mathrm{b}} \cdot \mathbf{J}^{\mathrm{a}} - \mathbf{H}^{\mathrm{b}} \cdot \mathbf{M}^{\mathrm{a}}\right) \mathrm{d}v$$

We call this

Reaction

$$\langle a, b\rangle = \iiint \left(\mathbf{E}^{\mathrm{a}} \cdot \mathbf{J}^{\mathrm{b}} - \mathbf{H}^{\mathrm{a}} \cdot \mathbf{M}^{\mathrm{b}}\right) \mathrm{d}v$$

and write

The Reciprocity Theorem

$$\langle a, b\rangle = \langle b, a\rangle$$

Reciprocity and Circuits

For a current source b

$$\langle a, b \rangle = \int \mathbf{E}^{\mathrm{a}} \cdot I^{\mathrm{b}}\, \mathrm{d}\mathbf{l} = I^{\mathrm{b}} \int \mathbf{E}^{\mathrm{a}} \cdot \mathrm{d}\mathbf{l} = -V^{\mathrm{a}} I^{\mathrm{b}}$$

By similar reasoning, for a voltage source b

$$\langle a, b \rangle = V^{\mathrm{b}} I^{\mathrm{a}}$$

Consider now a two-port network. Such a network can be characterized by a

Impedance Matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Reciprocity and Circuits

Let V_{ij} bet the voltage at port i due to a source at port j with all other ports open circuited. Then

$$Z_{ij} = \frac{V_{ij}}{I_j}$$

Now, we have seen that, in general

$$\langle j, i \rangle = -V_{ij} I_i$$

Thus, we have

Symmetry of the Impedance Matrix

$$Z_{ij} = -\frac{\langle j, i \rangle}{I_i I_j} = -\frac{\langle i, j \rangle}{I_i I_j} = Z_{ji}$$

More Important Applications of Reciprocity

Nothing on the previous slide restricted the result to

1. Two ports, or
2. "Circuits"

Indeed, our circuit might be composed of two antennas. Then reciprocity tells us that if we put a current source in the terminals of one antenna, and a voltmeter in the terminals of the other, the reading on the voltmeter does not change if we switch them.

Reciprocity also demonstrates that antennas next to perfect conductors do not radiate. (Why?)

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Reciprocity also demonstrates that antennas next to perfect conductors do not radiate. (Why?) This is true no matter how hard the military funding agencies wish otherwise.