$\{0,1\}$-Matrices: The Four Russians and the Mailman

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I am a proponent of LinBox,FFlas/FFpack, Givaro
C++ template libraries for exact linear algebra
Google: "Linbox team" to reach github project

ZO and ZOMO

$\{0,1\}$- and $\{0,1,-1\}$-matrices are ubiquitous.

- Graph adjacency matrix is ZO.
- Graph Laplacian is ZO + D.
- Boundary matrices of simplicial complex are ZOMO.
- Any matrix over GF2 is ZO, over GF3 is ZOMO.
- Many relations are expressed as ZO incidence matrices.
- ZO + very sparse is also seen in practice.
- Block Wiedemann gives opening to use ZO or ZOMO as projectors.

Matrix Multiplication

$$C = AB$$

$$\left(m \times p\right) = \left(m \times n\right) * \left(n \times p\right)$$

Using indices i, j, k in the dimensions m, n, p, respectively.

Definition: of matrix multiplication is that the i, j entry of C is the dot product of the i-th row of A times the j-th column of B.

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Definition: of matrix multiplication is that the i, j entry of C is the dot product of the i-th row of A times the j-th column of B. In the standard three nested loop presentation this is

for i in $[1..m]$
  for k in $[1..p]$
    for j in $[1..n]$
      $c_{i,k} = c_{i,k} + a_{i,j} b_{j,k}$.

Square Matrix Multiplication

- Matrix multiplication costs $O(n^3)$, classical.
- Matrix multiplication costs $O(n^{2.81})$, Strassen.
- Matrix multiplication costs $O(n^{2.38})$, in theory.
- Matrix multiplication costs $O(n^3/\lg(n))$ over GF2, method of 4 Russians.

Square matrix multiplication

BLAS gemm is really fast. How fast?

n	naive	blas	speedup
50	8.1e-05	6e-06	13.5
100	0.000848	3.2e-05	26.5
500	0.124	0.0025	49.6
1000	1.002	0.018	55.7
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How is it done?

Not by reduction in number of field operations but by attention to hardware (caches, pipelines, simd instructions, etc.)

Block Wiedemann algorithm

Matrix A is $n \times n$, $b \ll n$

- $n \times b$: $V_i = A^i V$, right projection
- $b \times b$: $S_i = UA^i V$, left projection
- $S_i \longrightarrow$ SigmaBasis $\longrightarrow$ MatrixMinpoly
- MatrixMinpoly $\longrightarrow$(whp) leading Frobenius invariants, particularly minpoly, perhaps charpoly.
- minpoly $\longrightarrow$ solve nonsingular,
  charpoly $\longrightarrow$ determinant
  (perhaps) leading invariants $\longrightarrow$ rank, nullspace,

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Block Wiedemann dominant steps

Matrix A is $n \times n$, $b \ll n$
$V_0 = V$ is $n \times b$, random. U is $b \times n$.
Wiedemann: $b = 1$, repeat $2n$ times:

1. $V_i = AV_{i-1}$
2. $s_i = UV_i$, s_i are scalars.

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(same number of A $\times$ column vector in steps 1.)

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(same number of A $\times$ column vector in steps 1. And simd instruction parallelism available!)
Wait, step 2 costs more. Does it have to? Make it $\{0,1\}$ or even $(I_b, I_b, \ldots, I_b)$.

focus on row operations in B and C

The order of the loops may be changed and a useful form is when the inner loop is ranging across a rows of B and C:

for i in [1..m]
  for j in [1..n]
    for k in [1..p]
      $c_{i,k} = c_{i,k} + a_{i,j} b_{j,k}.$

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  for j in [1..n]
    for k in [1..p]
      $c_{i,k} = c_{i,k} + a_{i,j} b_{j,k}$.

If A is a $\{0,1\}$-matrix, the inner loop is row addition.

for i in [1..m]
  for j in [1..n]
    if $a_{i,j} = 1$ then $C_i = C_i + B_j$.

Two ways to focus on row operations

for j in [1..n]
  for i in [1..m]
    if $a_{i,j} = 1$ then $C_i = C_i + B_j$.

or

for i in [1..m]
  for j in [1..n]
    if $a_{i,j} = 1$ then $C_i = C_i + B_j$.

Two methods to speed up multiplication by $\{0,1\}$ or $\{0,1,-1\}$ matrix.

V. Arlazarov, E. Dinic, M. Kronrod, I. Faradev, "On economical construction of the transitive closure of a directed graph", Dokl. Akad. Nauk SSSR, 194 (11). Original in Russian in Dokl. Akad. Nauk SSSR, 134 (3), 1970.

E. Liberty and S. W. Zucker, "The Mailman algorithm: A note on matrix - vector multiplication", Information Processing Letters, Volume 109 (3) 2009.

They have dual structures and complementary strengths vis a vis matrix shape.

4 Russians: look at column(s) of A

$$\begin{pmatrix} \\ C_2 += B_j \\ C_3 += B_j \\ \\ \end{pmatrix} = \begin{pmatrix} & 0 & \\ & 1 & \\ & 1 & \\ & 0 & \end{pmatrix} \times \begin{pmatrix} \\ \\ B_j \\ \\ \end{pmatrix}$$

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$$\begin{pmatrix} C_1 += B_{j+1} \\ C_2 += (B_j + B_{j+1}) \\ C_3 += B_j \\ C_4 += (B_j + B_{j+1}) \\ \\ \end{pmatrix} = \begin{pmatrix} & 01 & \\ & 11 & \\ & 10 & \\ & 11 & \\ & 00 & \end{pmatrix} \times \begin{pmatrix} \\ \\ B_j \\ B_{j+1} \\ \\ \end{pmatrix}$$

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$m + 1$ row adds instead of $2m$ row adds.

Mailman: look at row(s) of A

$$\begin{pmatrix} \\ \\ \\ C_4 = B_2 + B_3 \\ \\ \end{pmatrix} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 1 & 1 & 0 & 0 \\ & & & & \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

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$$\begin{pmatrix} \\ \\ \\ C_4 = B_2 + B_3 \\ \\ \end{pmatrix} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 1 & 1 & 0 & 0 \\ & & & & \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

$$\begin{pmatrix} \\ \\ \\ C_4 = B_1 + (B_2 + B_5) \\ C_5 = B_3 + (B_2 + B_5) \end{pmatrix} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

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$m + 2$ row adds instead of $2m$ row adds.

Back to four Russians

$$\begin{pmatrix} \\ C_2 {+}{=} B_j \\ C_3 {+}{=} B_j \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \\ \\ B_j \\ \\ \end{pmatrix}$$

$$\begin{pmatrix} C_1 {+}{=} B_{j+1} \\ C_2 {+}{=} (B_j + B_{j+1}) \\ C_3 {+}{=} B_j \\ C_4 {+}{=} (B_j + B_{j+1}) \\ \\ \end{pmatrix} = \begin{pmatrix} 01 \\ 11 \\ 10 \\ 11 \\ 00 \end{pmatrix} \times \begin{pmatrix} \\ \\ B_j \\ B_{j+1} \\ \\ \end{pmatrix}$$

t columns

Build table of 2^t B-row sums.

$$\begin{aligned} &\dots \\ T_{101} &= T_{001} + B_3 = B_1 + B_3 \\ T_{110} &= T_{010} + B_3 = B_2 + B_3 \\ T_{111} &= T_{011} + B_3 = B_1 + B_2 + B_3 \end{aligned}$$

Using table, sweep down col panel of A to update C row by row.

$$\begin{pmatrix} C_1 {+}{=} T_{110} \\ C_2 {+}{=} T_{010} \\ C_3 {+}{=} T_{101} \\ C_4 {+}{=} T_{110} \\ C_5 {+}{=} T_{011} \\ \dots \end{pmatrix} = \begin{pmatrix} 110 & & \\ 010 & & \\ 101 & & \\ 110 & & \\ 011 & & \\ \dots & & \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \\ \\ \\ \end{pmatrix}$$

Four Russians analysis

A is a $m \times n$ zero-one matrix.
Panel width is t.
The following two steps must be done n/t times:

1. Table construction, costing 2^t row additions ($2^t - t - 1$ to be precise).
2. Use table to put row combinations into C, costing m row adds.

$$\text{total cost in row additions is } mn/t + n2^t/t).$$

Back to Mailman

$$\begin{pmatrix} \\ \\ \\ C_4 = B_2 + B_3 \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \\ 0 & 1 & 1 & 0 & 0 \\ \\ \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

$$\begin{pmatrix} \\ \\ \\ C_4 = B_1 + B_2 + B_5 \\ C_5 = B_3 + B_2 + B_5 \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ \\ \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

t rows

Build table of 2^t B-row sums. Each row of B goes in exactly one sum, indexed by the pattern of C rows to which it contributes.. For instance, with $t = 3$, T_{101} includes B_j when B_j contributes to C_1 and C_3, but not C_2. Next, for each C_i, combine the entries of T that are sums that contribute to C_i (all those T entries for indices with i-th bit on.)

$$\begin{pmatrix} T_{001} + T_{011} + T_{101} + T_{111} \\ T_{010} + T_{011} + T_{110} + T_{111} \\ T_{100} + T_{101} + T_{110} + T_{111} \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ \\ \\ \\ \end{pmatrix} \times \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}$$

Table handling

$T[000]$
$T[001]$
$T[010]$
$T[011]$
$T[100]$
$T[101]$
$T[110]$
$T[111]$
Add last 4 entries to C_3.

Table handling

$T[000]$
$T[001]$
$T[010]$
$T[011]$
$T[100]$
$T[101]$
$T[110]$
$T[111]$
Add last 4 entries to C_3. Also add them to the first four entries.
$T[*00] = T[000] + T[100]$
$T[*01] = T[001] + T[101]$
$T[*10] = T[010] + T[110]$
$T[*11] = T[011] + T[111]$

Mailman analysis

A is a $m \times n$ zero-one matrix.
Panel width is t.
The following two steps must be done m/t times:

1. Build table using n row additions.
2. Use table to row combinations into C at cost 2×2^t row ops.

$$\text{total cost in row ops is } mn/t + 2m2^t/t).$$

Mailman analysis

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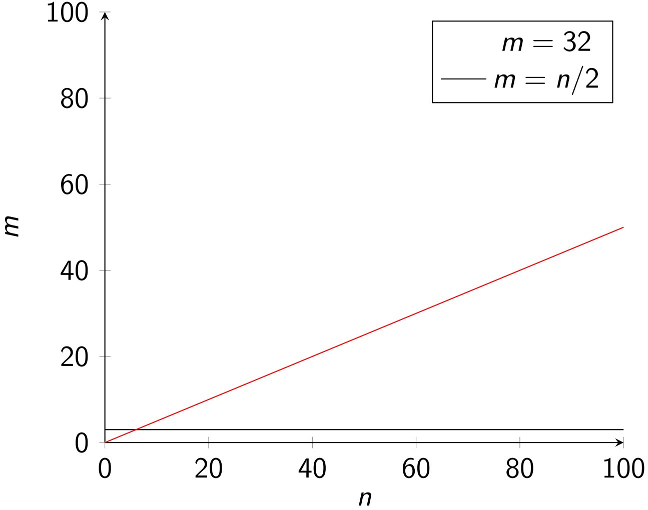
total cost in row ops is $mn/t + 2m2^t/t)$.
Compare 4 Russians: $mn/t + n2^t/t$.

Map for choosing method



Timing over Z_{10003}

A = time of 8×80000 ZO matrix times 80000×1000 matrix

vs

B = time of 8 reps of 1×80000 dense vector times 80000×1000 matrix.

11-fold speedup $B/A = 11$.

$C = B$ with ZO vector, speedup $C/A \approx 7$.

Example: Solve nonsingular system

1. Minpoly via Block Wiedemann using $U \in ZO^{4 \times n}$ and rational (poly) linear system solve with random rhs.
   m(x) $= \sum_{i=0}^{d} m_i x^i$. [$2d$ mv's]
2. $x = (-1/m_0 \sum_{i=1}^{d} m_i A^{i-1} b$. [$d - 1$ mv's]
3. Check $Ax = b$ [1 mv]. Go to 1 if fail, else return x.

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3. Check $Ax = b$ [1 mv]. Go to 1 if fail, else return x.

Block Wiedemann is faster than $b = 1$ Wiedemann because of simd in mv's and Mailman in panel products and tiny block size. Probability of success is adequate. Expected number of repetitions is $1 + \epsilon$. C

Method duality

	4 Russians	Mailman
matrix use in kernel		
A	$m \times t$ (few cols) each row a t bit index	$t \times n$ (few rows) each col a t bit index
C	update all rows	write t rows, done with
B	read t rows, done with	reread all rows
The table's two phases		
build it use it	$B \longrightarrow T$, indep of A scan A building is overhead	scan A $T \longrightarrow C$, indep of A using is overhead