Topology Control with a Limited Number of Relays

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***Abstract*—Network longevity and connectivity are key design goals in any wireless sensor network deployment. In this context, we consider the placement of relay nodes and individual transmission power assignments. Specifically, given a planar deployment of sensors and a base station, we seek the placement of a limited number of relays and optimal sensor power assignments such that the network is connected. We present a polynomial-time bicriteria approximation algorithm for this problem. We also provide an optimal $O(n^2 \log n)$-time algorithm for a restricted version where nodes lie on a simplified urban grid (that we call a comb-grid). We also study a related variant that assumes fixed transmission power values, with the goal of minimizing the number of relays. We provide extensive simulation results for the comb-grid case.**

I. Introduction

Wireless sensor networks (WSNs) comprise small, low-power devices that collect and communicate environmental data. Network managers must balance the inherent tension between network longevity goals and network connectivity constraints. One approach is to ensure base station connectivity through transmission power assignments that minimize power consumption. In many deployments, however, the combination of low-power radios and large-scale geographic distribution precludes such solutions. An alternative is to introduce a second-tier of more powerful observation relay nodes.

In this paper, we assume a two-tier network model in which sensor nodes communicate only with relays. In this context, we study the *Relay Node Placement* (RNP) problem in combination with topology control. Specifically, given a static sensor deployment, we seek a placement of relays and transmission power assignments that ensure connectivity while minimizing energy consumption. Our focus is motivated by practical deployment considerations: Duty cycle and bandwidth requirements dictate that relay nodes make use of more resource-intensive processors and radios, yielding concomitant increases in size, cost, and, most important, power consumption. Hence, in this paper, we focus primarily on the variant of the RNP problem in which the number of relays is limited to a specified maximum.

Problem Statement. Given a planar deployment of sensors and a base station, find a placement of at most k relays so that (*i*) each sensor is connected to some relay, (*ii*) the base station is connected to some relay, (*iii*) the relay network is connected, and (*iv*) the maximum transmission power used by any sensor is minimized. The problem is denoted as MinMax-kRNP. We also consider a restricted version of the problem involving placement within a simplified urban grid.

The restricted problem is motivated by an ongoing WSN deployment in Aiken, SC, designed to monitor a stormwater treatment overhaul [1]. The city streets follow a typical grid pattern, with *"green parkways"* between adjacent roads. An 802.15.4 collection network is deployed within a subset of these parkways, running approximately parallel to the adjacent roadways. Associated 802.11 relay nodes are installed at various points between the cells, along a focal track. All observation data is routed to a single base station that provides a high-speed link to the Internet. Architecturally, the system bears similarity to a number of other WSN deployments (e.g. [2]). Intuitively, these deployments can be modeled as a *comb-grid* in which a single horizontal line segment, the *comb-handle*, represents the main street, and vertical line segments, the *comb-teeth*, represent the side streets. The comb-teeth are separated by obstructions that make communication possible only between nodes located on the same street.

Paper Organization and Contributions. Section II surveys key elements of related work. Section III formally defines the network models and placement problems. Related RNP problems and solutions are also considered, including a linear-time algorithm for minimizing the number of relays on a comb-grid. Section IV presents an $O(n^2 \log n)$-time algorithm that solves the MinMax-kRNP problem on a comb-grid. Section V presents a polynomial-time *bicriteria* approximation algorithm for the general version of the MinMax-kRNP problem. The algorithm guarantees an approximation ratio of $(1 + \epsilon)$ for both the number of relays and the power assignments. Section VI presents an experimental evaluation of the number of relays on network performance. Section VII concludes with some directions for future research.

II. Related Work

A number of authors have considered RNP solutions to improve network lifetime while preserving connectivity [3]–[5]. Two communication models have been studied in the realm of RNP and the major difference is whether sensors are allowed to serve as routers. In the two-tiered model, the goal is to find the minimum number of relays that cover all sensors and that form a connected network of pure relays. This problem is NP-hard and a polynomial time approximation scheme (PTAS) is presented in [5].

Another approach to prolonging network lifetime is to use transmission power assignments to achieve a desired topology. This problem was first formulated in [6]. Since then, intensive work has been done on topology control in wireless sensor networks [7].

In this paper, we investigate an optimization problem that combines relay node placement and topology control. We seek not only an optimal relay placement, but also a minimized sensor range. This MINMAX-kRNP problem can be formulated as a two-tiered version of the geometric *p-center* (i.e., *central clustering*) problem [8]. A 2-approximation algorithm with an $O(n \log k)$ running time was provided in [8], and the authors showed that no approximation algorithm can have an approximation ratio better than 1.822 unless $P = NP$. Recently, [9] presented an $O(1)$-approximation algorithm under a two-tiered model assuming that $R \geq 2r$. In both the traditional version in [8] and the two-tiered version in [9], relays can only be placed at certain candidate locations, while in our two-tiered version there are *no* restrictions on the possible locations of the k relays. Furthermore, we use a less restrictive assumption, namely $R \geq r$. To the best of our knowledge, there is no prior work on a two-tiered version of p-center problems without any constraints on relay locations.

III. MODEL AND DEFINITIONS

We now formally define the network models and problems considered in the remainder of the paper.

A. Two-tiered Network Model

We consider relay node placement under a two-tiered model. There are three types of nodes: sensors, relays, and a base station. The sensors and the base station are pre-deployed at specific locations. Data is transmitted from sensors to relays and then through the relays to the base station. Sensors may only communicate with relays.

All relays have a uniform transmission range R. Each sensor has an adjustable transmission range with a uniform upper bound $U \leq R$. Without loss of generality, we neglect the bound U, taking R as the upper bound. If the minimum required sensor range for a certain problem instance is r and $r > U$, there is no solution to that instance.

Definition 3.1: (COVERED) A sensor is COVERED by a relay if the relay is within the transmission range of the sensor, i.e. the distance between them is at most r, where r is the transmission range of the sensor.

Definition 3.2: (CONNECTED) Two relays are CONNECTED if each is within the other's transmission range, i.e. the distance between them is at most R. The base station is CONNECTED to some relay if it is within the transmission range of the relay.

Definition 3.3: (RELAY NETWORK) The RELAY NETWORK is the network consisting of the relay nodes and the base station. The relay network is CONNECTED if relays are connected and the base station is connected to at least one relay.

B. Preliminaries on Comb-Grid

Definition 3.4: (COMB-GRID) A COMB-GRID consists of a single horizontal line segment, termed the COMB-HANDLE (or COMB), and a set of m vertical line segments that meet the comb-handle, each termed a COMB-TOOTH. (An example is shown in Figure 1.) Note that the comb-handle splits some vertical segments into two comb-teeth, and that the comb-teeth need not be equally spaced or equally long. Nodes may only communicate if they lie on the same line segment.

We now define two types of points that segment the comb into several *cells*; these points serve as the cell end-points.

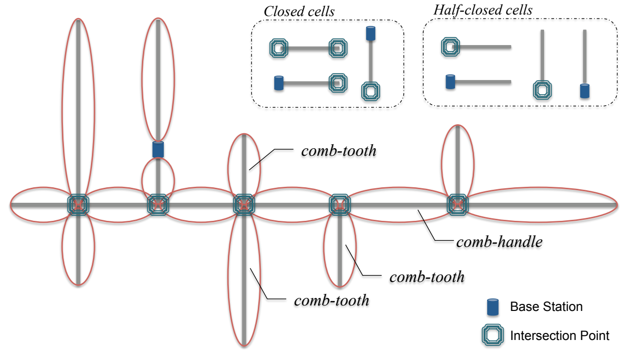

Fig. 1. Comb-grid with base station on comb-tooth (14 cells, circled)

Definition 3.5: (CELL) A CRITICAL ENDPOINT is either the base station point or an intersection point between the comb-handle and a comb-tooth. An END SENSOR is the leftmost (rightmost) sensor on the comb-handle (or the uppermost (lowermost) sensor on a comb-tooth) if there is no critical endpoint on or to the left (right) of (or above (below)) that sensor. A CELL is a line segment either between two adjacent critical endpoints or between a critical endpoint and an adjacent end sensor. Furthermore, a cell where both endpoints are critical endpoints is a CLOSED CELL, and a cell where one endpoint is an end sensor is a HALF-CLOSED CELL (see Figure 1).

The following claim provides upper and lower bounds for some parameters of a comb-grid.

Claim 3.1: Consider a comb-grid with one base station point and m distinct intersection points between the comb-handle and comb-teeth. The following results hold.

(a) Let τ denote the number of teeth. Then, $m \leq \tau \leq 2m$.
(b) The number of closed cells is m or $m - 1$.
(c) Let h denote the number of half-closed cells. Then, $m - 1 \leq h \leq 2m + 2$.

The reader is referred to [10] for the proof details.

Notice that almost all of the closed cells are located on the comb-handle. Hence, we introduce the following expository device: In the case of a closed cell located on a comb-tooth (when the base station lies on the same tooth), we refer to the intersection endpoint of the closed cell as the LEFT ENDPOINT of that cell, and to the other endpoint (where the base station is located) as the RIGHT ENDPOINT. Similarly, since most of the half-closed cells are located on comb-teeth, we will refer to the critical endpoint of a half-closed cell as the BOTTOM ENDPOINT and to the other endpoint (where the end sensor is located) as the TOP ENDPOINT.

C. Minimizing the Number of Relays: A Related Problem

In the RNP problems we consider, we are given a limited number of relays and are asked to place them so as to minimize the maximum sensor transmission power. A complementary problem assumes a fixed transmission power and asks us to place a minimum number of relays so that the relay network is connected and every sensor is covered by a relay. Formally:

Definition 3.6: (MIN-RNP) Given a set of n pre-deployed sensors and a base station, a sensor transmission range r, a relay transmission range $R \geq r$, find the minimum set of h relays that can be placed to satisfy both of the following conditions: (i) the relay network is connected, and (ii) each sensor is covered by at least one relay.

We consider two versions of this problem and provide an efficient algorithm for one version and an approximation algorithm for the other. These algorithms will be used as subroutines in our algorithms for the MINMAX-kRNP problems.

1) Min-RNP Problem on a Comb: In this restricted version, nodes are located on a comb. This version is solvable in time $O(n+h)$, assuming that for each cell, the sensors are already sorted. The detailed algorithm, denoted as OPTIMALMINR, is omitted here due to space limitations; it can be found in [10].

2) Min-RNP Problem in the Plane: In the general version of the problem, nodes are located in the Euclidean plane. Recall that the transmission range r is fixed (and uniform). Given r, we seek the minimum number of relays so that each sensor is covered and the relay network is connected.

It follows from previous results that MIN-RNP-2D is NP-hard. Reference [5] provides a PTAS for this problem without a base station. Their algorithm can be extended to our MIN-RNP-2D problem while preserving the approximation ratio:

Theorem 3.1: There exists a PTAS for MIN-RNP-2D.

Proof: (*sketch*) In [5], Efrat et al. present a structural lemma showing that an optimal set R^* of relays can be replaced by a Steiner tree T of "red" and "blue" edges with at most $(1+\epsilon)|R^*|$ relays, and all of the relays in T lie on a polynomial-size grid G which can be computed in polynomial time. There, all red edges are of length at most r, joining a sensor to a relay, and all blue edges are of length at most R, joining two relays[1]. We extend their structural lemma by adding a new type of edge to the tree T: a "purple" edge of length R, joining a relay to the base station. We then modify their proof. Specifically, their proof replaces the optimal set of relays by grid points using a "pin" process. In that process, they define an "iterated circle arrangement" that iteratively constructs a series of circles. In the first iteration, besides the circles from [5], we add circles of radii $R, 2R, \ldots, mR$ centered at the base station, where $m = \Theta(1/\epsilon)$, and then iteratively construct circles following the rules from [5]. These additional circles result in adding a constant (function of m) number of vertices to the arrangement. As in [5], after adding the new vertices, it is sufficient to perturb all of the relays in R^* (including those connected to the base station) onto the arrangement vertices. With this structural lemma in hand, we utilize the *m-guillotine theorem* [11] as in [5] to show that an optimal tree T^* with $|R^*|$ relays can be converted to a tree T_m which has at most $(1+\epsilon)|R^*|$ relays, and whose blue edges are m-guillotine. Finally, we use dynamic programming to explore the polynomial-size grid G and produce an optimal m-guillotine spanning tree lying on G. ■

D. MinMax-kRNP Problem

We now formally define the RNP problem that assumes a limited number of relays, the focus of the remaining document.

Definition 3.7: (MINMAX-kRNP) Given a set of pre-deployed n sensors and a base station, and a set of k relays each with a transmission range of R, place the k relays to satisfy both of the following conditions: (i) the relay network is connected, and (ii) the maximum distance (i.e., transmission range) of any sensor to the nearest relay is minimized.

In the MINMAX-kRNP problems, the goal is to minimize the maximum transmission power, which is uniformly assigned to all sensors. As in MIN-RNP, we assume that the maximum transmission range is no more than R. That minimized maximum sensor transmission range (denoted as MINMAX sensor range), is the global minimum rather than a "per node minimum" [6]. In fact, as is common in topology control algorithms, our results can be extended to minimize the sensor transmission range on a *per node* basis: After calculating the MinMax sensor range that guarantees connectivity, for each sensor, decrease its transmission range to the point where any additional reduction would result in the loss of connectivity.

[1] In [5], 1 and r are used as the ranges for sensors and relays, respectively.

IV. MINMAX-kRNP ON A COMB

In this section we provide a polynomial time algorithm for this restricted version where nodes are located on a comb. We begin with an observation:

Fact 4.1: Since nodes can only communicate with nodes located on the same line segment, it follows that any solution must place a relay at each intersection point. Thus, if $k < m$ (the number of distinct intersection points), no solution exists. In the remainder of this section, we assume $k \geq m$.

Note that even with this assumption, a solution is not guaranteed to exist, since the sensor transmission range cannot exceed R. In the algorithm that follows, we first test whether a solution exists when using R as the sensor transmission range. If one exists, we then find a solution with a smaller r. Our approach in that case is the following:

(a) First we determine a *sufficient set* of candidate values for r — that is, a finite set S of values such that the optimal solution will have a value for $r \in S$.
(b) Given S, we sort the values and perform a binary search over the list. For each potential value of r, we calculate the minimum number of relays required using algorithm OPTIMALMINR (Subsection III-C1). We compare the output of that algorithm to k (the limit on the number of relays) as part of a binary search on the list of candidate values for r. The final output will be a placement of k relays that yields the minimum value for r.

In the remainder of this section we define a sufficient set of candidate values for r, then provide the full algorithm. We use the following definition of a generalized *mod* operation for non-negative real numbers. Given two real numbers $x \geq 0$ and $y > 0$, we define $x \bmod y$ as follows:

$$x \bmod y \quad = \quad x - y * \lfloor x/y \rfloor .$$

We are now ready to define the notion of an EVEN DISTANCE.

Definition 4.1: (EVEN DISTANCE) Let $s_1, s_2, \ldots, s_{n_l}$ be sensors on a cell l, where n_l is the number of sensors on that cell. For half-closed cells, let V_0 be the critical endpoint, and for closed cells, let V_0 and V_1 be the left and right endpoints, respectively. Then the EVEN DISTANCES are:

$$0 \quad (1)$$
$$\frac{dist(s_i, s_j) \bmod R}{2}, \quad \text{for } 1<i<n_l, i+1<j<n_l \quad (2)$$
$$dist(s_i, V_0) \bmod R, \quad \text{for } 1<i< n_l \quad (3)$$
$$dist(s_i, V_1) \bmod R, \quad \text{for } 1<i<n_l \text{ in a closed cell} \quad (4)$$
$$R \quad (5)$$

The key to the correctness of our algorithm is the following:

Theorem 4.1: (Sufficiency) For MINMAX-kRNP-COMB, the optimal range r is an even distance.

Before proving the above theorem, we remark that the reasons for the several parts of Definition 4.1 will become clear as we proceed below. Here we clarify why zero is included as an even distance, since at first glance that might seem wrong. The reason is that in certain extreme cases, zero would be the relevant value for r. For example, consider the situation when n sensors are evenly distributed along a straight line starting from the base station and the distance between consecutive nodes (sensors or the base station) is R. And suppose we have n relays. Then the optimal solution places the n relays directly on top of those n sensors. Clearly the relay network is connected and each sensor is covered using a transmission range $r = 0$! With this in hand, we now start to prove Theorem 4.1.

Proof: We prove the theorem by contradiction. Assume that for a given problem instance, the optimal $r > 0$ is not an even distance. Among all solutions with that r, consider one having the fewest number of sensors whose shortest distance to a relay is r. In that solution, consider a sensor s_1 whose shortest distance to a relay is r, and let E_1 be the nearest relay that covers s_1, i.e. the distance from s_1 to E_1 is r. There are two cases to consider.

Case 1: s_1 is in a closed cell.

Without loss of generality, let E_1 be to the left of s_1. Then, let E_2 be the leftmost relay in a maximal sequence of relays that begins with E_1 and extends to the left such that the distance between consecutive relays is R. Notice that such a sequence could consist of only E_1, and in that case we let $E_2 = E_1$. Also, E_2 cannot be the left endpoint V_0 of the cell, since $r = dist(s_1, V_0) \bmod R$ and that is an even distance of type (3)[2]. Further, since E_2 is not the left endpoint of the cell, there is either a relay or the base station, say E_3, located to the left of E_2 by a distance of less than R.

Now let s_2 be the leftmost sensor that is covered by E_2. If s_2 is to the right of E_2 or on the same point as E_2, since E_3 is less than R away from E_2, E_2 can be shifted by a sufficiently small ϵ to its right. If s_2 is to the left of E_2, since r is not an even distance (i.e. the distance from E_2 to s_2 is less then r), then E_2 can be shifted by ϵ to its right. If there is no sensor covered by E_2, then E_2 can also be shifted since the distance from E_2 to E_3 is less than R. In each case, we then shift the entire sequence of relays from E_1 to E_2, inclusive, by ϵ to the right. The only possibility that we could not shift some relay in the sequence is if there is a sensor covered by that relay that is located distance r to the left of that relay. But then by the specification of even distance type (2) it follows that r is an even distance. Note that after this shift, the sensors and relays to the left of a relay in the sequence are ϵ further away from that relay, and the ones to the right are closer. But, with small enough ϵ, each relay in the sequence still covers the same set of sensors and connects to the same relay(s) as it did before the shift. This is a contradiction, since this is an optimal solution but with one fewer sensor whose shortest distance to a relay is r.

Case 2: s_1 is in a half-closed cell.

We first consider the subcase where E_1 lies below s_1. Let E_2 be the bottom relay in a maximal sequence of relays that begins with E_1 and extends downwards such that the distance between consecutive relays is R. As in the prior case, such a sequence cannot end at the bottom endpoint. And, as in the prior case, the entire sequence of relays can be shifted by ϵ upwards (towards s_1) to get a contradiction.

Then, consider the subcase where E_1 lies above s_1. Let E_2 be the top relay in a maximal sequence of relays that begins with E_1 and extends upwards such that the distance between consecutive relays is R. If there is at least one relay above E_2, then the distance from that relay to E_2 is less than R and the same analysis as used earlier shows that we can shift the entire sequence by ϵ downwards (towards s_1) to get a contradiction. On the other hand, if there is no relay beyond E_2, let s_2 be the end sensor of that cell and note that it must be covered by E_2. If s_2 is below, or coinciding with E_2, then the entire sequence of relays can be shifted downwards by ϵ. If s_2 is above E_2, the distance between s_2 and E_2 is less than r since r is not an even distance. Again, similarly to the earlier cases, it follows that the entire sequence of relays can be shifted downwards by ϵ. In either situation, with E_1 above s_1, we can shift the entire sequence by ϵ towards s_1. This yields an optimal solution but with one less sensor whose shortest distance to a relay is r, a contradiction. ∎

With the notion of even distances in hand, the details of our algorithm are shown in Algorithm 1. The correctness of the algorithm follows from Theorem 4.1. As far as running time is concerned, note that in Algorithm 1, Lines 6–19 take time at most $O(n \log n)$ to sort n sensors, and at most $O(n^2)$ to compute the set of *even distances*. Thus, in total, Step I takes time $O(n^2)$. After Step I, there are at most $O(n^2)$ values in set S. Line 22 then takes time $O(n^2 \log n)$ to sort those values. Line 32 takes time $O((n+k)\log n)$ to search over set S since there are at most $O(\log n)$ iterations and each iteration takes time $O(n+k)$ to test. Therefore, we have:

Lemma 4.1: (Running time) Algorithm 1 runs in time $O(n^2 \log n)$.

V. MINMAX-kRNP IN THE PLANE

In this section, we consider the 2D variant of the problem, in which nodes are located on a Euclidean plane. We present an algorithm that is approximate in terms of both r and the number of relays k used in the solution, which, in the literature, is referred to as a *bicriteria* approximation algorithm.

[2]Note that type (4) is analogously required for the situation when E_1 is to the right of s_1.

Algorithm 1 – ALGORITHM FOR MINMAX-kRNP-COMB

if OPTIMALMINR(R) $> k$ **then**
  return that no solution is possible
$S = \{0, R\}$;

/* Step I: Compute the set S of *even distances*. */
for cell i **do**
  Let $s_{i1}, s_{i2}, \ldots, s_{in_i}$ be the sensors in cell i, where n_i is the number of sensors in cell i;
  for $k = 1$ to $n_i - 1$ **do**
    for $j = k + 1$ to n_i **do**
      $S = S \cup \{(dist(s_{ik}, s_{ij}) \bmod R)/2\}$;
  if cell i is a half-closed cell **then**
    Let V_{i0} be the critical endpoint;
    for $k = 1$ to n_i **do**
      $S = S \cup \{dist(s_{ik}, V_{i0}) \bmod R\}$;
  else ▷ cell i is a closed cell
    Let V_{i0} and V_{i1} be the critical endpoints;
    for $k = 1$ to n_i **do**
      $S = S \cup \{dist(s_{ik}, V_{i0}) \bmod R\}$;
      $S = S \cup \{dist(s_{ik}, V_{i1}) \bmod R\}$;

/* Step II: Search over set S to find the MinMax r */
Sort the values in set S;
// *Binary search* for the least $r_{min} \in S$ such that the returned cardinality of OPTIMALMINR is $\leq k$
$low = 0$;
$high = n_s - 1$; ▷ n_s is the number of values in S
while $low < high$ **do**
  $mid = \lfloor (low + high)/2 \rfloor$;
  if OPTIMALMINR$(r_{mid}) \leq k$ **then** ▷ r_{mid} is the value in S at position mid
    $high = mid$;
  else
    $low = mid + 1$;
$r_{min} = r_{high}$; ▷ recall that r_{high} produces a valid solution with k relays
return r_{min} and the relay placement produced by OPTIMALMINR(r_{min}) ;

Recall that the goal of the MINMAX-kRNP-2D problem is to place k relays in the plane to satisfy the following conditions: (i) the relay network is connected, and (ii) each sensor is covered by at least one relay. We show that one can derive an approximation algorithm for the MINMAX-kRNP-2D problem from an appropriate approximation algorithm for the MIN-RNP-2D problem under the following assumption:

Assumption 5.1: Suppose r^* denotes the minimum sensor range for a given instance of the MINMAX-kRNP-2D problem. There are efficiently computable values r_{lb} and r_{ub} such that $0 < r_{\text{lb}} \leq r^* \leq r_{\text{ub}}$.

Recall that we use R as the upper bound of the sensor transmission range. Let's then consider the lower bound. Note that in the two-tiered model, sensors may only communicate with relays. Suppose the number of relays k is less than the number of sensors n (a reasonable assumption in practice). It follows that some relay must cover two or more sensors. Thus, the minimum sensor range must be at least $d_{\min}/2$, where $d_{\min}$ is the smallest distance between a pair of sensors. In other words, a possible value for r_{lb} is $d_{\min}/2$.

Suppose $\mathcal{A}$ is an approximation algorithm with a performance ratio guarantee of ρ for MIN-RNP-2D. An approximation algorithm for MINMAX-kRNP-2D that uses $\mathcal{A}$ is shown in Algorithm 2. The following lemma establishes the performance guarantee provided by Algorithm 2.

Algorithm 2 – APPROX. ALGORITHM FOR MINMAX-kRNP-2D

Input: An instance of the MINMAX-kRNP-2D problem and a fixed value $\epsilon > 0$. (It is assumed that an approximation algorithm $\mathcal{A}(r)$ with a performance guarantee of $\rho > 1$ for the MIN-RNP-2D problem is available.)
Output: The MinMax sensor range r and a placement of relays.
/* Step I */
Let $t = \lceil \log_{1+\epsilon}(r_{\text{ub}}/r_{\text{lb}}) \rceil$;

/* Step II: Binary search for the smallest integer $i \in [0, t]$ such that the number of relays returned by $\mathcal{A}(r_i)$ is $\leq \rho k$. */
$low = 0$; $high = t$;
while $low < high$ **do**
  $i = \lfloor (low + high)/2 \rfloor$;
  $r_i = (1+\epsilon)^i \, r_{\text{lb}}$;
  if Number of relays returned by Algorithm $\mathcal{A}(r_i) \leq \rho k$ **then**
    $high = mid$;
  else
    $low = mid + 1$;
$r_{min} = r_{high}$;
return r_{min} and the relay placement produced by $\mathcal{A}$ for $r = r_{min}$.

Lemma 5.1: Let $\epsilon > 0$ be a fixed value. For the given instance of the MINMAX-kRNP-2D problem, suppose we have values r_{lb} and r_{ub} that satisfy Assumption 5.1. Let the solution produced by Algorithm 2 use k' relays and have a sensor range of r'. Then, the following conditions hold:

(i) $r' \leq (1 + \epsilon) r^*$.
(ii) $k' \leq \rho k$, where ρ is the performance guarantee provided by Algorithm $\mathcal{A}$ for the MIN-RNP-2D problem.

Proof: We first argue that Algorithm 2 will always return a solution. To see this, note that $r^* \leq r_{\text{ub}} \leq (1+\epsilon)^t \, r_{\text{lb}}$ by Assumption 5.1 and by the choice of t in the algorithm. Therefore, when the sensor range is $(1+\epsilon)^t \, r_{\text{lb}}$, there is a solution to the MIN-RNP-2D instance with at most k relays; i.e., there is at least one integer i in the range $[0, t]$ for which Step II of the algorithm will be successfully completed.

To prove Part (i), notice that the sensor range r' returned by Algorithm 2 is given by $r' = (1+\epsilon)^i \, r_{\text{lb}}$ for some integer $i \geq 0$. If $i = 0$, then $r' = r_{\text{lb}} \leq r^*$, and Part (i) holds. So, assume that $i \geq 1$. We have the following claim:

Claim 5.1: $r^* > (1+\epsilon)^{i-1} \, r_{\text{lb}}$.

Proof of Claim 5.1: Assume for the sake of contradiction that $r^* \leq (1+\epsilon)^{i-1} \, r_{\text{lb}}$. When Algorithm 2 considered the sensor range $(1+\epsilon)^{i-1} \, r_{\text{lb}}$, the number k_{i-1} of relays used in the solution returned by $\mathcal{A}$ must satisfy the condition:

$$k_{i-1} > \rho k \tag{6}$$

since i is the smallest integer for which Algorithm 2 was successful in Step II. However, since $r^* \leq (1+\epsilon)^{i-1} \, r_{\text{lb}}$, when the sensor range is $(1+\epsilon)^{i-1} \, r_{\text{lb}}$, there is a solution to the MIN-RNP-2D problem with at most k relays. Since $\mathcal{A}$ provides a performance guarantee of ρ, the number of relays used in the solution returned by $\mathcal{A}$ is at most ρk. This contradicts Inequality (6) and establishes Claim 5.1.

Continuing the proof for Part (i), we note that $r' = (1+\epsilon)^i r_{\mathrm{lb}}$ and that $r^* > (1+\epsilon)^{i-1} r_{\mathrm{lb}}$ (Claim 5.1). These two inequalities together imply that $r' < (1+\epsilon)r^*$.

To prove Part (ii), we note that when the algorithm terminates, the number of relays used is at most ρk. ∎

The following theorem is a simple consequence of the above lemma and Theorem 3.1 in Subsection III-C2:

Theorem 5.1: Let $\epsilon > 0$ be a fixed value. For any instance of the MINMAX-kRNP-2D problem that satisfies Assumption 5.1, there is an approximation algorithm that uses at most $(1+\epsilon)k$ relays and chooses a sensor range of at most $(1+\epsilon)r^*$, where r^* is the minimum sensor range.

Proof: In Subsection III-C2, we showed that for any fixed $\epsilon > 0$, there is an approximation algorithm $\mathcal{A}$ with a performance guarantee of $1+\epsilon$ for the MIN-RNP-2D problem where sensors have a uniform transmission range $r \leq R$. Suppose this generated approximation algorithm is used in Algorithm 2. It can be seen from Lemma 5.1 that the resulting solution satisfies the conditions mentioned in the theorem. ∎

The running time of Algorithm 2 depends on that of Algorithm $\mathcal{A}$. It is reasonable to assume that the running time of the former algorithm is dominated by the calls to Algorithm $\mathcal{A}$. Note that the range $[0, t]$ has $t + 1 = O(\log(r_{\mathrm{ub}}/r_{\mathrm{lb}}))$ integers. We find the smallest integer i in Step II by doing a binary search over that range. Thus, the number of calls to Algorithm $\mathcal{A}$ is $O(\log t) = O(\log\log(r_{\mathrm{ub}}/r_{\mathrm{lb}}))$.

VI. SIMULATION RESULTS

We now study the impact of relay capacity in a comb-grid network via simulation. Recall that the number of relays is typically limited due to power consumption and budgetary constraints. This bound affects relay placement, sensor transmission power, and ultimately, network yield. In this regard, we study the impact of relay capacity on the MinMax sensor range and total packet reception rate (PRR).

For these studies, we consider a region of size $1000 \times 600\ m$, with one "main street" (i.e. the comb-handle), lying horizontally in the middle of the region, and several vertical "side streets" (i.e. the comb-teeth). To mimic an urban environment, the intersections between the main street and the side streets are evenly distributed, which divides the field into 20 blocks of size $100 \times 300\ m$. By comparison, the standard block size in Manhattan (New York) is approximately $80 \times 270\ m$.

We have considered an extensive set of scenarios using this topology, varying both the number of sensors and the number of available relays. Due to space constraints, we focus only on the results for scenarios involving 500 sensors. Given the relatively large deployment area, this qualifies as a semi-dense WSN deployment. The results from scenarios with more sensors exhibit similar trends.

A. Number of relays vs. MinMax sensor range

We first examine the effect of the number of relays on the MinMax sensor range. Initially, we randomly deploy 500 sensors and one base station on the comb-grid and set the relay transmission range to 100 meters. For each resulting deployment, we vary the number of relays, starting with the smallest number of relays (k) such that Algorithm 1 produces a solution. Finally, we increase the number of relays until the computed MinMax sensor range is 0 (i.e., there are sufficient relays to colocate one with every sensor). All results presented reflect averages computed over 100 experimental trials.

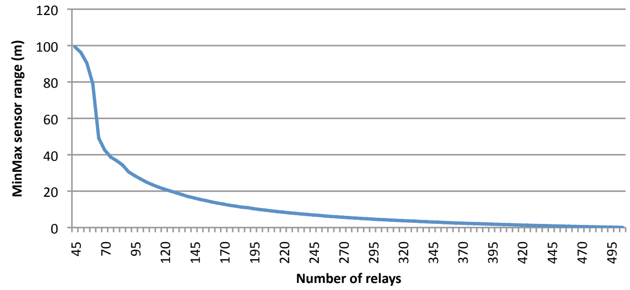

Fig. 2. Max. # of relays vs. MinMax sensor range

The results over our trials are summarized in Figure 2. The number of relays is represented on the horizontal axis, and the resulting MinMax range (measured in meters) is represented on the vertical axis. The MinMax sensor range decreases sharply as relays are added above the minimum required to establish connectivity — but the returns are diminishing. For a deployment consisting of 500 nodes, approximately 45 relays are required to connect the entire network. If the total number of relays is increased to 120, the MinMax sensor range drops dramatically from 99m to 20m. Beyond 120 relays, the MinMax sensor range continues to decline, but only gradually. These results reinforce our expectation that introducing a more expansive relay network assists in ensuring connectivity and reducing sensor power consumption. At the same time, the results serve as a guide for balancing cost (of relays) and network lifetime (via energy consumption). In this scenario, approximately $90 \sim 120$ relays appears to be optimal.

B. Number of relays vs. Packet reception rate

To study the impact of relay capacity on overall packet reception rate (at the base station), we use the popular QualNet [12] simulation platform. The QualNet simulation parameters are detailed in Table I.

Recall that sensor nodes may only communicate with relays. To implement this constraint in QualNet, the sensors in the bottom tier are configured as *reduced-function devices* (RFDs), and the second-tier relays are configured as *full-function devices* (FFDs). RFDs can only communicate with FFDs, and the communication links between FFDs (and the base station) are configured as WiFi connections.

We utilized the Friis transmission equation to calculate the transmission distance in free-space as a function of transmission power at the receiving node [13]:

$$d = \frac{10^{(P_t+G_t+G_r-P_r)/20.0}}{\frac{4\pi}{c} \cdot f},$$

where P_r and P_t are the transmission powers (in dBm) at the receiver and sender, G_r and G_t are the antenna gains (in dB)

TABLE I
QUALNET CONFIGURATION

General	
Application data	CBR (*constant bit rate*)
Topology	two-tiered network
Packet size	70 Bytes
Relay transmission range	100m
Routing protocol	AODV
Bottom layer	
Physical and MAC layer	802.15.4
Device type	RFD
Channel frequency	2.4 GHz
Tx power	varied
Rx sensitivity	-100dBm
Propagation limit	-100dBm
Second tier	
Physical and MAC layer	802.11a (*ad-hoc*)
Device type	FFD
Channel frequency	24.1 GHz
Tx power	23.39dBm
Rx sensitivity	-85 dBm
Propagation limit	-100dBm
Max propagation distance	400m
Data rate	6 Mbps
Path-loss model	free-space

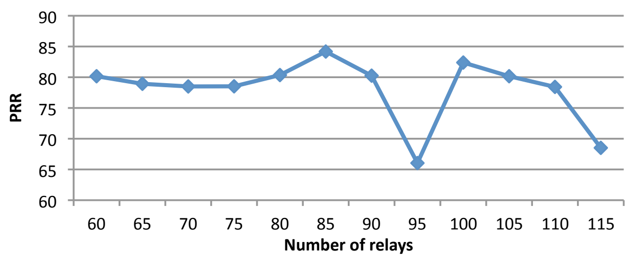

Fig. 3. Maximum number of relays vs. PRR

of the receiver and the sender, f is the signal frequency, and c is the speed of light in a vacuum, which is $2.998 \times 10^8 m/s$. Note that Friis's law only applies to the far field of the antenna (i.e., when the propagation distance is much larger than the square of the antenna size divided by the wavelength).

We consider a random deployment of 500 sensors and one base station on the comb-grid with a relay transmission range of 100 meters. For each resulting deployment, we again vary the number of relays within the solution space. Figures 3 and 4 summarize the results of our experimental trials. In both graphs, the horizontal axis represents the number of relays. In Figure 3, the vertical axis represents the effective packet reception rate at the base station. In Figure 4, the vertical axis represents the average packet delay (in seconds) from point of observation to delivery at the base station.

The key observation to be gleaned from Figure 3 is that beyond the minimum number of relays required to achieve connectivity, the impact on PRR is modest. We speculate that as we increase the number of relays, we are effectively trading congestion in the first tier of the network for congestion in the second tier. The results from Subsection VI-A indicate that introducing additional relays enables reduced transmission power in the collection network, with concomitant decreases in congestion. At the same time, the introduction of new relays increases congestion in the relay network. While congestion reduces exponentially in the collection network, the transmission power of the relays is significantly higher. We expect the two competing forces balance one another out.

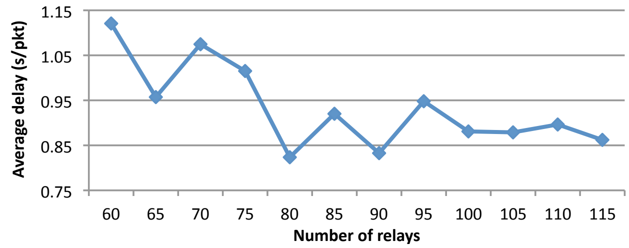

Fig. 4. Maximum number of relays vs. Delay

The results summarized in Figure 4 are somewhat similar. While introducing additional relays appears to reduce end-to-end observation delay, the reduction is modest. As the number of relays is increased from 60 to 115, the average delay falls by approximately 260 milliseconds.

VII. CONCLUSION

We conclude with two directions for future research. One is to consider other graph theoretic requirements for the relay network (e.g. higher connectivity). A second direction is to identify other practical sensor deployment structures and study the relay node placement problems for those structures.

ACKNOWLEDGMENTS

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