ELEG403 COMMUNICATIONS SYSTEMS ENGINEERING

ASSIGNMENT #7 due Wednesday November 12

1. Haykin Problem 8.3. Also, answer the following additional questions for this problem. For these added parts, let A be the event that $X < 0.5$, B be the event $X > 0.5$, and C be the event $0.25 < X < 0.75$.
   d) Find P(A), P(B), P(C), and P(A/C).
   e) Are A and C independent events?
   f) Find E(X), the mean value of X.

Let $a = 1/2$

2. Haykin Problem 8.8

3. Haykin Problem 8.18

4. Haykin Problem 8.32

5. Haykin Problem 8.33

SOLUTIONS TO ASSIGNMENT #7

1. Haykin Problem # 8.3

$$f_X(x) = a e^{-b|x|} \quad a, b \text{ positive}$$

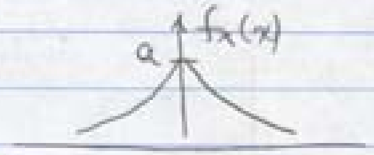
a.) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\therefore a\int_{-\infty}^{\infty} e^{-b|x|}dx = 2a\int_{0}^{\infty} e^{-bx}dx = \frac{2a}{b} = 1$$

$$a = b/2 \quad \text{or} \quad b = 2a$$

For $a = 1/2$, $b = 1$

$$f_X(x) = \frac{1}{2} e^{-|x|}$$

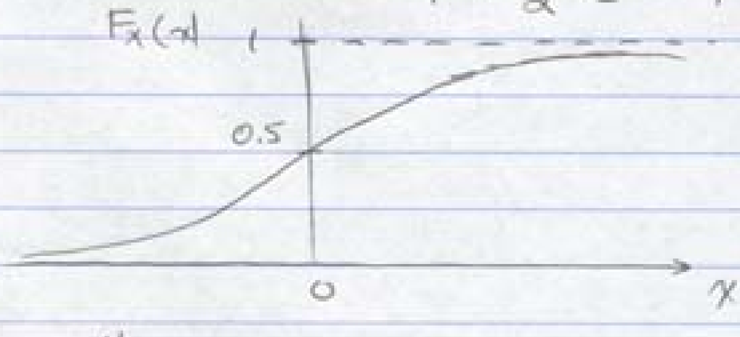


b.) $F_X(x) = P[X \le x]$

$$= \int_{-\infty}^{x} f_X(\alpha)\,d\alpha = a\int_{-\infty}^{x} e^{-2a|\alpha|}d\alpha$$

$$= \begin{cases} a\int_{-\infty}^{x} e^{2a\alpha}d\alpha = \frac{a}{2a} e^{2a\alpha}\Big|_{-\infty}^{x} = \frac{1}{2} e^{2ax}, & -\infty < x \le 0 \\ \underbrace{a\int_{-\infty}^{0} e^{2a\alpha}d\alpha}_{1/2} + a\int_{0}^{x} e^{-2a\alpha}d\alpha, & 0 \le x < \infty \end{cases}$$

$$= 1 - \frac{1}{2} e^{-2ax}, \quad 0 \le x < \infty$$



For $a = 1/2$,

$$F_X(x) = \begin{cases} \frac{1}{2} e^{x}, & -\infty < x \le 0 \\ 1 - \frac{1}{2} e^{-x}, & 0 \le x < \infty \end{cases}$$

1. cont'd) c.) $P(1 \le x \le 2) = \int_1^2 \frac{1}{2} e^{-|x|} dx$

$$= -\frac{1}{2}\left(e^{-2} - e^{-1}\right) = 0.116$$

d.) A: $x < 0.5$

$P(A) = P(x < 0.5) = F_x(0.5)$

$$= 1 - \frac{1}{2} e^{-1/2} = 0.6967$$

B: $x > 0.5$

$P(B) = P(x > 0.5) = 1 - P(x < 0.5) = 0.3033$

C: $0.25 < x < 0.75$

$P(C) = P(0.25 < x < 0.75)$

$$= \int_{1/4}^{3/4} \frac{1}{2} e^{-x} dx = -\frac{1}{2}\left(e^{-3/4} - e^{-1/4}\right) = 0.153$$

e.) If A and C are independent,

$P(AC) = P(A)P(C)$

$\therefore \quad P(A|C) = \dfrac{P(AC)}{P(C)} = P(A)$

$P(A|C) = \dfrac{P(A \text{ and } C)}{P(C)}$

But $P(A \underbrace{\text{and}}_{\text{intersection}} C) = P(0.25 < x < 0.5)$

$$= \int_{1/4}^{1/2} \frac{1}{2} e^{-x} dx = -\frac{1}{2}\left(e^{-1/2} - e^{-1/4}\right)$$

$$= 0.086$$

$\therefore \; P(A|C) = 0.086/0.153 = 0.562 \neq P(A)$

$\therefore$ A and C are <u>not</u> independent

1. cont'd) f.) $E(x) = \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|} dx$

$$= \int_{-\infty}^{0} \frac{x}{2} e^{x} dx + \int_{0}^{\infty} \frac{x}{2} e^{-x} dx$$

$u = x \quad du = dx$ $\quad$ $u = x \quad du = dx$

$dv = e^{x} \quad v = e^{x}$ $\quad$ $dv = e^{-x} \quad v = -e^{-x}$

$$= \cancel{\frac{x}{2} e^{x}}\Big|_{-\infty}^{0} - \frac{1}{2}\int_{-\infty}^{0} e^{x} dx - \cancel{\frac{x}{2} e^{-x}}\Big|_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} e^{-x} dx$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

2. Haykin Problem # 8.8

$X \rightarrow$ Gaussian random variable $\quad f_X(x) = \frac{1}{10\sqrt{2\pi}} e^{-x^2/200}$

$m_X = 0, \quad \sigma_X = 10V$

$Y = X + 5 \Rightarrow Y$ is Gaussian with $m_Y = 5, \sigma_Y = 10V$

$$f_Y(y) = \frac{1}{10\sqrt{2\pi}} e^{-(y-5)^2/200}$$

a.) $P(Y > 0) = \int_{0}^{\infty} f_Y(y) dy$

$$= \int_{0}^{\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{(y-5)^2}{200}} dy \qquad z = \frac{y-5}{\sqrt{200}}, \quad dz = \frac{1}{\sqrt{200}} dy$$

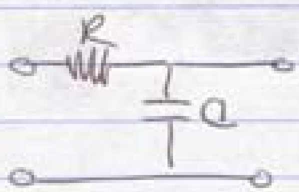
$$= \frac{1}{\sqrt{\pi}} \int_{\frac{-5}{10\sqrt{2}} = -\frac{1}{2\sqrt{2}}}^{\infty} e^{-z^2} dz$$ — tabulated function

$$= 0.5 + 0.5 \operatorname{erf}\left(\frac{1}{2\sqrt{2}}\right) = 0.69$$

b.) arithmetic mean

$Y = \frac{X_1 + X_2}{2}$, $\quad X_1, X_2$ independent

3. cont'd)



$$H(f) = \frac{1}{1 + j2\pi f RC}$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= \left(\frac{1}{1 + (2\pi f RC)^2}\right) \frac{\nu}{\nu^2 + \pi^2 f^2}$$

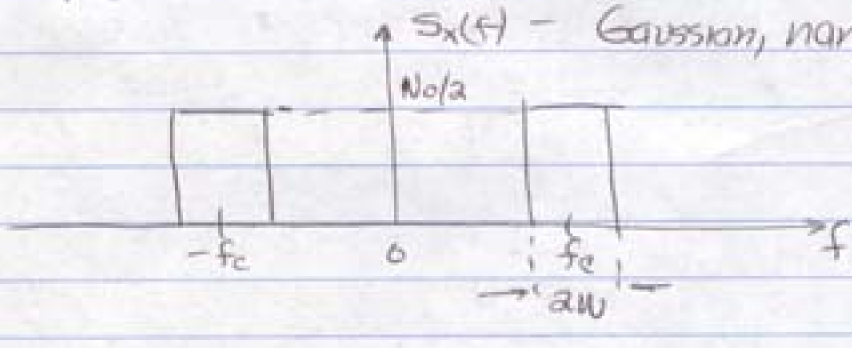
We need to compute the inverse transform. We can either use tables or expand $S_Y(f)$ using partial fractions

$$S_Y(f) = \frac{\nu}{1 - 4R^2C^2\nu^2}\left[\frac{-1}{\left(\frac{1}{2RC}\right)^2 + \pi^2 f^2} + \frac{1}{\nu^2 + \pi^2 f^2}\right]$$

$$\therefore R_Y(\tau) = \frac{\nu}{1 - 4R^2C^2\nu^2}\left[-2RC\, e^{-\frac{|\tau|}{RC}} + \frac{1}{\nu} e^{-2\nu|\tau|}\right]$$

4. Haykin Problem # 8.32

$S_X(f)$ - Gaussian, narrowband

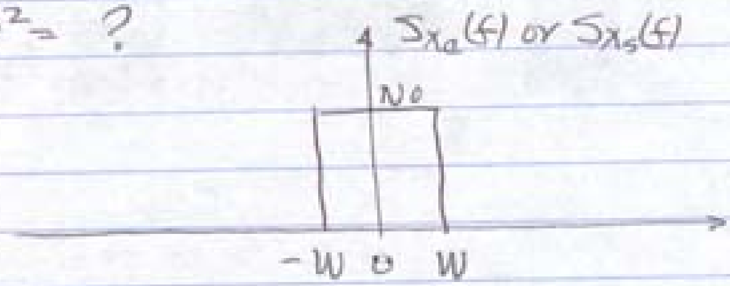


a) $x(t) = x_c(t)\cos W_c t - x_s(t)\sin W_c t$

$r_x(t) = \sqrt{x_c^2(t) + x_s^2(t)} \longrightarrow$ Rayleigh

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad r \geq 0$$

4. cont'd) $\sigma^2 = ?$



$$\therefore \sigma^2 = 2WN_o$$

$$f_R(r) = \frac{r}{2WN_o} e^{-\frac{r^2}{4N_oW}}, \quad r \geq 0$$

b.) Statistics of Rayleigh $E[R] = \sqrt{\frac{\pi}{2}}\,\sigma = \sqrt{\pi N_o W}$

$\sigma_R^2 = 0.858\, N_oW$

5. Haykin Problem #8.33

$$P(R \geq A_c) = \int_{A_c}^{\infty} \frac{r}{2N_oW} e^{-\frac{r^2}{4N_oW}} dr \qquad u = r^2/4N_oW$$

$$du = \cancel{2}r\,dr/\cancel{4}2N_oW$$

$$= \int_{A_c^2/4N_oW}^{\infty} e^{-u}\,du$$

$$= e^{-\left(\frac{A_c^2}{4N_oW}\right)} = \rho$$