UNIVERSITY of DELAWARE

数学与电子工程

Mathematics and Electrical Engineering

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提纲

我的单位

电子工程里的数学

数字通信里的数学

初等数学及其应用

高等数学及其应用

正交设计在现代通信里的应用

Outline:

My university

The math in EE

The math in communications

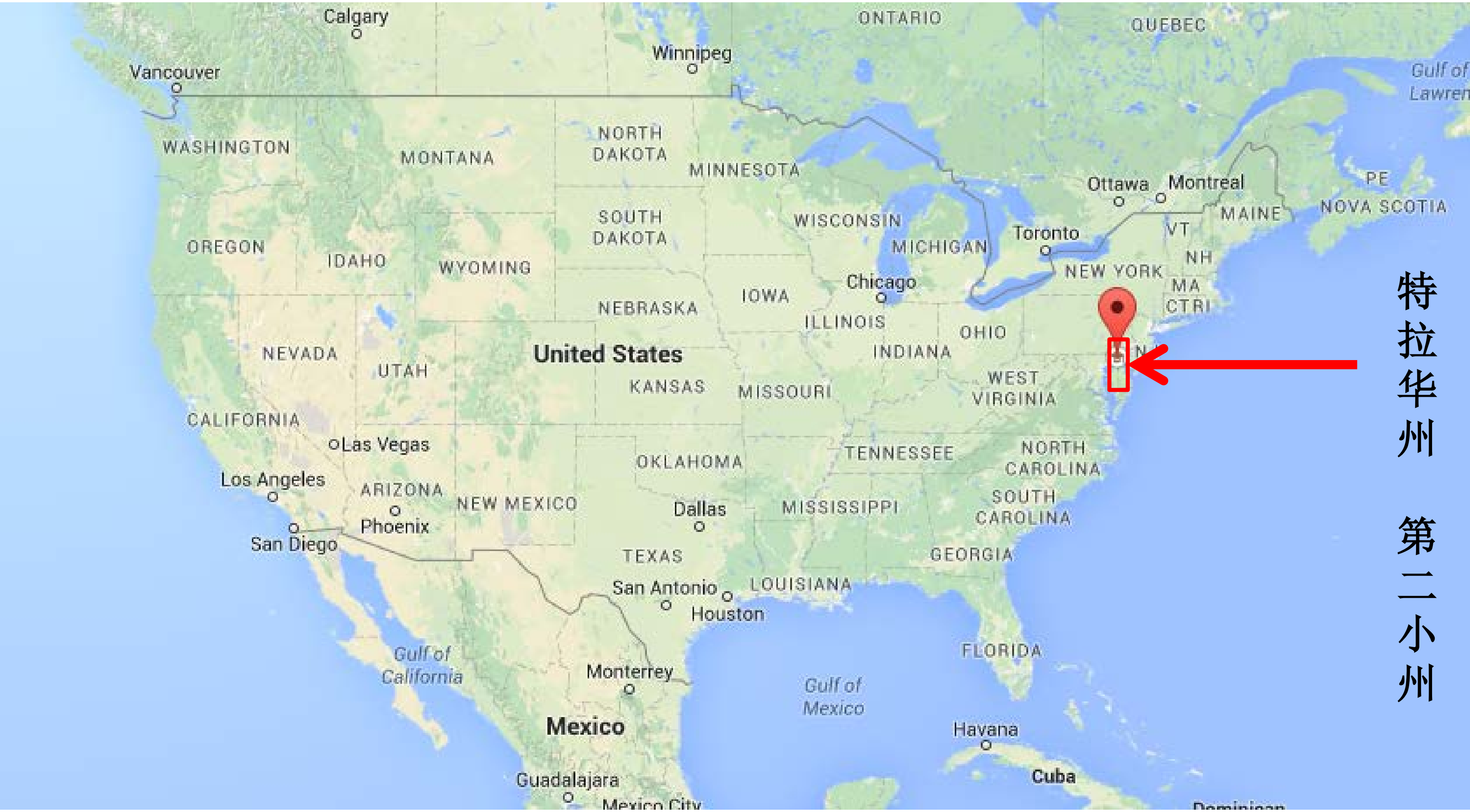
- Elementary math and its applications
- Advanced math and its applications

Orthogonal designs and applications in modern communications

美国特拉华州 (State of Delaware)

特拉华大学 (University of Delaware)

电子与计算机工程系 (Department of Electrical and Computer Engineering)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WEST VIRGINIA
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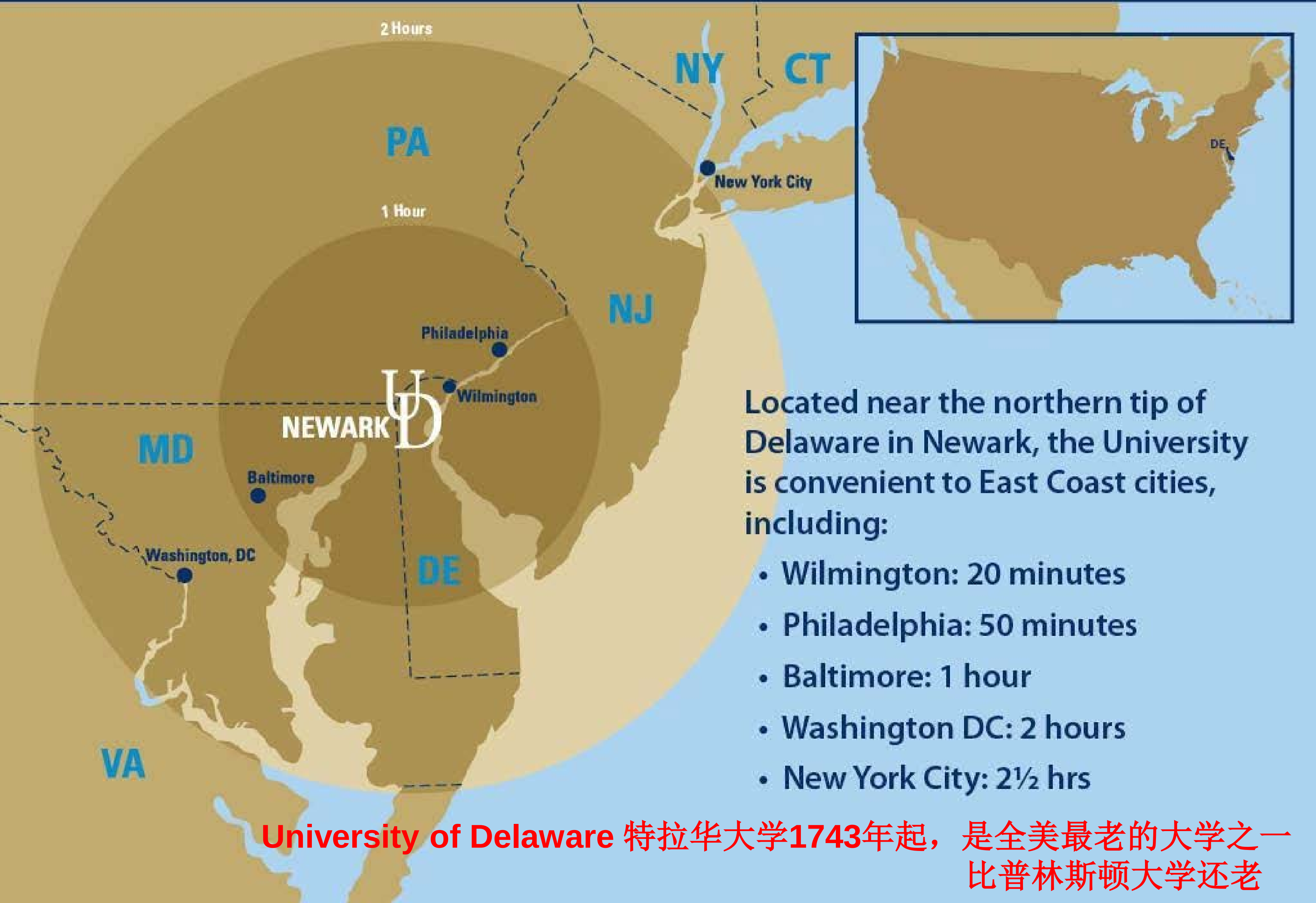
SOUTH CAROLINA
Phoenix
San Diego
TEXAS
GEORGIA
San Antonio
LOUISIANA
Houston
Gulf of California
FLORIDA
Monterrey
Gulf of Mexico
Mexico
Havana
Cuba
Guadalajara
Mexico City

特拉华州 第二小州

Williamsport
Wilkes-Barre
Danbury
New Haven
Milford
Hazleton
PENNSYLVANIA
Yonkers
New York
Long Island
Levittown
Pottsville
Bethlehem
Allentown
Edison
Altoona
Harrisburg
Reading
Trenton
Middletown
Lancaster
Brick
Philadelphia
Toms River
York
University of Delaware
Research-geared school
founded in 1743
Hagerstown
MARYLAND
Atlantic City
Dover
Germantown
Baltimore
Washington
DELAWARE
Manassas
Woodbridge
Stafford
Salisbury
Charlottesville
纽约市
费城
特拉华州
巴尔的摩市
华盛顿DC

Our Location at the Center of the East Coast of the United States

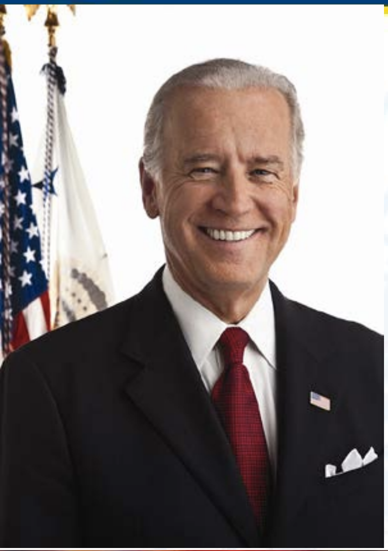


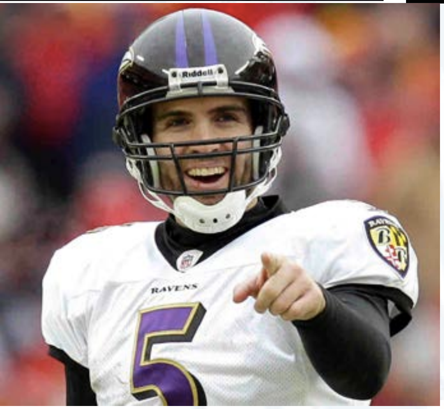
Located near the northern tip of Delaware in Newark, the University is convenient to East Coast cities, including:

- Wilmington: 20 minutes
- Philadelphia: 50 minutes
- Baltimore: 1 hour
- Washington DC: 2 hours
- New York City: 2½ hrs

University of Delaware 特拉华大学1743年起，是全美最老的大学之一 比普林斯顿大学还老

秋天的校园





Famous Alumni

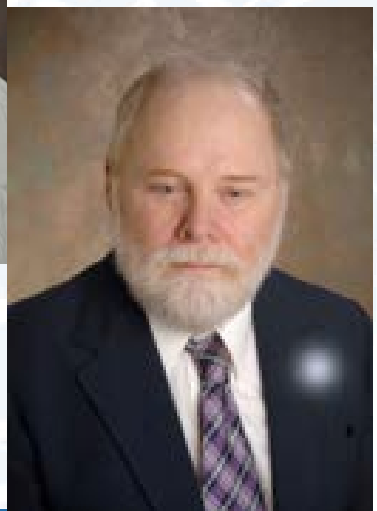
- Joe Biden, President of the USA.

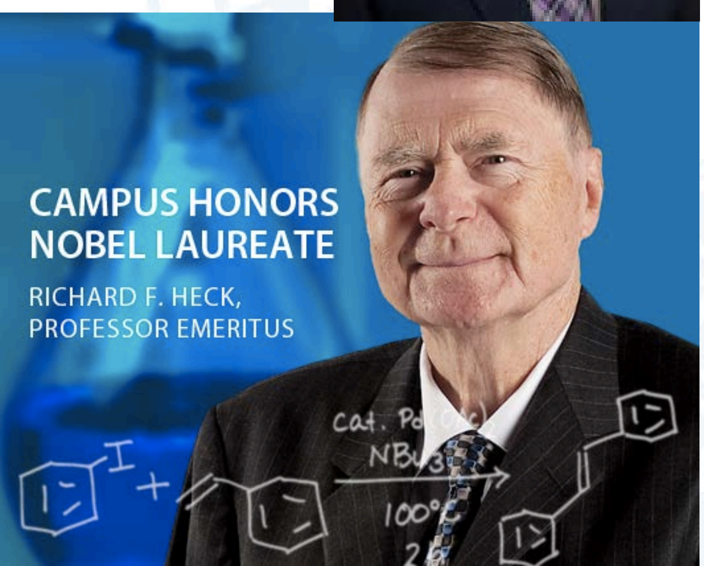
- Chris Christie, Governor of New Jersey and potential presidential candidate.

- Joe Flacco, NFL Super Bowl MVP (most valuable player).

- Xin Wang, builder of RenRen Net (人人网）

- Wayne Westerman, inventor of multi-touch interface.





Famous Faculty

- **Dave Farber, Internet pioneer.**
  Pioneer's Circle of Internet Hall of Fame
  网络先驱者名人墙

- **Dave Mills, Internet pioneer and inventor of the Network Time Protocol.**

- **Richard Heck, 2010 Nobel Prize in Chemistry.**


Evans Hall
Home of ECE Department
电子与计算机工程系

Innovating in leading tech sectors

FingerWorks, a company started by Electrical and Computer Engineering Professor John Elias and UD alumnus Wayne Westerman, developed the key technology in the iPhone's multi-touch interface.

"The iPhone would not have been possible without the engineering solutions of Professors John Elias and Wayne Westerman of the University of Delaware who developed multi-touch sensing capabilities" **--- Steve Jobs' biography**

2005年Apple公司买了FingerWorks公司后2007年才有iPhone

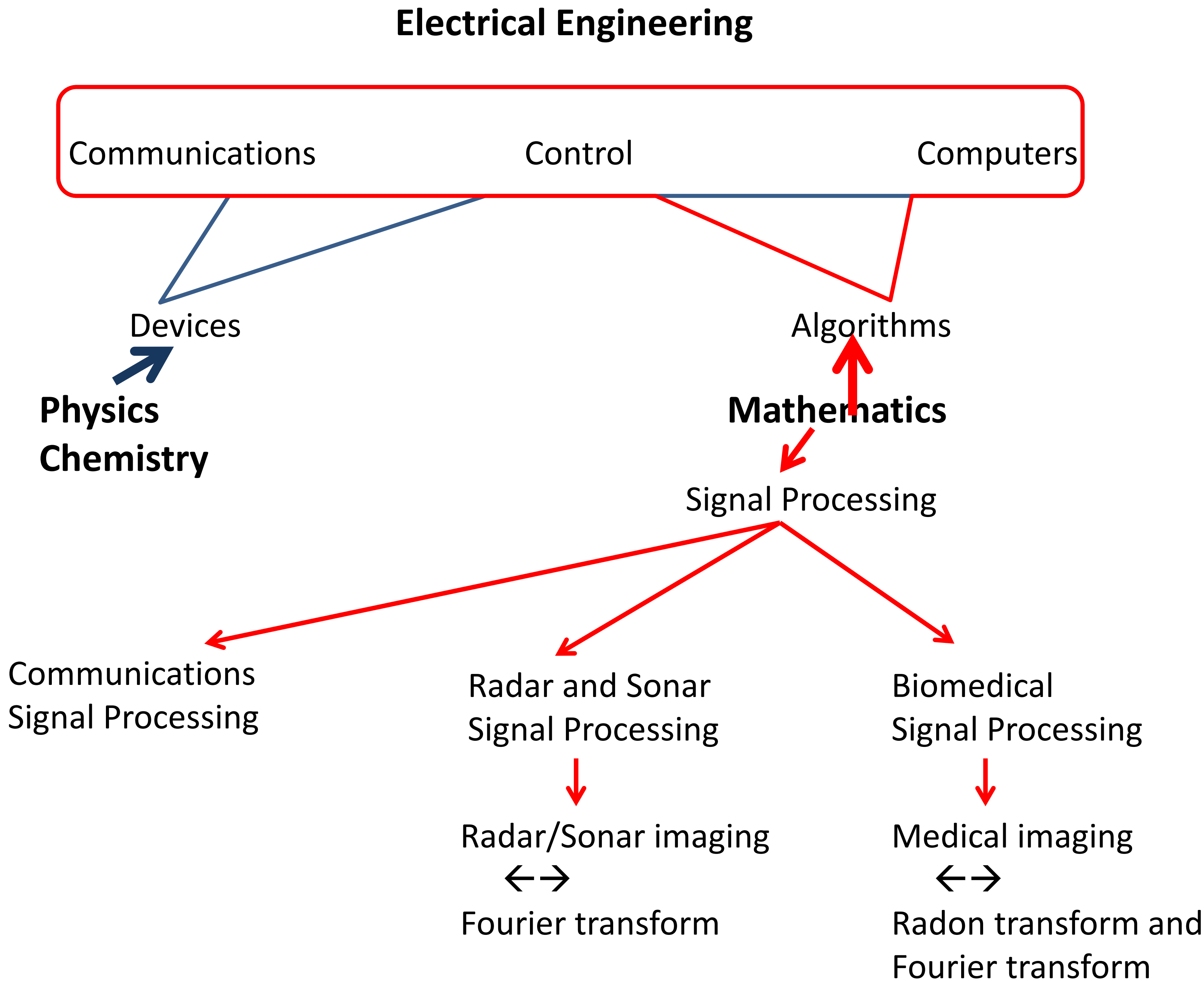
而真正的智能手机又是从**iPhone**开始的

从我在美国过去30年的生活来看，其它方面没有改进（衣，食，住，行），只有通信等电子产品改进了老百姓的日常生活。

在所有电子产品里，由于芯片速度的增加，通信/计算机改变得最大

智能手机的出现改变了人们的日常生活。。。

From my life in the past 30 years in USA, I do not see anything else is changed for better, but only electronics has changed dramatically, in particular, communications and computing devices.


Electrical Engineering
Communications
Control
Computers
Devices
Algorithms
Physics
Chemistry
Mathematics
Signal Processing
Communications
Signal Processing
Radar and Sonar
Signal Processing
Biomedical
Signal Processing
Radar/Sonar imaging
Fourier transform
Medical imaging
Radon transform and
Fourier transform

Communications and Math

模拟通信

Analog Communications

1G

Complex analysis
Differential equations
Linear algebra

数字通信

Digital Communications

2G, 3G, 4G → 5G

Transmitter

Receiver

$$\sum_n s_n p(t-nT)$$

How to design these signals to be transmitted

Modern algebra
Combinatorics
Geometry
Algebraic geometry
Number theory
Algebraic number theory

How to design them waveforms

Real analysis
Functional analysis
Harmonic analysis
Numerical analysis

A hot topic in 5G now

How to receive them

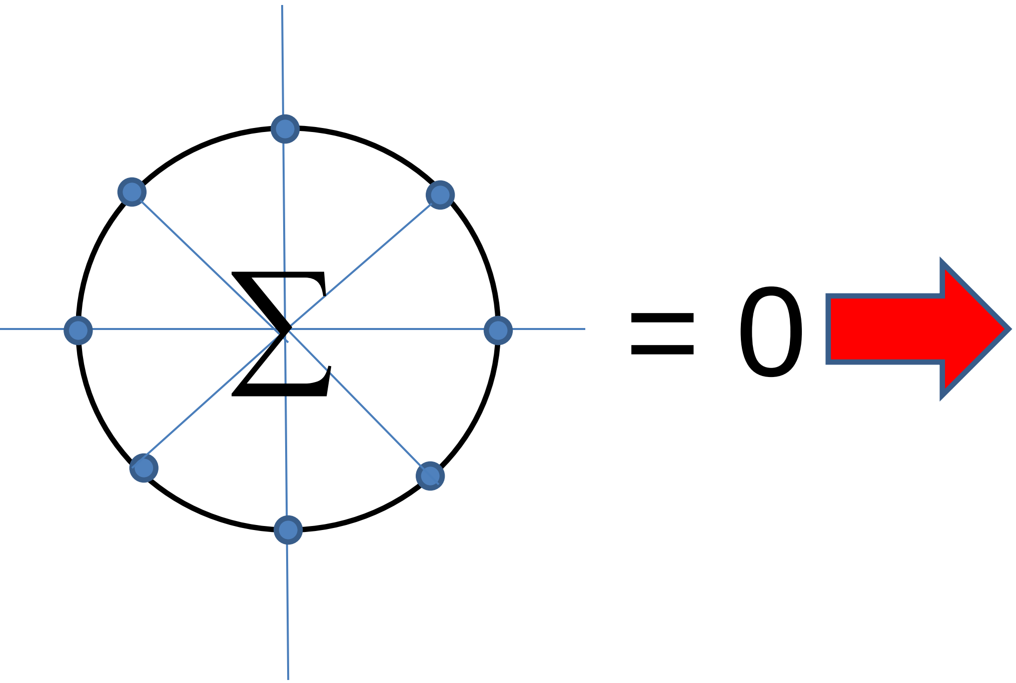
Probability theory
Statistics
Linear algebra

It is YOU to design both of transmitter and receiver: Math plays a perfect and truly useful role here

初等数学及现代通信

Elementary Math and Modern Communications

圆周上等分点之和等于零



= 0

Discrete Fourier Transform（DFT）

$$\sum_{n=0}^{N-1} e^{j2\pi\frac{kn}{N}} = N\delta(k),$$

$$0 \le k \le N-1$$

初等数学及现代通信

- 宽带信道：线性卷积 Channel: linear convolution

$$y(n)=h(0)x(n)+h(1)x(n-1)+\cdots+h(L-1)x(n-L)$$

- 加和去**CP** 后 线性卷积变成循环卷积
  After removing the CP, it becomes circular convolution

$$y(n)=h(n)\otimes x(n)$$

- **DFT 与OFDM**

  DFT

$$\mathrm{DFT}(h(n)\otimes x(n))=\mathrm{DFT}(h(n))\bullet\mathrm{DFT}(x(n))$$

  OFDM

$$Y(k)=H(k)X(k)$$

- 现在的 无线通信：**4G/LTE** 和 **WiFi**

问题：

Question:

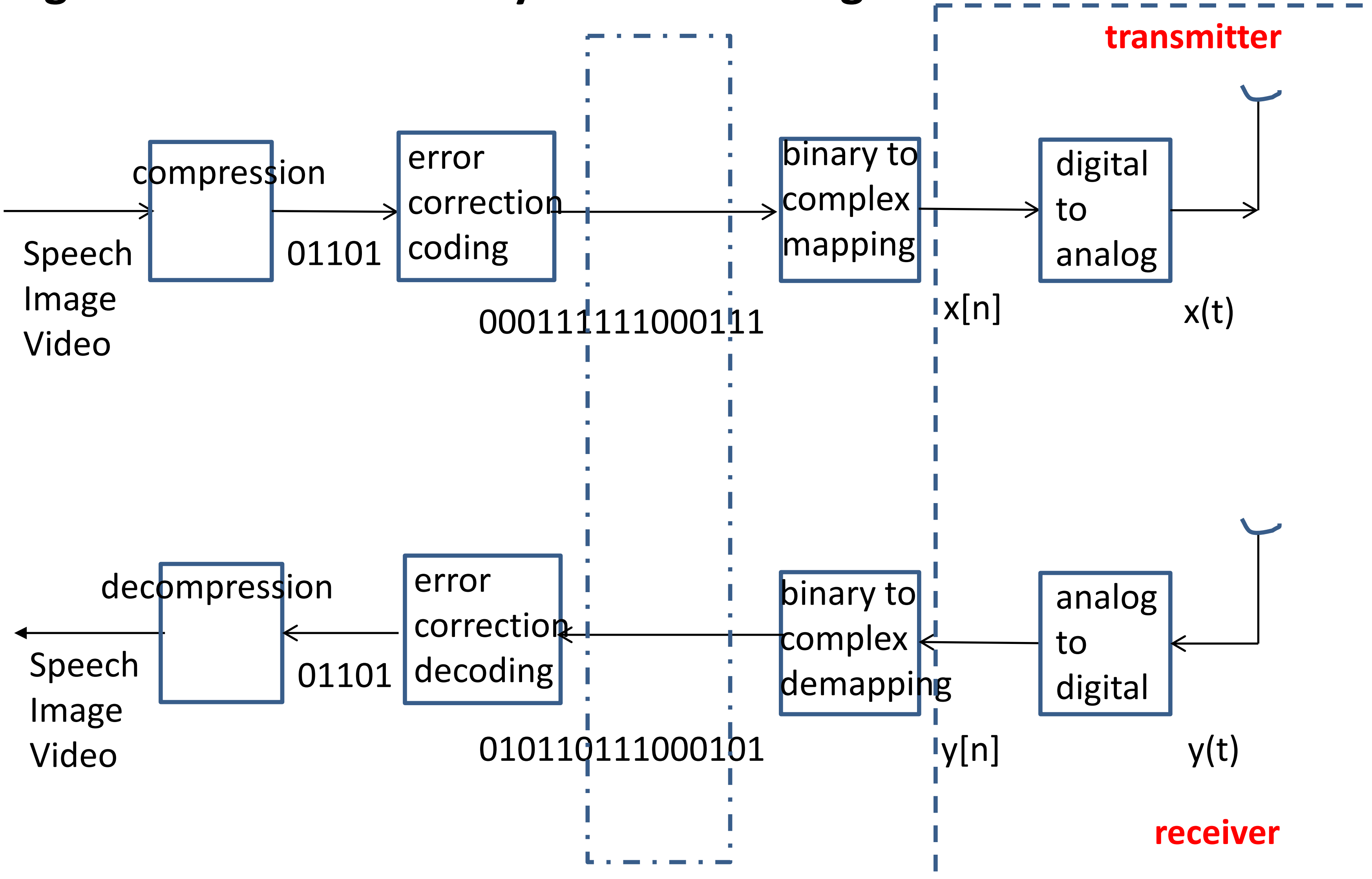
$$\sum_{n=0}^{N-1} e^{j2\pi\frac{kn^2}{N}} = ?, \quad 0 \le k \le N-1$$

请看我写的 refer to my paper on

Discrete Chirp-Fourier Transform （DCFT）

***IEEE Trans. on Signal Processing*, Nov. 2000.**

Digital Communication System Block Diagram



binary to binary channel

complex value to complex value channel
y = x + w

Digital Modulation and Demodulation

- Real data, such as speech/image/video, are collected and converted to binary sequences

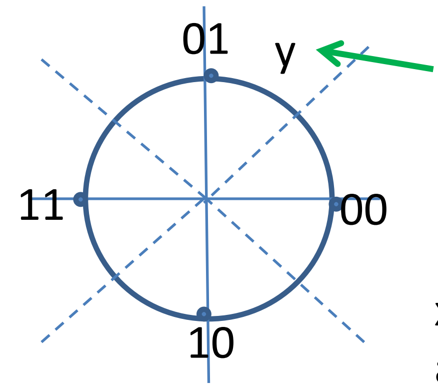
  0011010011100011010110011 0

- Binary sequences are mapped to complex numbers

- Received signal (the step of waveforms is skipped):

  y = x + w

  where x is a transmitted value and takes one of 1, i, -1, -i called a signal constellation (QPSK) and w is the noise

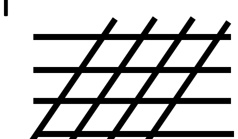


Received Value y

x is i and thus represents 01

- How to decide what x is and what x represents?
  - The half of the minimum distance between any two constellation points on the complex plane is the tolerable level of noise

- A signal constellation design: to find a finite set of a fixed number of complex numbers with a fixed sum of all the norms such that its minimum distance is maximized.
  - This is related to **sphere packing**: how to design 6 or more points is still open and it is conjectured that the equilateral triangular lattice points are optimal (this was shown asymptotically)



Error Correction Coding

- How to correct binary errors? This leads to error correction coding
  - The simplest error correction coding: repetition code

    1→111; 0→000

    Assumed 111 is transmitted but 101 is occurred at the receiver

    111
    101→111: also compare the distances with the two codewords 111 and 000
    find the one that is the closest to the received 101
    000
  - The distance between binary codewords is called Hamming distance
  - The decoding is also the minimum distance decoding
  - This simple code can correct one error but needs to expand three times

    NOT a good code (**code rate 1/3**)
- In practice, one prefers to simple encoding/decoding → This leads to linear codes

  x=Gs where **s** is a binary information vector, **G** is a binary encoding matrix called generator matrix, and **x** is a binary codeword

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

- Hamming code: input 4 bits, output 7 bits (**code rate is 4/7**)

  minimum Hamming distance 3

  correct one bit error
- **Can we do better??** (The above arithmetics are over the binary field)

Finite Fields: A Perfect Application

- In the binary coding **x=Gs**, the generator matrix **G** has less choices if all the elements in **G** are binary, for a fixed rate
  - How to include more choices for **G**?
    - To extend the components in **G** from binary to multi-ary
    - The linearity **Gs** is still kept for easy encoding and possibly easy decoding
    - For the decoding, not only the multi-ary elements form a ring but also need division, i.e., they need to form a domain, for convenience, a field.
    - For computational benefits, the multi-ary elements are represented by binary vectors
    - The question becomes: how to construct fields for binary vectors of a fixed size *m*:

$$\frac{\begin{pmatrix}0\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}} = ?$$

Galois field GF(2^m)

It can be understood that a finite field is to define addition/subtraction and multiplication/division for binary vectors

How to divide two binary vectors? use polynomials and modulo operation

$$\begin{bmatrix}\begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix}\\ \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix}\end{bmatrix}\begin{bmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ \begin{pmatrix}0\\1\end{pmatrix}\end{bmatrix}$$

is linear in terms of 4-ary operations

is NOT linear in terms of binary operations

its rate is 2/3 if it is treated as 4-ary arithmetics

its rate is 4/3 if it is treated as binary and clearly fails the decoding i.e., it is impossible in the binary case

→more possible valid code generator matrices **G**

Reed-Solomon Codes (RS Codes)

- Let α be a primitive element of GF(2^m) for a positive integer m (it is a non-zero binary vector of size m: for example, $[0,1,0,\ldots,0]^T$)

- An (n,k) RS code (1960) has the following generator matrix with *n*=$2^m - 1$

$$G = \begin{bmatrix} 1 & \alpha & \cdots & \alpha^{k-1} \\ 1 & \alpha^2 & \cdots & \alpha^{2(k-1)} \\ 1 & \alpha^3 & \cdots & \alpha^{3(k-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha^n & \cdots & \alpha^{n(k-1)} \end{bmatrix}$$

partial Vandermonde matrix

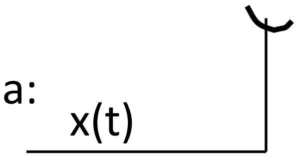
Its minimum Hamming distance is *n-k+1* that is optimal for (*n, k*) linear codes

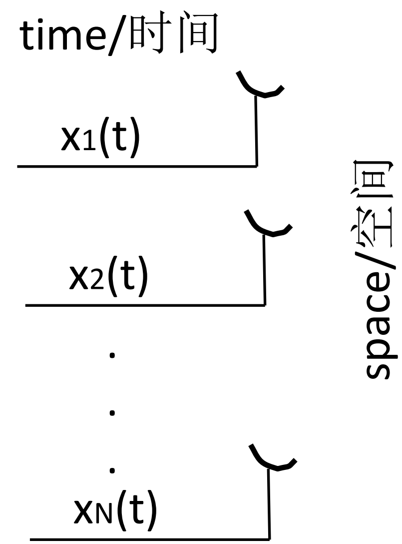
- RS codes are used in all computer memory and hard drivers, and also in many other communications systems: One of the most useful and famous error correction codes

- **Reed** received his **Ph.D. in mathematics** and was a USC professor (passed away in 2012) He has another famous code: Reed-Muller codes, where the concept of majority decoding was first used

Multiple Antennas (多天线系统）

- What we talked before is for single antenna:





- What to do for multiple antennas ? **4G, 5G, …**
  - Instead of designing a set of finite complex scalar values, such as, 1, i, -1, -i, to maximize its minimum distance, we need to design a set of matrices called a **space-time code (STC)** such that its minimum absolute value of the determinates of the difference matrices of any two distinguished matrices in the set is maximized, when the total energy is fixed:

$$X(L)=\{X_0, X_1, \cdots X_{L-1} : X_i \neq X_j \text{ are } N\times N \text{ matrices with } \|X_i\|_F^2 = N \text{ for } i \neq j\}$$

$$d_{X(L)} = \min\{|\det(X_i - X_j)| : 0 \le i \neq j \le L-1\}$$

The goal is to

$$\max\{d_{X(L)}\}$$

  - When *N*=2 and *L*<6, the optimal 2 by 2 STC can be easily obtained by using 2 by 2 orthogonal matrices with spherical packing points
    When L=6, it does not hold anymore

Bits to complex number mapping

Every time slot	00	01	11	10
One Tx antenna	1	j	-1	-j
Every two time slots Two Tx antennas	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

1st Tx antenna

2nd Tx antenna

1st time slot   2nd time slot

Optimal 2 by 2 Code of Six 2 by 2 Unitary Matrices

Wang-Xia'04

- Let

$$d=-5/2+\sqrt{22},\ \ a=\sqrt{1-3d/8},\ \ b=\sqrt{(1-a^2)/3}$$

$$\theta_1=2\arccos(d/2-a),\quad \theta_2=2\pi-\theta_1$$

$$X_0=\begin{bmatrix}-a-b\mathbf{i} & b-b\mathbf{i}\\ -b-b\mathbf{i} & -a+b\mathbf{i}\end{bmatrix},\ X_1=\begin{bmatrix}-a+b\mathbf{i} & b+b\mathbf{i}\\ -b+b\mathbf{i} & -a-b\mathbf{i}\end{bmatrix}$$

$$X_2=\begin{bmatrix}-a-b\mathbf{i} & -b+b\mathbf{i}\\ b+b\mathbf{i} & -a+b\mathbf{i}\end{bmatrix},\ X_3=\begin{bmatrix}-a+b\mathbf{i} & -b-b\mathbf{i}\\ b-b\mathbf{i} & -a-b\mathbf{i}\end{bmatrix}$$

$$X_4=e^{\mathbf{i}\theta_1/2}I_2,\qquad X_5=-e^{\mathbf{i}\theta_2/2}I_2$$

The best known 2 by 2 Unitary code of size 16 from parametric code family and is a subset of group of 32 elements.

Liang-Xia'02

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & e^{j\frac{15}{8}\pi} \\ e^{j\frac{13}{8}\pi} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} e^{j\frac{7}{4}\pi} & 0 \\ 0 & e^{j\frac{5}{4}\pi} \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & e^{j\frac{5}{8}\pi} \\ e^{j\frac{15}{8}\pi} & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e^{j\frac{3}{2}\pi} & 0 \\ 0 & e^{j\frac{1}{2}\pi} \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & e^{j\frac{11}{8}\pi} \\ e^{j\frac{1}{8}\pi} & 0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} e^{j\frac{5}{4}\pi} & 0 \\ 0 & e^{j\frac{7}{4}\pi} \end{pmatrix} \qquad A_7 = \begin{pmatrix} 0 & e^{j\frac{1}{8}\pi} \\ e^{j\frac{3}{8}\pi} & 0 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_9 = \begin{pmatrix} 0 & e^{j\frac{7}{8}\pi} \\ e^{j\frac{5}{8}\pi} & 0 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} e^{j\frac{3}{4}\pi} & 0 \\ 0 & e^{j\frac{1}{4}\pi} \end{pmatrix} \qquad A_{11} = \begin{pmatrix} 0 & e^{j\frac{13}{8}\pi} \\ e^{j\frac{7}{8}\pi} & 0 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} e^{j\frac{1}{2}\pi} & 0 \\ 0 & e^{j\frac{3}{2}\pi} \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & e^{j\frac{3}{8}\pi} \\ e^{j\frac{9}{8}\pi} & 0 \end{pmatrix}$$

$$A_{14} = \begin{pmatrix} e^{j\frac{1}{4}\pi} & 0 \\ 0 & e^{j\frac{3}{4}\pi} \end{pmatrix} \qquad A_{15} = \begin{pmatrix} 0 & e^{j\frac{9}{8}\pi} \\ e^{j\frac{11}{8}\pi} & 0 \end{pmatrix}$$

Another Way to Construct Space-Time Codes

Binary bits are first mapped to complex valued symbols x_i and these x_i are embedded into an N by N matrix : Example, the well-known Alamouti code:

$$\mathbf{C} = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} : \text{scalars } \; x_1, x_2 \in S \right\}$$

$$X^H X = (|x_1|^2 + |x_2|^2) I_2 \quad \text{for any } x_1 \text{ and } x_2$$

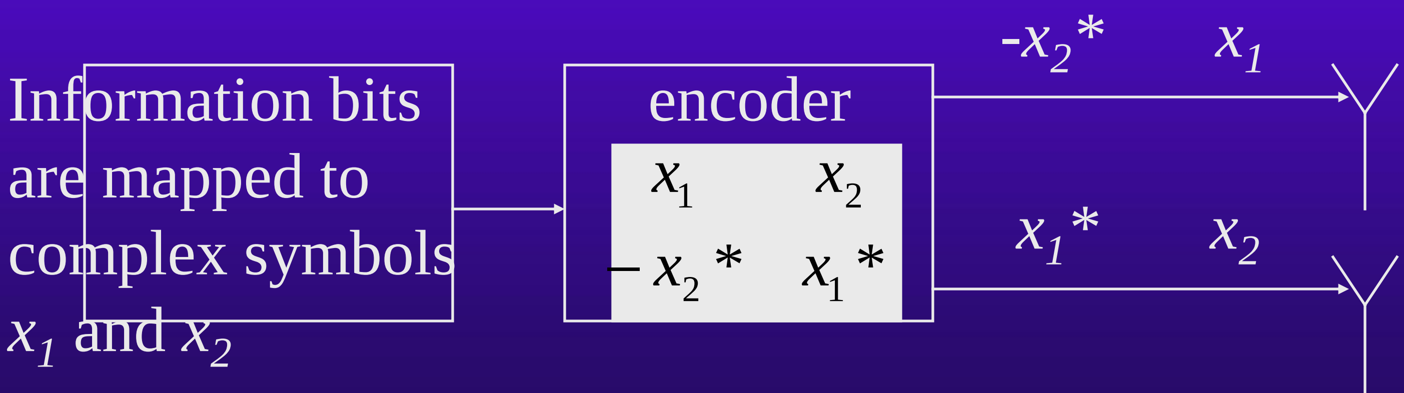
- Use **orthogonal designs** (**compositions of quadratic forms**)
  - This leads to orthogonal space-time codes
- Use cyclic division algebra
  - This leads to non-vanishing determinate codes
  - Heavily involve with algebraic number theory, such as cyclotomic fields

Alamouti Code for 2 Transmit Antennas (1998)

Alamouti code from 2 by 2 orthogonal design

$$\begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix}$$

It is an option in 3G



Alamouti Scheme: Fast ML Decoding and Full Diversity

Signal Model:

$$Y = CA + W,$$

where

$$C \in \mathcal{C} = \left\{ \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} : x_1, x_2 \in \boldsymbol{S} \right\}$$

$\boldsymbol{S}$ is a signal constellation, for example

$$\boldsymbol{S} = \{\pm 1, \pm j\}$$

Alamouti Code: Fast ML Decoding

- ML decoding is to minimize

$$\| Y - CA \|_F^2 = tr\{(Y - CA)^H (Y - CA)\}$$
$$= tr\{Y^H Y\} - tr\{Y^H CA - A^H C^H Y\} + tr\{AA^H C^H C\}$$

- Orthogonality: $C^H C = (|x_1|^2 + |x_2|^2) I_2$

  for any values x_1 and x_2.

  - ✓ The cross term $x_1 x_2$ can be canceled and x_1 and x_2 can be separated: $\| Y - CA \|_F^2 = f_1(x_1) + f_2(x_2)$

  - ✓ x_1 and x_2 can be decoded separately:

$$\min_{(x_1, x_2) \in S^2} = \min_{x_1 \in S} f_1(x_1) \text{ and } \min_{x_2 \in S} f_2(x_2)$$

  - ✓ The decoding complexity is reduced from $|S|^2$ to $2|S|$ i.e., complex symbol-wise decoding

Alamouti Code: Full Rank Property

- For any two different matrices

$$C = C(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix}, \tilde{C} = \tilde{C}(y_1, y_2) = \begin{pmatrix} y_1 & y_2 \\ -y_2^* & y_1^* \end{pmatrix}$$

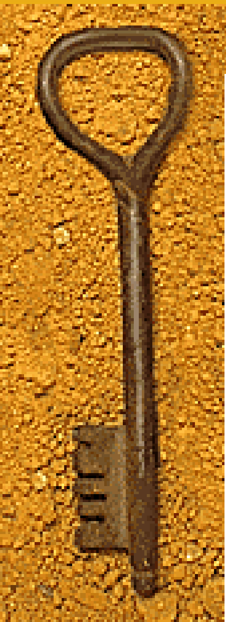
$$C \neq \tilde{C} \Leftrightarrow (x_1, x_2) \neq (y_1, y_2)$$

- Their difference matrix is also orthogonal

$$B(C, \tilde{C}) = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 \\ -(x_2 - y_2)^* & (x_1 - y_1)^* \end{pmatrix} = C(x_1 - y_1, x_2 - y_2)$$

- Because of the orthogonality, *B* has full rank

$$(B(C, \tilde{C}))^H B(C, \tilde{C}) = (|x_1 - y_1|^2 + |x_2 - y_2|^2) I_2$$



General Size

- For n transmit antennas and k information symbols $x_1, x_2, \cdots, x_k$, the orthogonality $C^H C = (|x_1|^2 + \cdots + |x_k|^2) I_n$ provides
  - **Fast ML decoding**:

$$\| Y - CA \|_F^2 = \sum_{i=1}^{k} f_i(x_i),$$

then the minimization can be done separately

$$\min_{(x_1, x_2, \ldots, x_k) \in \mathcal{A}^k} \| Y - CA \|_F^2 = \sum_{i=1}^{k} \min_{x_i \in \mathcal{A}} f_i(x_i).$$

In this case, the decoding complexity of the left term is $|\mathcal{A}|^k$, and that of the right term is $k|\mathcal{A}|$.

  - **Full diversity**:

$$(B(C, C'))^H (B(C, C')) = (|x_1 - y_1|^2 + \cdots + |x_k - y_k|^2) I_n.$$

- Motivated by the Alamouti's scheme, Tarokh-Jafarkhani-Calderbank (1999) proposed space-time block codes from orthogonal designs for any number of transmit antennas.
  - Real orthogonal designs for real constellations, such as PAM.
  - Complex orthogonal designs for complex constellations, such as PSK and QAM.

Space-Time Block Codes from Real Orthogonal Designs

- A *real orthogonal design* in variables $x_1, x_2, \cdots, x_k$ is a $p \times n$ matrix C such that:
  - The entries of C are $\pm x_1, \pm x_2, \cdots, \pm x_k$, or their linear combinations of real coefficients.
  - $C^{\mathcal{T}} C = (x_1^2 + x_2^2 + \cdots + x_k^2) I_n.$
- n is the number of transmit antennas.
- p is the time delay in the en/decoding.
- k is the number of information symbols transmitted.

- The *rate* of C is defined as $R = k/p$, which means that p transmission time slots carry k information symbols.
  - For a given delay p, one wants k as large as possible to have the throughput (or transmission rate) as large as possible.
  - It is not hard to prove that $R \leq 1$, i.e., $k \leq p$.

Its proof is given in next slide.

- For a given n, a small delay p is also desired.

For $L=2$ transmit antennas:
$k=p=2$
Rate $R=k/p=1$

$$L_2 = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$

- Proof of rate $R \le 1$, i.e., $k \le p$:

Let $C = \sum_{i=1}^{k} A_i x_i$, where A_i, $i = 1,2,\ldots,k$, are real $p \times n$ real constant matrices. *Consider* the first column of C, $(C)_1$:

$(C)_1 = \sum_{i=1}^{k} (A_i)_1 x_i$, where $(A_i)_1$ are the first columns of A_i. Due to the orthogonality of C,

$(C)_1^T (C)_1 = \sum_{i=1}^{k} |x_i|^2$. Thus

$(C)_1^T (C)_1 = 0 \;\leftrightarrow\; x_i = 0, i = 1,2,\ldots,k.$

This proves that vectors $(A_i)_1$, $i = 1,2,\ldots,k$, of size $p \times 1$ are linearly independent. There are at most p linearly independent vectors in p dimensional space, so $k \le p$.

- For 4 transmit antennas:

$$L_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}_{4\times 4}.$$

  - Number of transmit antennas $n = 4$
  - $k = p = 4$
  - Rata $R = k/p = 1$

- For 8 transmit antennas:
  - $k=p=8$
  - Rate $R=k/p=1$

$$L_8 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$$

Hurwitz Matrix Equations

- The real orthogonal space-time block code design is equivalent to the design of real matrices satisfying the Hurwitz matrix equations:
  - Write C as $C = A_1x_1 + A_2x_2 + \cdots + A_kx_k$, where A_i are some $p \times n$ real constant matrices and x_i are indeterminates.
  - Then, the orthogonality is equivalent to the following *Hurwitz matrix equations*:

$$\begin{aligned} A_i^T A_i &= I_n \text{ for } 1 \leq i \leq k \\ A_i^T A_j + A_j^T A_i &= 0_n \text{ for } 1 \leq i \neq j \leq k. \end{aligned}$$

Composition of Quadratic Forms

- The orthogonality is also equivalent to the following *[k,n,p] composition formula*:

$$(x_1^2 + \cdots + x_k^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_p^2,$$

where $X = (x_1, \cdots, x_k)$ and $Y = (y_1, \cdots, y_n)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y.

- View X, Y, Z as column vectors. Then $Z = CY$ and C is a $p \times n$ matrix whose entries are linear forms of X. Then

$$(x_1^2 + \cdots + x_k^2)Y^T Y = Z^T Z = Y^T C^T C Y.$$

- Since Y consists of indeterminates, the above equation is equivalent to

$$C^T C = (x_1^2 + \cdots + x_k^2) I_n.$$

Mathematical Historical Background

- n-square identities – $[n, n, n]$ composition formula, $(p = k = n)$
  - Complex numbers: $\alpha = x_1 + \mathbf{i}x_2$ and $\beta = y_1 + \mathbf{i}y_2$ then $|\alpha|^2|\beta|^2 = |\alpha\beta|^2$:

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = z_1^2 + z_2^2,$$

where $z_1 = x_1y_1 + x_2y_2$ and $z_2 = x_1y_2 - x_2y_1$.

– Quaternionic numbers (Hamilton's quaternions 1843) – Euler's formula (1748): $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2,$$

where

$$\begin{aligned}
z_1 &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \\
z_2 &= x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 \\
z_3 &= x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2 \\
z_4 &= x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1
\end{aligned}$$

- Legendre (1830) showed that 3-square identity is impossible.
- 8-square identity by Degen (1818) – Octonionic numbers – Cayley numbers (1845), also Graves (1843).
- Hurwitz (1898) showed that an n-square identity exists if and only if $n = 1, 2, 4, 8$, which is called "**1,2,4,8 Theorem**."
- Radon (1922) constructively showed that, an [p,n,p] formula exists over the real field if and only if $n \leq \rho(p)$, where $\rho(p)$ is the Hurwitz-Radon function defined as follows: if $p = 2^{4a+b}p_0$ for an odd p_0 and $0 \leq b \leq 3$ then $\rho(p) = 8a + 2^b$.
  * This result implies that, for any transmit antenna number n, there exist a delay p and $k = p$ such that the $p \times n$ real orthogonal design of rate 1 exists.
  * For n-identities, $k = n = p$.
  * $n = \rho(p) = p$ if and only if $p = n = 1, 2, 4, 8$. This means that real $n \times n$ square orthogonal C of n variables has and only has sizes $1, 2, 4, 8$.

* For a given transmit antenna number n, the minimum time delay p is given by

$$p = \min 2^{4c+d},$$

where the minimization is taken over the set

$$\{c, d : 0 \leq d < 4,\ c \geq 0 \text{ and } 8c + 2^d \geq n\}.$$

For example, When $n = 2$, p is 2; When $n = 3, 4$, p is 4; When $n = 5, 6, 7, 8$, p is 8; and so on.

- $A_1, A_2, \ldots, A_n$ of size $p \times k$ and $B_1, B_2, \ldots, B_k$ of size $p \times n$ are two families of Hurwitz matrices **if and only if** the following two C are real orthogonal designs $C = [A_1\mathbf{x}, \cdots, A_n\mathbf{x}]$ where $\mathbf{x} = [x_1, \cdots, x_k]^T$ and $C = B_1 x_1 + \cdots + B_k x_k$ **Two different representations**

- There are n square Hurwitz matrices $A_1, A_2, \ldots, A_n$ of size $p \times p$ by using Clifford algebra with $n \leq \rho(p)$ → $k=p$ → rate=1
- There are p Hurwitz matrices $B_1, B_2, \ldots, B_k$ of size $p \times n$ with

$$n \leq \rho(p)$$

The basic problem for real orthogonal designs or real space-time block codes for PAM signals is solved.

- Hurwitz (1923) independently showed the above result for [k,n,n] composition formulas over the complex field, where $p = n$, i.e., square complex orthogonal designs as we shall see later.
- New proofs of the Hurwitz-Radon theory using representation theory of Clifford algebra: Eckmann (1943), Lee (1948), Albert (1942), Dubisch (1946).

- By relaxing the orthogonality of C into the linear independency, i.e., full rankness, Adams (1962) showed that the Hurwitz-Radon function bound still holds for the size of matrix C by using the Chern class over the vector fields.
- **Classic book**: A. V. Geramita and J. Seberry, *Orthogonal Designs, Quadratic Forms and Hadamard Matrices*, Lecture Notes in Pure and Appl. Math. 45, Marcel Dekker, New York, 1979.
- A recent book: D. B. Shapiro, *Compositions of Quadratic Forms*, New York, De Gruyter, 2000.
- Tarokh-Jafarkhani-Calderbank (1999) nicely linked it to the space-time coding.

Space-Time Block Codes from Complex Orthogonal Designs

- A *complex orthogonal design* (Tarokh-Jafarkhani-Calderbank 1999) in variables $x_1, x_2, \cdots, x_k$ is a $p \times n$ matrix G such that:
  - The entries of G are complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \cdots, x_k, x_k^*$.
  - $G^{\mathcal{H}} G = (|x_1|^2 + \cdots + |x_k|^2) I_n$.
- A *generalized complex orthogonal design* (Tarokh-Jafarkhani-Calderbank 1999) in variables $x_1, x_2, \cdots, x_k$ is a $p \times n$ matrix G such that:
  - The entries of G are complex linear combinations of $x_1, x_1^*, x_2, x_2^*, \cdots, x_k, x_k^*$.
  - $G^{\mathcal{H}} G = D$, where D is an $n \times n$ diagonal matrix with (i, i)th diagonal

element of the form

$$l_{i,1}|x_1|^2 + l_{i,2}|x_2|^2 + \cdots + l_{i,k}|x_k|^2,$$

where all the coefficients $l_{i,1}, l_{i,2}, \cdots, l_{i,k}$ are strictly positive numbers.

- This orthogonality also guarantees the fast ML decoding and the full diversity as in the real case.

- It is not hard to prove that $R \leq 1$, i.e., $k \leq p$.

- When the diagonal elements in matrix D are the same, its equivalent composition formula of quadratic forms becomes the *[k,n,p] Hermitian composition formula*:

$$(|x_1|^2 + \cdots + |x_k|^2)(|y_1|^2 + \cdots + |y_n|^2) = |z_1|^2 + \cdots + |z_p|^2.$$

- Hurwitz (1923) constructively showed: If $p = n = 2^a b$ for an odd b, then $k \leq a + 1$ and $k = a + 1$ can be achieved.

- It was later generalized by Wolfe (1976) to the **amicable orthogonal designs**.
- It implies that the complex $n \times n$ **square** orthogonal G of n complex variables exists only when $n = 1, 2$, which is different from the real orthogonal designs even by excluding the one of size 8. This implies that the rate of a complex square orthogonal space-time code is less than 1 for more than 2 transmit antennas.
- There is not much known in the literature for *non-square* complex orthogonal designs (based on the communications with Shapiro).

- It constructively implies that the complex 4×4 square orthogonal G of 3 complex variables exists. The construction was explicitly re-given in Tarokh-Jafarkhani-Calderbank (1999)'s paper. This implies that the rates of complex orthogonal space-time codes for 3 and 4 transmit antennas can be 3/4.

– From the amicable designs (the basic idea is similar to the Hurwitz's), the optimal rates and constructions of *square* codes ($p = n$) are obtained:

* $R = k/p = 1$ when $n = 1, 2$
* $R = k/p = 3/4$ when $n = 3, 4$
* $R = k/p = 1/2$ when $n = 5, 6, 7, 8$
* $R = k/p = 5/16$ when $9 \leq n \leq 16$
* $R \to 0$ as $n \to \infty$

– The rate 3/4 codes for 3 and 4 transmit antennas can be simplified as (a few groups have presented in the literature):

$$G_4 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2^* & x_1^* & 0 & x_3 \\ x_3^* & 0 & -x_1^* & x_2 \\ 0 & x_3^* & -x_2^* & -x_1 \end{bmatrix},$$

which is equivalent to the one given by Hurwitz in the 20's.

- Using the Hurwitz (1923) construction, the code rate is **too small** for more than 4 transmit antennas.

- Half rate, $R = 1/2$, complex orthogonal space-time codes from rate 1 real orthogonal designs (Tarokh-Jafarkhani-Calderbank (1999)):

$$G_n = \begin{bmatrix} C_n \\ C_n^* \end{bmatrix},$$

$$\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \\ x_1^* & x_2^* \\ -x_2^* & x_1^* \end{bmatrix}$$

where C_n is a rate 1 real $k \times n$ orthogonal design. Thus, G_n is a $2k \times n$ design of rate $1/2$, i.e., $p = 2k$.

- Space-Time Codes don't have to be square!

- Amicable family: family of matrices of the same size $\{A_1,\cdots,A_s,B_1,\cdots,B_t\}$ forms an amicable family if

$$\begin{array}{lll} (i) & A_i^H A_i = B_j^H B_j = I, & 1 \le i \le s, 1 \le j \le t \\ (ii) & \begin{cases} A_i^H A_j + A_j^H A_i = 0, & 1 \le i, j \le s \\ B_i^H B_j + B_j^H B_i = 0, & 1 \le i, j \le t \end{cases} & \\ (iii) & A_i^H B_j = B_j^H A_i, & 1 \le i \le s, 1 \le j \le t \end{array}$$

- $C(x_1,\cdots,x_k) = \sum_{i=1}^{k} \{A_i \operatorname{Re}(x_i) + \mathbf{i}\, B_i \operatorname{Im}(x_i)\}$ is a complex orthogonal design iff $\{Ai, Bi, i=1,\ldots,k\}$ is an Amicable design.

- However the other representation

$$C(x_1,\cdots,x_k)=[A_1\mathbf{x},\cdots,A_n\mathbf{x}],$$

$$\text{where } \mathbf{x}=(\mathrm{Re}(x_1),\mathrm{Im}(x_1),\cdots,\mathrm{Re}(x_k),\mathrm{Im}(x_k))^T$$

does not work!

- Use a different representation

$$[A_1\mathbf{x}+B_1\mathbf{x}^*,\cdots,A_n\mathbf{x}+B_n\mathbf{x}^*]$$

where $\mathbf{x}=(x_1,\cdots,x_k)^T$

Two Questions

- Can a non-square *p* by *n* complex orthogonal design have rate 1, i.e., $k=p$, when $n>2$? If not, what is the bound?
- How to construct rate over ½ complex orthogonal designs?

Rate Upper Bounds for Complex Orthogonal Designs

- Liang-Xia'03 showed that their symbol rates, *k/p*, is strictly less than 1 for more than 2 transmit antennas.
- H.Wang-Xia'03 showed that their symbol rates, *k/p*, can not be above ¾ when *n>2*, and conjectured that their symbol rates are upper bounded by

$$\frac{k}{p} \le \frac{\left\lceil \frac{n}{2} \right\rceil + 1}{2\left\lceil \frac{n}{2} \right\rceil}$$

For 3 and 4 transmit antennas Rates $\frac{k}{p} = \frac{3}{4}$

  - ✓ Su-Xia'03 **first** showed that ¾ holds for n>2 when no linear processing is allowed.
  - ✓ Liang'03 showed that this conjecture holds when no linear processing is allowed.
- H.Wang-Xia'03showed that for a *p* by *n* generalized complex orthogonal design, the rate is upper bounded by 4/5 when *n>2*.

Two Main Lemmas

- **Lemma** (Liang-Xia): $\mathcal{G}$ is a generalized complex orthogonal design if and only if there exist diagonal positive definite matrices E_i, $i = 1, 2, \cdots, n$, such that its associated matrices A_i and B_i, $i = 1, \cdots, n$, satisfy the following conditions:

$$A_i^H A_j + B_j^t B_i^* = \delta_{ij} E_i, \quad A_i^H B_j + B_j^t A_i^* = 0, \quad B_i^H A_j + A_j^t B_i^* = 0, \tag{2}$$

or equivalently,

$$\begin{pmatrix} A_i & B_i \\ B_j^* & A_j^* \end{pmatrix}^H \begin{pmatrix} A_j & B_j \\ B_i^* & A_i^* \end{pmatrix} = \delta_{ij} \begin{pmatrix} E_i & 0 \\ 0 & E_j^* \end{pmatrix}, \tag{3}$$

for all $i, j = 1, \cdots, n$, where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

In particular, $\mathcal{G}$ is a complex orthogonal design if and only if (2) or (3) holds for $E_i = I$ for $1 \leq i \leq n$.

- Diagonalization Lemma (Wang-Xia): Let A and B be two $p \times k$ matrices and satisfy conditions: $A^H A + B^t B^* = I$ and $A^H B + B^H A = 0$. Then, there exist a unitary matrix V of size $p \times p$ and a unitary matrix U of size $2k \times 2k$ such that the $p \times 2k$ matrix (A, B) can be diagonalized as follows

$$V(A,B)U = \Sigma \triangleq \begin{pmatrix} D_\lambda & 0 & 0 & 0 \\ 0 & I_{k-s} & 0 & 0 \\ 0 & 0 & D_\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{p \times 2k}, \qquad (4)$$

where $k - s \geq 2k - p$, $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_s)$,
$D_\mu = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_s)$ and $\lambda_i^2 + \mu_i^2 = 1$,
$1 > \lambda_i \geq \sqrt{1/2} \geq \mu_j > 0, i, j = 1, 2, \cdots, s$, $k + s = \kappa$, and
$\kappa = \mathrm{rank}(A, B) \geq k$, and furthermore the $2k \times 2k$ unitary matrix U has the following form

$$U = \begin{pmatrix} W_1 & W_2 \\ W_2^* & W_1^* \end{pmatrix},$$

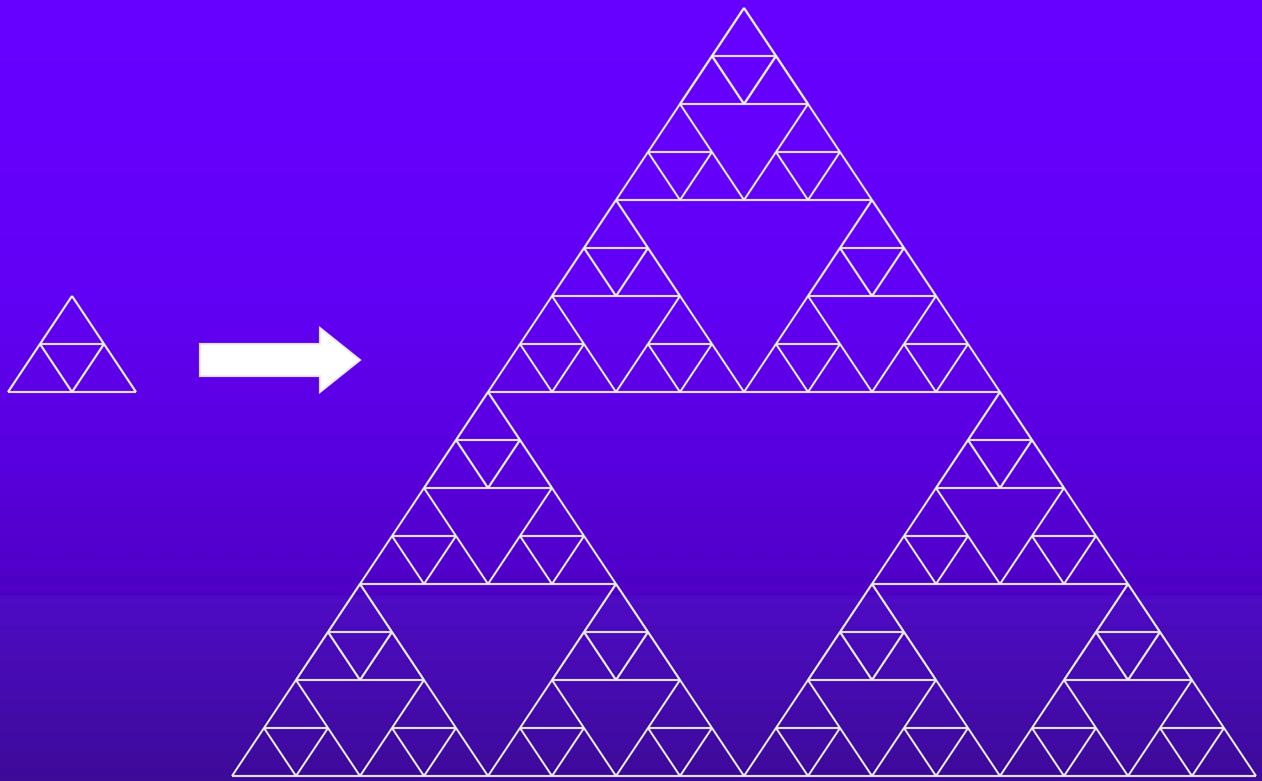
where W_i, $i = 1, 2$, are $k \times k$ matrices.

- (Wang-Xia’03) The above rate upper bounds hold for a finite QAM (excluding PSK or PAM) signal constellation.

- (Liang 2003, Su-Xia-Liu 2004, Lu-Fu-Xia 2005) constructed complex orthogonal designs with the above rates and **the constructions by Lu-Fu-Xia 2005 have closed-forms**.

Closed Form for COD

Self-Similarity



Construct COD B_{n+2}, B_{n+1} from B_n

Construction Unites for *n=2k-1*

B_n : $p_n \times n$ COD, number of nonzero complex variables is d_n

$\overline{B}_n$: $p_n \times 1$ vector, same set of complex variables as B_n

$\hat{B}_n$: $q_{1,n} \times 1$ vector, same set of complex variables as B_n

$Q_{m,n}$: $q_{m,n} \times n$ COD, $Q_{0,n} = B_n$

number of nonzero complex variables $v_{m,n} \le d_n$

$\overline{Q}_{m,n}$: $q_{m-1,n} \times 1$ vector, $\overline{Q}_{0,n} = \overline{B}_n$

same set of complex variables *as* $Q_{m,n}$

$\hat{Q}_{m,n}$: $q_{m+1,n} \times 1$ COD, $\hat{Q}_{0,n} = \hat{B}_n$

same set of complex variables *as* $Q_{m,n}$

A Theorem (Lu-Fu-Xia'05)

Theorem 1: Let A be a complex orthogonal design and has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where (i) A_{11} and A_{22} have the same set of nonzero complex variables; (ii) A_{21} and A_{12} have the same set of nonzero complex variables; (iii) A_{11} and A_{12} do not share any common nonzero complex variable. Then, the following $\overline{A}$

$$\overline{A} = \begin{bmatrix} (-1)^k A_{11} & (-1)^l A_{12} \\ (-1)^m A_{21} & (-1)^n A_{22} \end{bmatrix}$$

is also a complex orthogonal design if $k + l + m + n$ is even.

Orthogonality among Units

$$\begin{bmatrix} B_n(i) & \overline{B}_n(j) \\ B_n(j) & (-1)^k\,\overline{B}_n(i) \end{bmatrix} \qquad \begin{bmatrix} \overline{B}_n(i) & \overline{B}_n(j) \\ \hat{B}_n(j) & -\hat{B}_n(i) \end{bmatrix} \quad \text{where } n = 2k\text{-}1$$

$$\begin{bmatrix} B_n(i) & \overline{Q}_{1,n}(j) \\ Q_{1,n}(j) & -\hat{B}_n(i) \end{bmatrix} \qquad \begin{bmatrix} \overline{B}_n(i) & \overline{Q}_{1,n}(j) \\ \overline{Q}_{1,n}(j) & -\overline{B}_n(i) \end{bmatrix}$$

$$\begin{bmatrix} \overline{Q}_{m,n}(i) & \overline{Q}_{m,n}(j) \\ \hat{Q}_{m,n}(j) & -\hat{Q}_{m,n}(i) \end{bmatrix} \qquad \begin{bmatrix} \overline{Q}_{m,n}(i) & \overline{Q}_{m+1,n}(j) \\ Q_{m+1,n}(j) & -\hat{Q}_{m,n}(i) \end{bmatrix}$$

$$\begin{bmatrix} \hat{B}_n(i) & \overline{Q}_{2,n}(j) \\ \overline{Q}_{2,n}(j) & -\hat{B}_n(i) \end{bmatrix} \qquad \begin{bmatrix} \hat{Q}_{m-1,n}(i) & \overline{Q}_{m+1,n}(j) \\ \overline{Q}_{m+1,n}(j) & -\hat{Q}_{m-1,n}(i) \end{bmatrix}$$

All the above matrices are COD

Orthogonal Property among Units

$B_n(i)$ **has the same structure of *B_n*, but the indices of nonzero complex variables in *$B_n(i)$* are from *$(i-1)d_n+1$* to *id_n*, where *d_n* is the number of nonzero complex variables in *B_n*.**

B_4 with $d_4=3$

$$\begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_3^* \\ x_3 & 0 & -x_1^* & x_2^* \\ 0 & x_3 & -x_2 & -x_1 \end{bmatrix}$$

$B_4(i)$

$$\begin{bmatrix} x_{3(i-1)+1} & x_{3(i-1)+2}^* & x_{3(i-1)+3}^* & 0 \\ x_{3(i-1)+2} & -x_{3(i-1)+1}^* & 0 & x_{3(i-1)+3}^* \\ x_{3(i-1)+3} & 0 & -x_{3(i-1)+1}^* & x_{3(i-1)+2}^* \\ 0 & x_{3(i-1)+3} & -x_{3(i-1)+2} & -x_{3(i-1)+1} \end{bmatrix}$$

Inductive Construction for *n+1* and *n+2*

$$B_{n+1} = \begin{bmatrix} B_n(1) & \overline{B}_n(2) \\ B_n(2) & (-1)^k \overline{B}_n(1) \end{bmatrix}$$

$$B_{n+2} = \begin{bmatrix} B_n(1) & \overline{B}_n(2) & \overline{B}_n(3) \\ B_n(2) & (-1)^k \overline{B}_n(1) & \overline{Q}_{1,n}(4) \\ B_n(3) & -\overline{Q}_{1,n}(4) & (-1)^k \overline{B}_n(1) \\ Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) \end{bmatrix}$$

$$\overline{B}_{n+2} = \begin{bmatrix} (-1)^k \overline{Q}_{1,n}(4) \\ \overline{B}_n(3) \\ -\overline{B}_n(2) \\ \hat{B}_n(1) \end{bmatrix}$$

$$\hat{B}_{n+2} = \begin{bmatrix} (-1)^k \overline{B}_n(1) \\ \hat{B}_n(2) \\ \hat{B}_n(3) \\ -\hat{Q}_{1,n}(4) \end{bmatrix}$$

$$Q_{m,n+2} = \begin{bmatrix} Q_{m-1,n}(1) & \overline{Q}_{m,n}(2) & \overline{Q}_{m,n}(3) \\ Q_{m,n}(2) & -\hat{Q}_{m-1,n}(1) & \overline{Q}_{m+1,n}(4) \\ Q_{m,n}(3) & -\overline{Q}_{m+1,n}(4) & -\hat{Q}_{m-1,n}(1) \\ Q_{m+1,n}(4) & \hat{Q}_{m,n}(3) & -\hat{Q}_{m,n}(2) \end{bmatrix}$$

$$\overline{Q}_{m,n+2} = \begin{bmatrix} -\overline{Q}_{m-1,n}(1) \\ \overline{Q}_{m,n}(2) \\ \overline{Q}_{m,n}(3) \\ -\overline{Q}_{m+1,n}(4) \end{bmatrix}$$

$$\hat{Q}_{m,n+2} = \begin{bmatrix} \hat{Q}_{m-1,n}(1) \\ -\hat{Q}_{m,n}(2) \\ -\hat{Q}_{m,n}(3) \\ \hat{Q}_{m+1,n}(4) \end{bmatrix}$$

Orthogonality for New Units

$$\begin{bmatrix} B_n(i) & \overline{B}_n(j) \\ B_n(j) & (-1)^k \overline{B}_n(i) \end{bmatrix} \rightarrow \begin{bmatrix} B_{n+2}(i) & \overline{B}_{n+2}(j) \\ B_{n+2}(j) & (-1)^k \overline{B}_{n+2}(i) \end{bmatrix}$$

COD

?

$$\begin{bmatrix} B_{n+2}(i) & \overline{B}_{n+2}(j) \\ B_{n+2}(j) & (-1)^k \overline{B}_{n+2}(i) \end{bmatrix} = \left[\begin{array}{ccc|c} B_n(1) & \overline{B}_n(2) & \overline{B}_n(3) & (-1)^k \overline{Q}_{1,n}(8) \\ B_n(2) & (-1)^k \overline{B}_n(1) & \overline{Q}_{1,n}(4) & \overline{B}_n(7) \\ B_n(3) & -\overline{Q}_{1,n}(4) & (-1)^k \overline{B}_n(1) & -\overline{B}_n(6) \\ Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) & \hat{B}_n(5) \\ \hline B_n(5) & \overline{B}_n(6) & \overline{B}_n(7) & -\overline{Q}_{1,n}(4) \\ B_n(6) & (-1)^k \overline{B}_n(5) & \overline{Q}_{1,n}(8) & (-1)^{k+1} \overline{B}_n(3) \\ B_n(7) & -\overline{Q}_{1,n}(8) & (-1)^k \overline{B}_n(5) & (-1)^k \overline{B}_n(2) \\ Q_{1,n}(8) & \hat{B}_n(7) & -\hat{B}_n(6) & (-1)^{k+1} \hat{B}_n(1) \end{array}\right]$$

COD

Rate Formula

$$\begin{cases} q_{0,n+2} = 3q_{0,n} + q_{1,n} \\ q_{m,n+2} = q_{m-1,n} + 2q_{m,n} + q_{m+1,n}, \quad m > 0 \\ q_{0,n} = p_n \end{cases}$$

$$\begin{cases} v_{0,n+2} = 3v_{0,n} + v_{1,n} \\ v_{m,n+2} = v_{m-1,n} + 2v_{m,n} + v_{m+1,n}, \quad m > 0 \\ v_{0,n} = d_n \end{cases}$$

$$\Rightarrow \begin{cases} q_{m,2k-1} = \dfrac{(2k)! \times [k + \frac{m(m+1)}{k}]}{(k+m+1)!(k-m)!} \\ v_{m,2k-1} = \dfrac{(2k-1)!}{(k+m)!(k-m-1)!} \end{cases}$$

$$R_{2k-1} = \frac{v_{0,2k-1}}{q_{0,2k-1}} = \frac{k+1}{2k}$$

Design Examples

$$B_1 = [x_1] \qquad \overline{B}_1 = [x_1^*] \qquad \hat{B}_1 = [x_1] \qquad Q_{1,1} = [0] \qquad \overline{Q}_{1,1} = [0] \qquad \hat{Q}_{1,1} = [\phi]$$

$$B_2 = \begin{bmatrix} B_1(1) & \overline{B}_1(2) \\ B_1(2) & (-1)^k \overline{B}_1(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* \\ x_2 & -x_1^* \end{bmatrix}$$

$$B_3 = \begin{bmatrix} B_1(1) & \overline{B}_1(2) & \overline{B}_1(3) \\ B_1(2) & (-1)^k \overline{B}_1(1) & \overline{Q}_{1,1}(4) \\ B_1(3) & -\overline{Q}_{1,1}(4) & (-1)^k \overline{B}_1(1) \\ Q_{1,1}(4) & \hat{B}_1(3) & -\hat{B}_1(2) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* \\ x_2 & -x_1^* & 0 \\ x_3 & 0 & -x_1^* \\ 0 & x_3 & -x_2 \end{bmatrix}$$

Design Examples

$$B_4 = \begin{bmatrix} B_3(1) & \overline{B}_3(2) \\ B_3(2) & (-1)^k \overline{B}_3(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_6^* \\ x_3 & 0 & -x_1^* & -x_5^* \\ 0 & x_3 & -x_2 & x_4 \\ x_4 & x_5^* & x_6^* & 0 \\ x_5 & -x_4^* & 0 & x_3^* \\ x_6 & 0 & -x_4^* & -x_2^* \\ 0 & x_6 & -x_5 & x_1 \end{bmatrix}$$

COD for 4 antennas

p = 6, d = 8, R = 3/4

$$B_5 = \begin{bmatrix} B_3(1) & \bar{B}_3(2) & \bar{B}_3(3) \\ B_3(2) & (-1)^k \bar{B}_3(1) & \bar{Q}_{1,3}(4) \\ B_3(3) & -\bar{Q}_{1,3}(4) & (-1)^k \bar{B}_3(1) \\ Q_{1,3}(4) & \hat{B}_3(3) & -\hat{B}_3(2) \end{bmatrix} = \left[\begin{array}{ccc|c|c} x_1 & x_2^* & x_3^* & 0 & 0 \\ x_2 & -x_1^* & 0 & x_6^* & x_9^* \\ x_3 & 0 & -x_1^* & -x_5^* & -x_8^* \\ 0 & x_3 & -x_2 & x_4 & x_7 \\ \hline x_4 & x_5^* & x_6^* & 0 & -x_{10}^* \\ x_5 & -x_4^* & 0 & x_3^* & 0 \\ x_6 & 0 & -x_4^* & -x_2^* & 0 \\ 0 & x_6 & -x_5 & x_1 & 0 \\ \hline x_7 & x_8^* & x_9^* & x_{10}^* & 0 \\ x_8 & -x_7^* & 0 & 0 & x_3^* \\ x_9 & 0 & -x_7^* & 0 & -x_2^* \\ 0 & x_9 & -x_8 & 0 & x_1 \\ \hline x_{10} & 0 & 0 & -x_7^* & x_4^* \\ 0 & -x_{10} & 0 & x_8 & -x_5 \\ 0 & 0 & -x_{10} & x_9 & -x_6 \end{array}\right]$$

COD for 5 antennas

$p = 15$, $d = 10$, $R = 2/3$

Smaller Size COD for *n=4l*

n = 2k - 1, if k is odd, there exists one smaller (half) size COD B'_{n+3} than B_{n+3}

$$B'_{n+3} = \begin{bmatrix} B_n(1) & \bar{B}_n(2) & \bar{B}_n(3) & -\bar{Q}_{1,n}(4) \\ B_n(2) & -\bar{B}_n(1) & \bar{Q}_{1,n}(4) & \bar{B}_n(3) \\ B_n(3) & -\bar{Q}_{1,n}(4) & -\bar{B}_n(1) & -\bar{B}_n(2) \\ Q_{1,n}(4) & \hat{B}_n(3) & -\hat{B}_n(2) & \hat{B}_n(1) \end{bmatrix}$$

$$p'_{n+3} = \frac{1}{2} p_{n+3}$$

Smaller Size COD for *n=4l*

$$B'_4 = \begin{bmatrix} B_1(1) & \overline{B}_1(2) & \overline{B}_1(3) & -\overline{Q}_{1,1}(4) \\ B_1(2) & -\overline{B}_1(1) & \overline{Q}_{1,1}(4) & \overline{B}_1(3) \\ B_1(3) & -\overline{Q}_{1,1}(4) & -\overline{B}_1(1) & -\overline{B}_1(2) \\ Q_{1,1}(4) & \hat{B}_1(3) & -\hat{B}_1(2) & \hat{B}_1(1) \end{bmatrix} = \begin{bmatrix} x_1 & x_2^* & x_3^* & 0 \\ x_2 & -x_1^* & 0 & x_3^* \\ x_3 & 0 & -x_1^* & -x_2^* \\ 0 & x_3 & -x_2 & x_1 \end{bmatrix}$$

COD from our design for 4 antennas: ***d = 3, p = 4, R = 3/4***

----- ***coincides with the existing one***

Liang's and Su-Xia-Liu's: ***d = 6, p = 8, R = 3/4***

$$B_8^{'} = \left(\begin{array}{ccccc|c|c|c}
x_1 & x_2^* & x_3^* & 0 & 0 & -x_{20}^* & -x_{30}^* & 0 \\
x_2 & -x_1^* & 0 & x_6^* & x_9^* & 0 & 0 & x_{33}^* \\
x_3 & 0 & -x_1^* & -x_5^* & -x_8^* & 0 & 0 & -x_{32}^* \\
0 & x_3 & -x_2 & x_4 & x_7 & 0 & 0 & x_{31} \\
x_4 & x_5^* & x_6^* & 0 & -x_{10}^* & 0 & 0 & x_{34}^* \\
x_5 & -x_4^* & 0 & x_3^* & 0 & x_{19}^* & x_{29}^* & 0 \\
x_6 & 0 & -x_4^* & -x_2^* & 0 & -x_{18}^* & -x_{28}^* & 0 \\
0 & x_6 & -x_5 & x_1 & 0 & x_{17} & x_{27} & 0 \\
x_7 & x_8^* & x_9^* & x_{10}^* & 0 & 0 & 0 & x_{35}^* \\
x_8 & -x_7^* & 0 & 0 & x_3^* & -x_{16}^* & -x_{26}^* & 0 \\
x_9 & 0 & -x_7^* & 0 & -x_2^* & x_{15}^* & x_{25}^* & 0 \\
0 & x_9 & -x_8 & 0 & x_1 & -x_{14} & -x_{24} & 0 \\
x_{10} & 0 & 0 & -x_7^* & x_4^* & -x_{11}^* & -x_{21}^* & 0 \\
0 & -x_{10} & 0 & x_8 & -x_5 & x_{12} & x_{22} & 0 \\
0 & 0 & -x_{10} & x_9 & -x_6 & x_{13} & x_{23} & 0 \\
\hline
x_{11} & x_{12}^* & x_{13}^* & 0 & 0 & x_{10}^* & 0 & -x_{30}^* \\
x_{12} & -x_{11}^* & 0 & x_{16}^* & x_{19}^* & 0 & -x_{33}^* & 0 \\
x_{13} & 0 & -x_{11}^* & -x_{15}^* & -x_{18}^* & 0 & x_{32}^* & 0 \\
0 & x_{13} & -x_{12} & x_{14} & x_{17} & 0 & -x_{31} & 0 \\
x_{14} & x_{15}^* & x_{16}^* & 0 & -x_{20}^* & 0 & -x_{34}^* & 0 \\
x_{15} & -x_{14}^* & 0 & x_{13}^* & 0 & -x_9^* & 0 & x_{29}^* \\
x_{16} & 0 & -x_{14}^* & -x_{12}^* & 0 & x_8^* & 0 & -x_{28}^* \\
0 & x_{16} & -x_{15} & x_{11} & 0 & -x_7 & 0 & x_{27} \\
x_{17} & x_{18}^* & x_{19}^* & x_{20}^* & 0 & 0 & -x_{35}^* & 0 \\
x_{18} & -x_{17}^* & 0 & 0 & x_{13}^* & x_6^* & 0 & -x_{26}^* \\
x_{19} & 0 & -x_{17}^* & 0 & -x_{12}^* & -x_5^* & 0 & x_{25}^* \\
0 & x_{19} & -x_{18} & 0 & x_{11} & x_4 & 0 & -x_{24} \\
x_{20} & 0 & 0 & -x_{17}^* & x_{14}^* & x_1^* & 0 & -x_{21}^* \\
0 & -x_{20} & 0 & x_{18} & -x_{15} & -x_2 & 0 & x_{22} \\
0 & 0 & -x_{20} & x_{19} & -x_{16} & -x_3 & 0 & x_{23} \\
\hline
x_{21} & x_{22}^* & x_{23}^* & 0 & 0 & 0 & x_{10}^* & x_{20}^* \\
x_{22} & -x_{21}^* & 0 & x_{26}^* & x_{29}^* & x_{33}^* & 0 & 0 \\
x_{23} & 0 & -x_{21}^* & -x_{25}^* & -x_{28}^* & -x_{32}^* & 0 & 0 \\
0 & x_{23} & -x_{22} & x_{24} & x_{27} & x_{31} & 0 & 0 \\
x_{24} & x_{25}^* & x_{26}^* & 0 & -x_{30}^* & x_{34}^* & 0 & 0 \\
x_{25} & -x_{24}^* & 0 & x_{23}^* & 0 & 0 & -x_9^* & -x_{19}^* \\
x_{26} & 0 & -x_{24}^* & -x_{22}^* & 0 & 0 & x_8^* & x_{18}^* \\
0 & x_{26} & -x_{25} & x_{21} & 0 & 0 & -x_7 & -x_{17} \\
x_{27} & x_{28}^* & x_{29}^* & x_{30}^* & 0 & x_{35}^* & 0 & 0 \\
x_{28} & -x_{27}^* & 0 & 0 & x_{23}^* & 0 & x_6^* & x_{16}^* \\
x_{29} & 0 & -x_{27}^* & 0 & -x_{22}^* & 0 & -x_5^* & -x_{15}^* \\
0 & x_{29} & -x_{28} & 0 & x_{21} & 0 & x_4 & x_{14} \\
x_{30} & 0 & 0 & -x_{27}^* & x_{24}^* & 0 & x_1^* & x_{11}^* \\
0 & -x_{30} & 0 & x_{28} & -x_{25} & 0 & -x_2 & -x_{12} \\
0 & 0 & -x_{30} & x_{29} & -x_{26} & 0 & -x_3 & -x_{13} \\
\hline
x_{31} & x_{32}^* & x_{33}^* & -x_{34}^* & -x_{35}^* & 0 & 0 & 0 \\
x_{32} & -x_{31}^* & 0 & 0 & 0 & x_{23}^* & -x_{13}^* & x_3^* \\
x_{33} & 0 & -x_{31}^* & 0 & 0 & -x_{22}^* & x_{12}^* & -x_2^* \\
0 & x_{33} & -x_{32} & 0 & 0 & x_{21} & -x_{11} & x_1 \\
x_{34} & 0 & 0 & x_{31}^* & 0 & -x_{24}^* & x_{14}^* & -x_4^* \\
0 & -x_{34} & 0 & -x_{32} & 0 & x_{25} & -x_{15} & x_5 \\
0 & 0 & -x_{34} & -x_{33} & 0 & x_{26} & -x_{16} & x_6 \\
x_{35} & 0 & 0 & 0 & x_{31}^* & -x_{27}^* & x_{17}^* & -x_7^* \\
0 & -x_{35} & 0 & 0 & -x_{32} & x_{28} & -x_{18} & x_8 \\
0 & 0 & -x_{35} & 0 & -x_{33} & x_{29} & -x_{19} & x_9 \\
0 & 0 & 0 & x_{35} & -x_{34} & -x_{30} & x_{20} & -x_{10}
\end{array}\right)$$

- A design example for n=8 transmit antennas.
- In this case, d=35, p=56.
- Rate =d/p=5/8
- This construction is inductive for all n with closed-forms
- **Liang's and Su-Xia-Liu's:** ***$d = 70$, $p = 112$, $R = 5/8$***
  - ***These constructions do not have closed-forms and computer-aid or manual help is needed***

COD Construction Comparison

n	Liang &Su-Xia-Liu		Lu-Fu-Xia		
	d	p	d	p	Rate=d/p
1	1	1	1	1	1
2	2	2	2	2	1
3	3	4	3	4	3/4
4	6	8	3	4	3/4
5	10	15	10	15	4/6
6	20	30	20	30	4/6
7	35	56	35	56	5/8
8	70	112	35	56	5/8
9	126	210	126	210	6/10
10	252	420	252	420	6/10
11	462	792	462	792	7/12
12	924	1584	462	792	7/12
13	1716	3003	1716	3003	8/14
14	3432	6006	3432	6006	8/14

结束语 Conclusion

代数： Binary field → Finite fields → Complex number field
Algebraic number fields
Algebra

→ Quaternionic numbers → Octonionic numbers
四元数体 八元数体

→ Norm identities (Composition formulas)

$$\|x \bullet y\| = \|x\| \bullet \|y\|$$

Perfect application at the transmitter side

Counting: 数准了就是代数，数不准就是分析 ☺
count accurately is algebra, count nonaccurately is analysis,
根本就数不清楚 是几何拓扑
cannot count is topology/geometry

分析：
Analysis

$$\|x \bullet y\| \le \|x\| \bullet \|y\|$$

When dot is inner product, it is the Schwarz inequality
→ matched filter (匹配滤波器）
Optimal receiver

When dot is addition, it is the triangular inequality

When dot is multiplication, it is the norm inequality

结束语

- 电子工程中有材料和算法两大块， 而算法就是应用数学

  纵观过去几十年，人们在生活上最大的变化就是在通信上的变化

  数学在通信里的应用起着非常重要的作用

- 所有的数学都是有 用的，你不知道哪天就会用上

Thank You