CISC 320 Algorithms: hw1 solutions to problems 5 and 7

5. (2-2) What is the value returned by the following function? What is the runtime in big-O?

```
int pesky(int n) {
int r = 1;
for (int i = 1; i <= n; ++i)
for (int j = 1; j <= i; ++j)
for (int k = j; k <= j+i; ++k)
r += 1;
return r;
}
```

The innermost loop (indexed by k) increases r by 1 a total of i times, so the loop could be replaced by `r += i;`. Then the second loop (indexed by j) increases r by i a total of i times, so can be replaced by `r += i^2;`. Finally the outer loop (indexed by i) increases r by i^2 for i running from 1 to n. [Remark: For the first time the body of the loop depends on the index of the loop.] This is problem 1-11 which we solved in class using induction. The loop computes $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$, which is $\mathrm{O}(n^3)$.

The runtime has the same $\mathrm{O}(n^3)$ complexity since the total number of steps is proportional to the number of times 1 is added to r. To be pedantic about it, the loop index k is compared and incremented for each increment of r, thus directly in proportion. The other loop index steps occur less often. QED.

7. Show n^2 is $\mathrm{O}(2^n)$.

We all sort of know that 2^n grows much faster than n^2 (exponential vs quadratic growth). How can we show it using only elementary algebra?

I suggest using proof by induction (but depending on the $n/2$ case, not the $n-1$ case in the inductive step).

I'll show that $n^2 \leq 2^n$ (ie. I'll use C = 1).

Base cases:

n	n^2	2^n
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128

We see that the assertion is true for $n = 4$ through 7. My plan is that the inductive step for n will be based on a previous case of $n/2$ or $(n+1)/2$, whichever is an integer. Thus we'll have a basis for induction starting at $n = 8$ and using the inequality starting at $k = 4$.

Inductive step: Suppose $n = 2k$ is even. Then
$n^2 = 4 \times k^2 \leq 4 \times 2^k$ (using inductive hyp.)
$= 2^{k+2}$
$< 2^{2k}$, if $k \geq 2$. Done with the even n case.

Now suppose n is odd, $n = 2k - 1$.
$n^2 = 4k^2 - 4k + 1 \leq 4 \times 2^k - 4k + 1$ (by inductive hyp.)
$< 2^{k+2}$ (using that $4k - 1$ is pos for pos integer k.)
$< 2^{2k-1}$, if $k \geq 3$. Done with the odd n case.

QED.