C1 **Theory** (25 points)

a. (12.5 points)
Let

$$L_1 = \{w \in \{a,b\}^* \mid \text{6th from last symbol of } w \text{ is } b \ \wedge \ |w| \text{ is a multiple of } 2\}. \tag{1}$$

Show that $L_1$ is regular.

b. (12.5 points)
Employ an appropriate pumping lemma to show that

$$L_2 = \{a^m b^n \mid m, n \in N \ \wedge \ m > n\} \tag{2}$$

is *not* regular.

**C2 Theory** (25 points)

The purpose of this problem is to take you through an argument showing that there is a *non*-regular language that can't be shown non-regular by pumping.

For any language $L$, let

$$P(L) = (\exists n)(\forall x_1, x_2, x_3 \mid x_1x_2x_3 \in L \wedge |x_2| = n)(\exists u, v, w)\big[uvw = x_2 \wedge |v| \geq 1 \wedge (\forall i)[x_1uv^iwx_3 \in L]\big].$$

Then Pumping can be stated as follows.

PUMPING. ($\forall$ regular $L$)[$P(L)$].

For all $w \in \{a, b\}^*$, let $\#_a(w)$ = the number of $a$'s in $w$.

Let

$$L_3 = \big\{w \in \{a, b\}^* \mid \#_a(w) \text{ is prime } \vee \ aa \text{ is a substring of } w\big\}.$$

a. (12.5 points)

*Show* that $L_3$ is not regular.

**Hint for C2a:** Suppose, by way of contradiction, otherwise.

Let

$$L' = \big\{w \in \{a, b\}^* \mid aa \text{ is not a substring of } w\big\}.$$

*Show* that $L'$ is regular.

Let

$$L'' = \big\{w \in \{a, b\}^* \mid \#_a(w) \text{ is prime } \wedge \ aa \text{ is not a substring of } w\big\}.$$

*Show, by citing the assumption that $L_3$ is regular and relevant, standard result(s) in automata theory*, that $L''$ is, then, itself regular

Hence, by Pumping, $P(L'')$.

Let $n$ be as in the statement $P(L'')$. Let $p$ be any prime such that $2p > n$. Let $x = (ab)^p$. Let $x_1 = \varepsilon$. Let $x_2$ = the first $n$ symbols of $x$. Let $x_3$ be such that $x_1x_2x_3 = x$. Clearly, then, $x_1x_2x_3 \in L''$ and $|x_2| = n$. Let $u$, $v$, and $w$ be from $P(L'')$ for this choice of $x_1$, $x_2$, and $x_3$. Consider the following cases and get a contradiction in each case.

CASE OF $v$ begins *and* ends in $b$.

CASE OF $v$ begins *or* ends in $a$.

b. (12.5 points)

*Prove* that $P(L_3)$.

**Hint for C2b:** Choose $n = 3$. Let $x_1$, $x_2$, $x_3$ be such that $x_1x_2x_3 \in L_3$ and $|x_2| = n$. Consider the following cases, the first with subcases to consider.

CASE OF $x_2$ contains a $b$. Choose $v = b$. Choose $u$ and $w$ such that $uvw = x_2$. *To show that, for all $i$, $x_1uv^iwx_3 \in L_3$*, consider the following subcases.

SUBCASE OF $\#_a(x_1x_2x_3)$ is prime.

SUBCASE OF $aa$ is a substring of $x_1x_2x_3$.

CASE OF $x_2 = aaa$. Choose $u = aa$, $v = a$, and $w = \varepsilon$. *Show that, for all $i$, $x_1uv^iwx_3 \in L_3$.*

C3 Theory (25 points)

Let $N =$ the set of non-negative integers.

We write $(f_i \mid i \in N)$ for the infinite sequence of functions $(f_0, f_1, f_2, \ldots)$.

**Definition** *A sequence of functions* $(f_i \mid i \in N)$ *is said to be* uniformly computable $\overset{\text{def}}{\Leftrightarrow}$ *the function* $\lambda i, x.f_i(x)$ *is computable.*

**Example 1** *For each* $i, x$, *let*

$$f_i(x) = i^2x^3 + 4i. \tag{3}$$

*Then* this $(f_i \mid i \in N)$ is *clearly uniformly computable.*

**Definition** *A sequence of functions* $(f_i \mid i \in N)$ *is* uniformly primitive recursive $\overset{\text{def}}{\Leftrightarrow}$ *the function* $\lambda i, x.f_i(x)$ *is primitive recursive.*

**Example 2** *Define* $\lambda i, x.f_i(x)$ *as in (3) of Example 1 above. Then* $(f_i \mid i \in N)$ is, *in fact, uniformly* primitive recursive.

a. (12.5 points)

*Prove, employing the Hint just below* that there is a sequence of functions $(F_i \mid i \in N)$ such that

1. $(\forall i)[F_i$ is computable] *and*
2. $(F_i \mid i \in N)$ is *not* uniformly computable.

**Hint for C3a:** Let $A$ be an r.e. *not* computable set. Write $A$ as $\{a_0 < a_1 < a_2 < \ldots\}$. For each $i, x$, let $F_i(x) \overset{\text{def}}{=} a_i$.

*Show* that, for *each, fixed* $i \in N$, $\lambda x.F_i(x)$ is a primitive recursive (hence, computable) function.

Suppose for contradiction $(F_i \mid i \in N)$ *is* uniformly computable. Then $\lambda i, x.F_i(x)$ is computable.

*Show*, then, that $\lambda i.F_i(0)$ is computable, monotone increasing, and has range $A$.

*Show how to obtain a contradiction from this.*

b. (12.5 points)

*Prove, employing the Hint just below* that there is a sequence of functions $(G_i \mid i \in N)$ such that

1. $(\forall i)[G_i$ is primitive recursive],
2. $(G_i \mid i \in N)$ is *not* uniformly primitive recursive, and
3. $(G_i \mid i \in N)$ *is* uniformly computable.

**Hint for C3b:** Fix a standard algorithmic coding of the finite sets of equations each defining a one argument primitive recursive function 1-1 onto $N$. Let $G_i$ be the one argument primitive recursive function defined by the finite set of such equations with code number $i$. Do *not* waste time providing details about such a coding.

Trivially, $(\forall i)[G_i$ is primitive recursive].

Suppose for contradiction $(G_i \mid i \in N)$ is uniformly primitive recursive. Hence, $\lambda i, x.G_i(x)$ is primitive recursive. Define $g(x) = 1 + G_x(x)$. To get a contradiction, *show that* $g$ *is both primitive recursive and not primitive recursive.*

*Argue* **very informally and briefly** that $\lambda i, x.G_i(x)$ is computable.

C4 **Theory** (25 points)

Let $N$ denote the set of non-negative integers.

Fix a standard programming formalism $\varphi$ for computing all the *one-argument* partial computable functions which map $N$ into $N$.[1] Code (Gödel) number the $\varphi$-programs *onto* $N$. Let $\varphi_p$ denote the partial function computed by program (number) $p$ in the $\varphi$-system.

Let $\Phi_p(x)$ = the number of steps $\varphi$-program $p$ executes on input $x$ if $p$ on $x$ halts and undefined if $p$ on $x$ does not halt. You may assume: $\Phi_p(x)$ defines a *partially* computable function of $p, x$; $\Phi_p(x)$ is defined exactly when $\varphi_p(x)$ is defined; and

$$\{(p, x, t) \mid \Phi_p(x) \leq t\} \text{ is a computable set.} \tag{4}$$

We write $\Phi_p(x) > t$ to mean *either* $\Phi_p(x)$ is defined to be some number $> t$ *or* $\Phi_p(x)$ is undefined.

You may assume with*out* proof that, in the $\varphi$-system, Universality, S-m-n, and the Kleene Recursion Theorem (KRT) hold.

Suppose

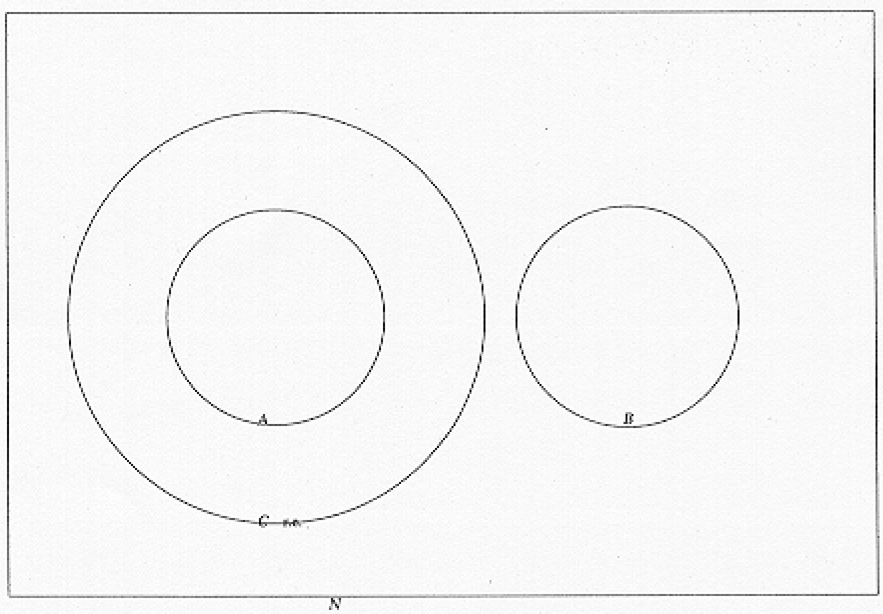
$$A = \{x \mid \varphi_x \text{ is total and strictly increasing}\} \tag{5}$$

and

$$B = \{x \mid \varphi_x \text{ is total and not strictly increasing}\}. \tag{6}$$

*Prove by explicit application of KRT* that there is no r.e. set $C$ such that $A \subseteq C \subseteq \overline{B}$.

**Hint for C4:** Suppose for contradiction otherwise. Then we have the following picture.




Explain why there is a $c$ such that domain$(\varphi_c) = C$.

Employ KRT to obtain a self-referential $e$ (for computing $\varphi_e$) and which, on input $x$, considers whether or not $\Phi_c(e) \leq x$.

Obtain a contradiction.

[1] N.B. We are using a *lower* case Greek phi for this notation. Below we employ the upper case of this Greek letter but for a different purpose.