How to Make Product Zero: Define

Xiang-Gen Xia, University of Delaware, 2023.1.10

Let S be a set of elements where the multiplications of elements are well defined. If one arbitrarily takes elements x_1, x_2, …, x_k from set S and these elements may be repeated, i.e., it may be possible that $x_1=x_2$. Then, one multiplies these k elements together $x_1x_2\cdots x_k$. The most interesting outcome of this product would be 0. Although in general, it will not happen but there are two cases to make the above product 0 as follows.

One is when the multiplication of the elements in S is not commutative, such as matrices. In this case, to make most (or as many as possible) of these products 0 is a method used in free probability theory, see for example [1], and its detailed discussions can be found in [2]. The reason to make these products 0 as many as possible is to calculate the high order moments of a sum of variables. It is to have a limit theorem of noncommutative random variables as the central limit theorem of the conventional commutative random variables. If most of the cross products of these non-commutative random variables are 0 in mean (called the freeness in analogous to the independence of the conventional commutative random variables), then, the high order moments of their sum can be dominated and estimated as that of the semicircle distribution, as the number of the elements in the sum goes to infinity. For more details, see [1,2].

The other is when the multiplication of the elements in S is commutative. In this case, to make these products 0 as much as possible if one can make $x^2=0$. If an addition is defined for every two elements in set S and x+x=0 as well, S is called a Wang algebra [3] that is a nice and recent article written by Bob Ross and Cong Ling. I would like to thank Cong to send his paper to me. It comprehensively covers the story about Wang and the related results. The Wang in Wang algebra is Ki-Tung Wang (王季同), one of the Chinese pioneers in electronics. Wang observed Wang algebra from circuit theory and applied it to circuit theory as well. Wang algebra has applications in fast calculating the determinants of symmetric matrices. Interestingly, another Chinese pioneer in electronics, Yu-Hsiu Ku (顾毓琇)， also did some similar works. For more details, see [3]. Wang is the father of the former life-time Editor-in-Chief of Acta Electronica Sinica (电子学报) Shou Jue Wang (王守觉).

One can see that the first above is for cross products and the second above is for auto-products of elements. Both of them define products as zeros but fortunately in practice there are indeed such products (approximately and/or exactly).

References

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