C1 **Theory** (25 points)

Let N denote the set of non-negative integers.

Let $\Sigma = \{a, b\}$.

In this problem, C1, we will make reference to the languages L_1, L_2, and L_3 defined just below.

Definition

(i) $L_1 \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \textit{in at least one prefix of } w \textit{ there are exactly 4 more } b\textit{'s than } a\textit{'s}\}$.[1]

(ii) $L_2 \stackrel{\text{def}}{=} \{a^i b^j \mid i, j \in N \ \wedge \ i \neq j\}$.

(iii) $L_3 \stackrel{\text{def}}{=}$ *the set of all $w \in \Sigma^*$ such that every other symbol of w is an a, but starting at the first symbol if $|w|$ is odd and starting at the second symbol for $|w|$ even.*[2]

a. (8 points)
Explicitly use Myhill-Nerode to show that L_1 is *not* regular.
For credit, you must use Myhill-Nerode for this problem, C1(a).

b. (8 points)
Explicitly use the Hint below for this problem, C1(b), and use the Generalized Pumping Lemma (GPL) *directly* to show that L_2 is not regular. *No fair* working with $\overline{L_2}$ or some such.

Hint for C1(b): Consider the word $a^n b^{n+n!}$ for *suitable* n.
Be sure to say what YOU mean by n.
Explicitly employ an *upper* bound on $|v|$, where v comes from a *suitable* application of GPL.

c. (9 points)
Prove that L_3 *is* regular. N.B. If you do this by programming in *relevant* system(s), also *document* your "code." *Your documentation should be to help the graders understand what you had in mind.*

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[1] For example, $aaaabbbaabbbbbbbbabba \in L_1$ since the prefix $aaaabbbaabbbbbbb$ has exactly 10 b's and 6 a's.

[2] For example, $\varepsilon, abaaa, baba \in L_3$, where '$\varepsilon$' denotes the empty word. ε is in L_3 since it contains no counterexamples.

C2 Theory (25 points)

For this problem, C2, you should assume with*out* proof the following *constructive* form of Pumping for Context Free Languages.

Theorem (Constructive Pumping for CFLs) There is an algorithm for transforming any push-down automaton $\mathcal{M}$ into a corresponding positive integer $n_{\mathcal{M}}$ such that

$$(\forall z \mid z \in L(\mathcal{M}) \wedge |z| \geq n_{\mathcal{M}})(\exists u, v, w, x, y \mid uvwxy = z \wedge |vwx| \leq n_{\mathcal{M}} \wedge vx \neq \varepsilon)(\forall i \geq 0)[uv^iwx^iy \in L].$$

a. (6.25 points)

Suppose that $\mathcal{M}$ is a pda. Prove that

$$L(\mathcal{M}) \neq \emptyset \Leftrightarrow (\exists x \in L(\mathcal{M}))[|x| < n_{\mathcal{M}}]. \tag{1}$$

Hint for C2(a): Don't forget to prove *each* direction. For one direction use Constructive Pumping for CFLs.

b. (6.25 points)

Use without proof the result of C2(a) just above to *devise* an algorithm for deciding of an arbitrary pda $\mathcal{M}$, over a fixed alphabet Σ, whether or not

$$L(\mathcal{M}) = \emptyset. \tag{2}$$

c. (6.25 points)

Suppose that $\mathcal{M}$ is a pda. *Show* that

$$L(\mathcal{M}) \text{ is infinite} \Leftrightarrow (\exists x \in L(\mathcal{M}))[n_{\mathcal{M}} \leq |x| < 2n_{\mathcal{M}}]. \tag{3}$$

Hint for C2(c): Use Constructive Pumping for CFLS for *each* direction!

d. (6.25 points)

Use without proof C2(c) just above to *devise* an algorithm for deciding of an arbitrary pda $\mathcal{M}$, over a fixed alphabet Σ, whether or not $L(\mathcal{M})$ is infinite.

C3 **Theory** (25 points)

Fix a standard programming formalism φ for computing all the *one-argument* partial computable functions which map the non-negative integers into themselves. Code (Gödel) number the φ-programs *onto* the entire set of non-negative integers. Let φ_p denote the partial function computed by program (number) p in the φ-system. You may assume without proof that, in the φ-system, Universality and S-m-n hold. *You may not need all these assumptions — just say which ones you are using where.* Let $W_p \stackrel{\text{def}}{=}$ the domain of φ_p.[3]

In this problem, C3, you may proceed *in*formally and assume with*out* proof standard characterizations of the r.e. sets. Just be sure to say *which* results you are using.

a. (10 points)

Prove there *is* a computable function f such that

$$\{W_{f(x)} \mid x \in N\} = \text{the collection of all finite subsets of } N. \tag{4}$$

Hint for C3(a): The range of such an f would be r.e.

Be sure *not* to assume or try to show the *false* statement that

$$\{p \mid W_p \text{ is finite}\} \text{ is r.e.}. \tag{5}$$

Of course it is *ir*relevant to show the falsity of (5), so don't do it.

b. (15 points)

Prove there is *no* computable function g such that

$$\{W_{g(x)} \mid x \in N\} = \text{the collection of all infinite r.e. subsets of } N. \tag{6}$$

Hint for C3(b): Suppose for contradiction otherwise. Create an *infinite r.e.* set A differing from each of $W_{g(0)}, W_{g(1)}, W_{g(2)}, \ldots$. To do this, think *in*formally about how to enumerate algorithmically such an A from dovetailed algorithmic enumerations of $W_{g(0)}, W_{g(1)}, W_{g(2)}, \ldots$. Make sure you list into A an element from each $W_{g(i)}$ *and* that you do *not* list into A at least one element from each $W_{g(i)}$. Be sure to *show why this is possible.*

[3] Then $W_0, W_1, W_2, \ldots$ provides a standard listing of *all* the r.e. sets (of non-negative integers).

C4 Theory (25 points)

Fix a standard programming formalism φ for computing all the *one-argument* partial computable functions which map the non-negative integers into themselves. Code (Gödel) number the φ-programs *onto* the entire set of non-negative integers. Let φ_p denote the partial function computed by program (number) p in the φ-system. Let Φ denote a standard Blum step-counting measure associated with φ.[4] Let $W_p \stackrel{\text{def}}{=}$ the domain of φ_p.[5] You may assume *without* proof that in the φ-system Universality, S-m-n, *and the Kleene Recursion Theorem (KRT)* hold.

The first two parts of this question will lead you (with *very* useful hints) through a proof of the following

Theorem *Suppose* Δ is a collection of r.e. sets. *Let*

$$P_\Delta \stackrel{\text{def}}{=} \{p \mid W_p \in \Delta\}. \tag{7}$$

Suppose P_Δ is r.e.
Then

$$(\forall p)[W_p \in \Delta \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{8}$$

The third and fourth parts of this question each asks you to apply the theorem and also provides very useful hints. In that interest and for later use, let

$$A = \{p \mid W_p = \{0\}\}. \tag{9}$$

a. (6.25 points)
Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[W_p \in \Delta \Rightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta]]. \tag{10}$$

Hint for C4(a): Suppose that $W_p \in \Delta$. Suppose for contradiction that ($\forall$ finite sets $D \subseteq W_p$)[$D \notin \Delta$]. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether or not "e appears in P_Δ within x steps." Make this precise, figure out what to have e do in each case, etc., and get a contradiction.

b. (6.25 points)
Assume all the hypotheses of the Theorem. *Explicitly use KRT* in the φ-system (formally or informally — as you choose) to prove that

$$(\forall p)[(\exists \text{ a finite set } D \subseteq W_p)[D \in \Delta] \Rightarrow W_p \in \Delta]. \tag{11}$$

Hint for C4(b): Suppose ($\exists$ a finite set $D \subseteq W_p$)[$D \in \Delta$]. Let D be an example. Suppose for contradiction that $W_p \notin \Delta$. Apply KRT to obtain an self-referential e which determines its I/O behavior on input x *in part* according to whether it eventually discovers that "[$x \in D \vee e$ appears in P_Δ]." Make this precise, figure out what to have e do if it makes this discovery, *etc.*

c. (6.25 points)
Explicitly use the Theorem stated above in this question, C4, to show that A is *not* r.e., *where* A is defined in (9) above.

Hint for C4(c): Suppose for contradiction otherwise. Clearly $A = P_\Delta$ for $\Delta = \{\{0\}\}$. Therefore, from (8) above, we have that $(\forall p)[W_p = \{0\} \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D = \{0\}]]$. Pick D and W_p so that $D = \{0\} \subseteq W_p \neq \{0\}$. Get a contradiction.

d. (6.25 points)
Explicitly use the Theorem stated above in this question, C4, to show that $\overline{A}$ is *not* r.e., *where* A is defined in (9) above.

Hint for C4(d): Suppose for contradiction otherwise. Clearly $\overline{A} = P_\Delta$ for $\Delta = \{W_p \mid W_p \neq \{0\}\}$. Therefore, from (8) above, we have that $(\forall p)[W_p \neq \{0\} \Leftrightarrow (\exists \text{ a finite set } D \subseteq W_p)[D \neq \{0\}]]$. Pick a D and a W_p to get a contradiction.

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[4] Hence, (i) $(\forall p)[\text{domain}(\Phi_p) = \text{domain}(\varphi_p)]$, *and* (ii) $[\{(p, x, t) \mid \Phi_p(x) \leq t\}$ is an algorithmically decidable set$]$.

[5] Then $W_0, W_1, W_2, \ldots$ provides a standard listing of *all* the r.e. sets (of non-negative integers).