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13. [Individual Problem]

The Music Album Problem: Before the days of iTunes (and after the days of “45s”), it was not possible to purchase music one song at a time. If you wanted a song you had to buy an album that included that song. Consider then that:

- U is the set of all songs available on any album
- $L = (S_1, S_2, \dots, S_n)$ are the songs that you want to own
- Albums A_1, A_2, \dots, A_m are available for purchase where album A_i consists of a subset of the songs in U
- All albums cost the same amount
- You have enough money in your budget to purchase B albums

Question: Are there B albums that together include all of the songs in your list L ?

Show that the Music Album Problem is NP-complete.

Proof:

The instance of Music Album Problem (MAP) is:

- U : the set of all songs;
- L : the set of songs we want to own;
- A_1, A_2, \dots, A_m , the set of albums, each containing some songs from U
- B

The question is whether we can find B albums that contain all songs in L .

First we need to show $MAP \in NP$. A certificate to this problem is a set of albums: $A_{i1}, A_{i2}, \dots, A_{iB}$. It is easy for us to check whether these B albums contain all songs we want to own by doing the following: we can put all songs contained in these albums into a single set, then we search in this set for each song we want to own. If all songs are found in this set, then this is a solution to MAP .

The running time for checking this certificate depends on both the number of songs we want to own (say n) and the number of songs contained in these albums (say m). But a naïve algorithm can do this task in time $O(n * m)$. So the checking can be done in polynomial time and thus $MAP \in NP$.

We can reduce an instance of vertex cover to MAP . An instance of vertex cover is:

Graph $G = (V, E)$ and a positive integer k .

We can transfer each vertex in V into an album in MAP and each edge in E can be transferred into a song in MAP . For $\forall (u, v) \in E$, then album A_u and A_v contains this song. Also B is set to k .

Obviously this construction can be done in polynomial time.

Proof that the construction is correct:

- **Vertex cover => MAP**

Suppose we have a solution to the instance of Vertex Cover. That is a set of K vertices that any edge in this graph is incident to at least one vertex in this set.

So knowing this set of K vertices, we also know a solution to MAP which is a set of K albums. Each album corresponds to one vertex in the solution to Vertex cover. During our construction process, each edge is turned into a song we want to possess and is contained in two albums. As all edges are covered by the vertex set, all songs are contained in the set of K albums.

Thus, there is a solution to the MAP.

- **MAP => Vertex cover**

Also assume that there is a solution to the instance of MAP. So we have a set of albums and we put the corresponding vertices into the solution of vertex cover, e.g. if we should buy album A_v , we put vertex v into vertex cover. As any song on these albums represents an edge in the graph of Vertex Cover and an album represents a vertex. Any edge in the vertex cover is incident to at least one vertex. Thus there is a solution to the vertex cover.

In all, we can reduce an instance of vertex cover to an instance of MAP, which proves MAP is NP-C.