A Quick Introduction to Complex Numbers

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1 Introduction

Complex numbers are widely used in science, mathematics, and engineering—but it is in electrical engineering that complex numbers really shine. They are used in circuit theory, electromagnetic wave studies, antenna and waveguide design, signal processing, electric power designs, and electric motors. It is fair to say that had complex numbers not been invented, electrical engineering would not be where it is today.

2 Elementary Arithmetic

2.1 Rectangular Coordinates

A *complex number*, z, is made up from two *real numbers*, x and y, and is usually written as $z = x + jy$. (Beware that mathematicians and physicists usually use $z = x + iy$ and that in some contexts a complex number is written in "tuple" notation, $z = (x, y)$.) x is the *real* part of z; y is the *imaginary* part. The real part is sometimes written as $x = \mathrm{Re}(z)$ and the imaginary part as $y = \mathrm{Im}(z)$. As we shall see below, j is usually considered to be the square root of -1, i.e., $j = \sqrt{-1}$ or $j^2 = -1$.

Addition or subtraction of two complex numbers and mutiplication of a complex number by a real constant are done componentwise:

$$z_1 \pm z_2 = x_1 \pm x_2 + j(y_1 \pm y_2), \tag{1}$$

$$cz = cx + jcy. \tag{2}$$

If a and b are real constants, then we can derive the following, useful formulas:

$$a\mathrm{Re}(z_1) + b\mathrm{Re}(z_2) = \mathrm{Re}(az_1 + bz_2) \tag{3}$$

$$a\mathrm{Im}(z_1) + b\mathrm{Im}(z_2) = \mathrm{Im}(az_1 + bz_2) \tag{4}$$

Unfortunately, multiplication cannot be defined in the obvious way.[1] Define multiplication of two complex numbers by

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1). \tag{5}$$

For example, $(1 + 2j)(3 + 4j) = (3 - 8) + j(4 + 6) = -5 + 10j$. An easy way to remember this formula—and this formula must be remembered—is to do ordinary polynomial multiplication and replace j^2 by -1,

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 + j^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned}$$

Before division can be accomplished, we need to define the *conjugate* of z, denoted z^*, as $z^* = x - jy$. (Note the conjugate of a real number is itself.) Now multiply z by z^*,

$$\begin{aligned} zz^* &= xx - y(-y) + j(xy - xy) \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

where $|z| = \sqrt{x^2 + y^2}$ is the *magnitude* of z. Note $|z|$ is a nonnegative, real number and $|z| = 0$ if and only if $z = 0$. Now, division is easy:

$$1/z = z^*/zz^* = (1/|z|^2)z^* \tag{6}$$

2.2 Polar Coordinates

If is often convenient to think of a complex number, z, as representing a point or a vector from the origin to the point in the xy or *complex* plane. (Think of the tuple representation as defining a point in the plane!) The complex number, $z = x + jy$, is sometimes written as $R\angle\theta$ with the distance to the point as $R = |z|$

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[1] $z_1 z_2 = x_1 x_2 + jy_1 y_2$ does not work because if $z_1 z_2 = 0$ then we want $z_1 = 0$ or $z_2 = 0$. However, this may not true.

and the angle from the x axis measured in a counterclockwise direction, θ can be found as $\theta = \tan^{-1}(\mathrm{Im}(z)/\mathrm{Re}(z))$. For example, $1 + 1j = \sqrt{2}\angle(\pi/4)$ and $1 - 1j = \sqrt{2}\angle(-\pi/4)$.[2] There is a difficulty with finding the angle, however. If $z \to -z$, then $\theta \to \theta \pm \pi$. But, $\mathrm{Re}(z) \to -\mathrm{Re}(z)$ and $\mathrm{Im}(z) \to -\mathrm{Im}(z)$. The ratio of real and imaginary parts remains the same! So, one must be careful when computing the angle to predetermine which quadrant the complex number is in. However, it is easy to go from polar coordinates to rectangular coordinates:

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned}$$

The great virtue of using polar coordinates is that multiplication and division is much easier. We will state the result as a theorem:

Theorem 1. If $z_1 = R_1\angle\theta_1$ and $z_2 = R_2\angle\theta_2$, then $z_1 z_2 = R_1 R_2 \angle(\theta_1 + \theta_2)$.

Proof:

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1) \tag{7}$$

First, show the magnitude part:

$$\begin{aligned} |z_1 z_2|^2 &= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \\ &= x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2 \\ &= x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 \\ &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Now, show the angle part:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{\mathrm{Im}(z_1 z_2)}{\mathrm{Re}(z_1 z_2)}\right) \\ &= \tan^{-1}\left(\frac{x_1 y_2 + x_2 y_1}{x_1 x_2 - y_1 y_2}\right) \\ &= \tan^{-1}\left(\frac{x_1 y_2/R_1 R_2 + x_2 y_1/R_1 R_2}{x_1 x_2/R_1 R_2 - y_1 y_2/R_1 R_2}\right) \\ &= \tan^{-1}\left(\frac{\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2}{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}\right) \\ &= \tan^{-1}\left(\frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)}\right) \\ &= \tan^{-1}\left(\tan(\theta_1 + \theta_2)\right) \\ &= \theta_1 + \theta_2. \end{aligned}$$

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[2] We will measure all angles in radians and will normalize them so that $-\pi \le \theta \le \pi$.

One can use this formula to show: $z^{-1} = R^{-1}\angle(-\theta)$, $z^2 = R^2\angle(2\theta)$, and $z^n = R^n\angle(n\theta)$ where n is an integer.

What happens when n is not an integer? For instance, what does $z^{\frac{1}{2}}$ mean? Take $z^{\frac{1}{n}}$ to be *any* complex number such that $z^{(\frac{1}{n})n} = z$. In particular, if $z = R\angle\theta$ then $z^{\frac{1}{2}} = R^{\frac{1}{2}}\angle\theta/2$ works. However, $z^{\frac{1}{2}} = R^{\frac{1}{2}}\angle(\theta/2 \pm \pi)$ also works! When this number is squared, one gets $R\angle(\theta \pm 2\pi)$ which is identical to $R\angle\theta$.

In general, a complex number z has exactly n complex n^{th} roots. We can write these as $z_k^{\frac{1}{n}} = R^{\frac{1}{n}}\angle(\theta + k2\pi)/n$ where $k = 0, 1, 2, \ldots, n-1$. When one raises any of these numbers to the n^{th} power, one gets $R\angle(\theta + k2\pi)$. The extra $k2\pi$ is irrelevant, so each of these are indeed an n^{th} root of z.

3 Functions of a Complex Variable

3.1 Polynomials and Rational Functions

Let z now refer to an unknown complex number or, simply, a complex variable. The simplest functions of z are polynomials:

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k \tag{8}$$

When the a_k are real numbers, a very famous and strong result about these polynomials exists and is usually called "The Fundamental Theorem of Algebra": Every n^{th} degree polynomial in a complex variable, z, with real coefficients, a_k, has exactly n roots.

We have seen a special case of this result already in computing n^{th} roots of complex numbers. Consider the polynomial, $z^n - a = 0$. Then the numbers, $z_k = |a|^{\frac{1}{n}}\angle(\theta + k2\pi)/n$ are the roots of this polynomial.

The fundamental theorem of algebra says that every polynomial with real roots can be factored in one and only one way as[3]:

$$\begin{aligned} P(z) &= \sum_{k=0}^{n} a_k z^k \\ &= a_n(z - z_1)(z - z_2)(z - z_3)\cdots(z - z_n) \\ &\equiv a_n \prod_{k=1}^{n}(z - z_k) \end{aligned}$$

The complex numbers, z_k, are the *zeroes* of the polynomial, i.e., $P(z_k) = 0$.

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[3] The factorization is unique except for trivial reorderings of the terms.

A complex polynomial, like the other complex functions discussed below, has a real part and an imaginary part. For example, the polynomial $z-1 = x-1+jy$ has real part equal to $x-1$ and imaginary part equal to y. We can talk about its magnitude, $|P(z)| = \sqrt{(x-1)^2+y^2}$, and its angle, $\angle P(z) = \tan^{-1}(y/(x-1))$. As expected each of these are functions of x and y. As another example, we will often need to determine the magnitude of a polynomial. It is usually most convenient to factor the polynomial and use the formula

$$\begin{aligned} |P_n(z)| &= |a_n(z-z_1)(z-z_2)\cdots(z-z_n)| \\ &= |a_n|\cdot|z-z_1|\cdot|z-z_2|\cdots|z-z_n|. \end{aligned}$$

After polynomials, the next simplest complex functions are rational functions which are simply ratios of polynomials. Let $N(z)$ and $D(z)$ be polynomials in z. Then a rational function, $R(z)$, can be written:

$$R(z) \equiv \frac{N(z)}{D(z)} \tag{9}$$

Factor $N(z)$ and $D(z)$ and denote the zeroes of $N(z)$ as z_k and the zeroes of $D(z)$ as p_j. (For convenience, we will assume that no $z_k = p_j$.) Then $R(z_k) = 0$ and $R(p_j) = \infty$. As above, the z_k are the *zeroes* of $R(z)$, but the p_j are called the *poles* of $R(z)$. The location of the poles and zeroes in the complex plane can determine a great deal about the properties of the system under consideration.

As an example, consider the power sum:

$$R(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \tag{10}$$

when $|z| < 1$. $R(z)$ has no zeroes, but it has a pole at $z = 1$. We see that the series convergence is limited by the location of the pole. This is a general result, but not one we will pursue further.

3.2 Exponential Functions and Euler's Formula

After polynomials and rational functions come exponentials and trigonometric functions in terms of complexity and usefulness. The exponential function, e^z, is defined through its power series:

$$e^z \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{11}$$

where $k!$ (pronounced k *factorial*) equals the product, $1\cdot2\cdot3\cdots(k-1)\cdot k$ and, for convenience, $0! = 1$. It turns out that all the usual properties of exponentials still work for complex exponentials, in particular, $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

Consider a purely imaginary $z = 0 + jy$. Then,

$$\begin{aligned} e^{jy} &= \sum_{k=0}^{\infty} \frac{(jy)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{j^k y^k}{k!} \end{aligned}$$

Now, use the properties that $j^2 = -1$, $j^3 = -j$ and $j^4 = 1$ and separate into real and imaginary parts:

$$\begin{aligned} e^{jy} &= 1 - y^2/2! + y^4/4! - \cdots \\ &\quad + j(y - y^3/3! + y^5/5! - \cdots) \end{aligned}$$

The real part is the power series for $\cos(y)$ and the imaginary part is the power series for $\sin(y)$!! Thus, we get the incredibly useful Euler's formula:

$$e^{jy} = \cos(y) + j\sin(y) \tag{12}$$

If z is not purely imaginary we can write

$$e^z = e^x e^{jy} = e^x(\cos(y) + j\sin(y)) \tag{13}$$

A little manipulation of Euler's formula yields some interesting results:

$$\begin{aligned} e^{-jy} &= \cos(-y) + j\sin(-y) \\ &= \cos(y) - j\sin(y) \end{aligned}$$

Combining this formula with Euler's gives

$$\cos(y) = \frac{(e^{jy} + e^{-jy})}{2} \tag{14}$$

and

$$\sin(y) = \frac{(e^{jy} - e^{-jy})}{2j}. \tag{15}$$

Careful use of these formulas allows one to manipulate trigonometric expressions without knowing any trigonometric identities. For example, let's derive the formula for $\cos^2(y)$:

$$\begin{aligned} \cos^2(y) &= \frac{(e^{jy} + e^{-jy})}{2}\frac{(e^{jy} + e^{-jy})}{2} \\ &= \frac{e^{2jy} + 2 + e^{-2jy}}{4} \\ &= \frac{1 + \cos(2y)}{2} \end{aligned}$$

There is a strong connection between the polar coordinates representation of a complex number and Euler's formula. Consider

$$\begin{aligned} z &= R\angle\theta \\ &= R\cos\theta + jR\sin\theta \\ &= R(\cos\theta + j\sin\theta) \\ &= Re^{j\theta}. \end{aligned}$$

So $R\angle\theta$ is just shorthand for $Re^{j\theta}$.

4 Homework

1. For $z_1 = -1 + 1j$, $z_2 = 3j$, and $z_3 = 3 - 5j$, find
   (a) the polar representation of each.
   (b) $z_1 z_2$, $z_1 z_3$, z_1/z_3, and z_2/z_3 in both rectangular and polar coordinates. (You should do this with minimal use of calculators.)
   (c) e^{z_1} and e^{z_3}.
2. Let $P(z) = z^2 - 2z + 1$, find
   (a) $|P(z)|$ and $\angle P(z)$. What does the curve $|P(z)| = 1$ look like in the complex plane?
   (b) the zeroes of $P(z)$.
3. 3. Using Euler's formula, prove
   (a) $\cos^2\alpha + \sin^2\alpha = 1$.
   (b) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.
4. 4. Using the result $\cos(\omega t + \theta) = \mathrm{Re}(e^{j(\omega t+\theta)})$ and complex algebra—not trigonometric identities, find
   (a) $\cos(\omega t) + 2\cos(\omega t + \pi/4)$.
   (b) $\cos(\omega t) + \cos(\omega t + 2\pi/3) + \cos(\omega t + 4\pi/3)$.