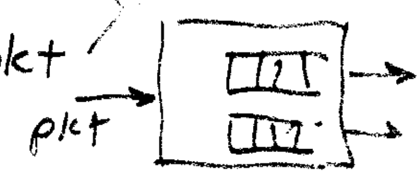
Delay models in computer networks

! Delays due following causes

1. processing delay frm last bit of pkt to time it is put on outgoing queue (note - this is not only model -)

2. queueing delay in output queue (low speed)

3. transmission delay to send pkt



6. propagation delay ~~v~~ of light

Little's Theorem

Let $N(t)$ number of pkts in system at time t

$\alpha(t)$ number pkts arrived in interval $[0, t]$

$\beta(t)$ number pkts departed in interval $[0, t]$

T_i time spent in system by ith pkt

Then avg # pkts in system $= N_t = \frac{1}{t}\int^{t} N(\tau)\, d\tau$

In steady state: $N = \lim_{t \to \infty} N_t$ time average (pkts)

Let $\lambda_t = \frac{\alpha(t)}{t}$ the time-average arrival rate up to time t

and $\lambda = \lim_{t \to \infty} \lambda_t$

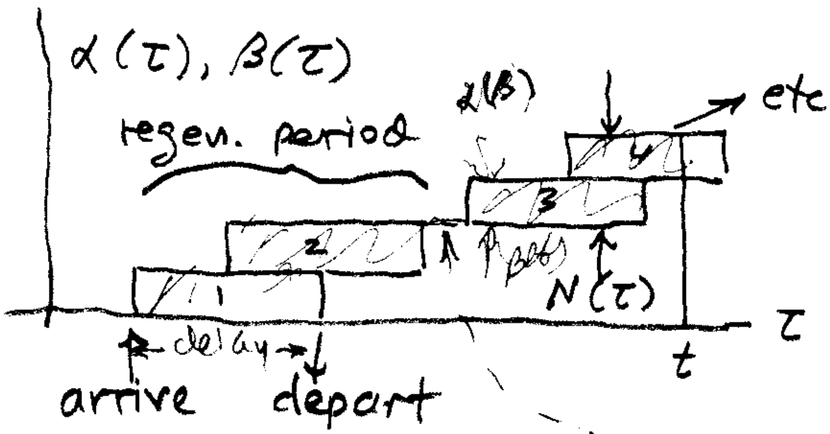
Now, time average ~~arrival~~ pkt delay $T_t = \frac{\sum_{i=0}^{\alpha(t)} T_i}{\alpha(t)}$

and $T = \lim_{t \to \infty} T_t$

Little's Result : $\boxed{N = \lambda T}$

Proof (enlightening)

Assume $N(0) = 0$ and FIFO service



The shaded area is $\int_0^t N(\tau)\,d\tau$ in seconds

However, if choose t when $N(t) = 0$, this area is $\sum_{i=1}^{\alpha(t)} T_i$. Then $\frac{1}{t}\int_0^t N(\tau)\,d\tau = \frac{1}{t}\sum_{i=1}^{\alpha(t)} T_i$

but the RH side $= \frac{\alpha(t)}{t} \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)} \equiv \lambda_t T_t$

and LH side $\frac{1}{t}\int_0^t N(\tau)\,d\tau \equiv N_t$

qwed.
in limit
$t \to \infty$

Now probabilistic form

Replace time averages with ensemble averages

Let $p_n(t)$ = prob of n pkts in system at time t
(waiting or in service)

We are given $p_n(0)$ for all n, then run the system to steady state. This is a Markov process.

Then the avg # of pkts in system

ensemble $$\bar{N}(t) = \sum_{n=0}^{\infty} n p_n(t)$$

but, $\lim_{t \to \infty} p_n(t) = p_n$, $n = 0, 1, \dots$ indep. of initial distrib. Thus, the avg # in system at steady state is

$$\bar{N} = \sum_{n=0}^{\infty} n p_n = \lim_{t \to \infty} \bar{N}(t)$$

stationary process

Now, consider the avg delay of each customer

$\bar{T}_k$ typically converges as $k \to \infty$

$$\bar{T} = \lim_{k \to \infty} \bar{T}_k$$

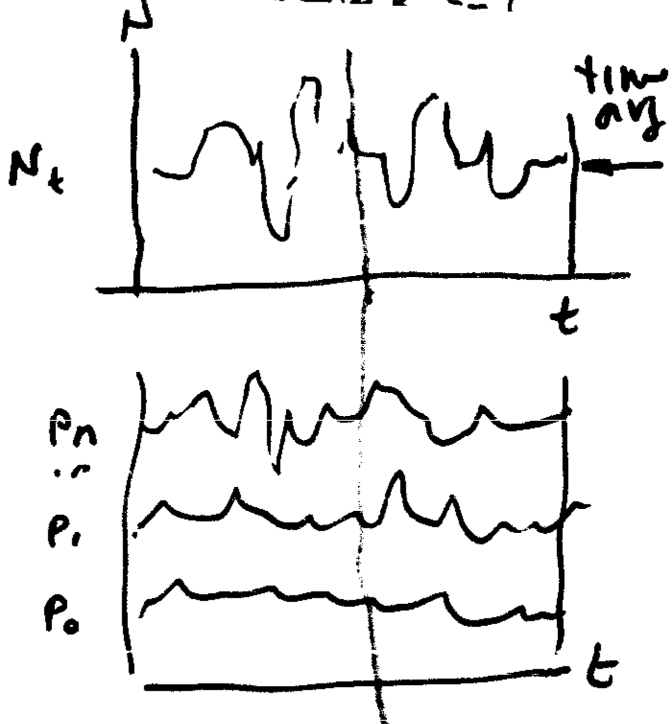
Systems of interest here are ergodic in that

If $N_t = \frac{1}{t}\int_0^t N(\tau)\,d\tau = \bar{N}(t)$, then $N = \lim_{t\to\infty} N_t$
time avg

But $\sum_{n=0}^{\infty} n P_n(t) = \bar{N}(t)$ and the two averages are equal
ensemble avg

We can also say if T_k is the delay for kth pkt

$\bar{T}_k = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} T_k$ and $\lim_{k\to\infty} \bar{T}_k = T$ avg time in system



$\frac{1}{t}\int_0^t N(\tau)\,d\tau = N_t$
time avg

$\to \infty$ are equal

$\sum_{n=0}^{\infty} n P_n(t) = N(t)$
ensemble

Note $\sum_{n=0}^{\infty} P_n(t) = 1$

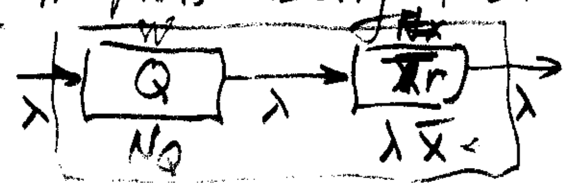
then $\lambda = \lim_{t\to\infty} \frac{\#\text{ arrivals in interval } [0,t]}{t}$ and $\boxed{N = \lambda T}$

Exomple 3.1 13 Apr 4

Let λ arrival rate; N_Q avg # pkts waiting in queue (not including the ones in transmission) and W the time waiting in que (not incl. ones in transmission)

Then $N_Q = \lambda W$. If $\bar{X}$ is avg transmission time, then avg # pkts being transmitted

is $N_X = \underbrace{\lambda \bar{X}}_{\text{this must } < 1}$



$N_Q + \lambda \bar{X} = N$ then $T = \frac{}{\lambda}$

Queueing Systems $M/M/1 \leftarrow$ # servers
service ~~time~~ process (exponential)
arrival process (exponential)
M = memoryless
G = general
D = deterministic

Note that, if p_n is prob. of n pkts in system and $N = \sum_{n=0}^{\infty} n p_n$, then by Little: $T = \frac{N}{\lambda}$

Need characterize arrival process - Poisson

Let $A(t)$ a counting process that represents the total number of arrivals up to time t.

i.e.

1. $A(0) = 0$ at times t and $s < t$,

$A(t) - A(s) =$ # arrivals in $[s, t]$.

Poisson arrival statistics

Let the number of arrivals between t and $t+\tau$ $(\tau > 0)$ be poisson distributed with parameter λ, then

$$P\{A(t+\tau) - A(t) = n\} = \frac{\lambda\tau}{n!} e^{-\lambda\tau}$$

and the average number of arrivals in τ is $\lambda\tau$

Note λ is arrival <u>rate</u>.

Properties

1. <u>Interarrival times</u> are independent and exponentially distributed. with parameter λ

   if $\tau_n = t_{n+1} - t_n$, $\boxed{P\{\tau_n \le s\} = 1 - e^{-\lambda s} \quad (s \geq 0)}$ interarrival

   and are mutually independent

2. for every $t \geq 0$ and $\delta \geq 0$ prob. interval (plots/s)

no arrive $P\{A(t+\delta) - A(t) = 0\} = 1 - \lambda\delta + o(\delta)$

one arrive $P\{A(t+\delta) - A(t) = 1\} = \lambda\delta + o(\delta)$

two arrive $P\{A(t+\delta) - A(t) = 2\} = o($

where $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$

13 Apr 6

Service statistics

pkt service times are exponentially distributed with parameter μ. If S_n is the time for the nth pkt,

$$\boxed{P\{S_n \leq s\} = 1 - e^{-\mu s} \quad (s \geq 0)}$$

Service time μ ~~is service rate~~ sec/pkt

Note the <u>memoryless</u> charact.   (pkts/s)

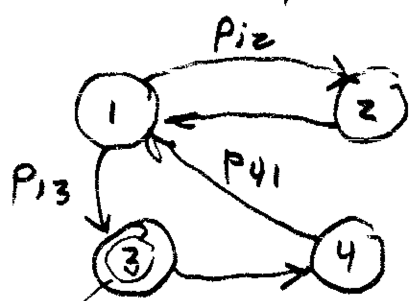
arrival $\quad P\{\tau_n > r+t \mid \tau_n > t\} = P\{\tau_n > r\} \quad (r, t \geq 0)$

service $\quad P\{S_n > r+t \mid S_n > t\} = P\{S_n > r\} \quad$ "

e.g. The prob that an arrival will occur in the next t sec. is independent of the time since the last arrival.

Markov Chains

The key is that the future behavior of the system depends only on the present state and trans. probabilities, not on how it got there.



etc.

Thus $\{N(t) \mid t \geq 0\}$ is a ~~continuous~~ discrete time Markov chain

in system here

reachable

ergodic ... (simpler than continuous-time)

now shrink transition times to limit as $\delta \to 0$

Markov

consider transition matrix

state vector

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{matrix} s_1 \\ s_2 \end{matrix}$$

$$w = w^{(n)} = (w^{(n)}_{s_1}, w^{(n)}_{s_2})$$

$$w^{(0)} = \text{set of initial states}$$

so $w^{(n+1)}_{s_j} = \sum_k w^{(n)}_{s_k} p_{kj}$

or $w^{(n+1)} = w^{(n)} P$

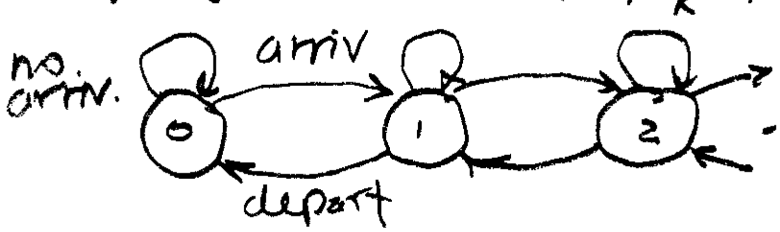
In a regular Markov chain some power of $P^{(n)}$ has only positive elements and $P^{(n+1)} = P^{(n)}$. This is denoted P^*. Alternatively the chain can oscillate and never converge

This implies a finite chain; however we can have an unbounded number of states

We do this like round-robin approx ~~continuous~~ in time-sharing

Consider a progression $0, \delta, 2\delta, \ldots, k\delta$

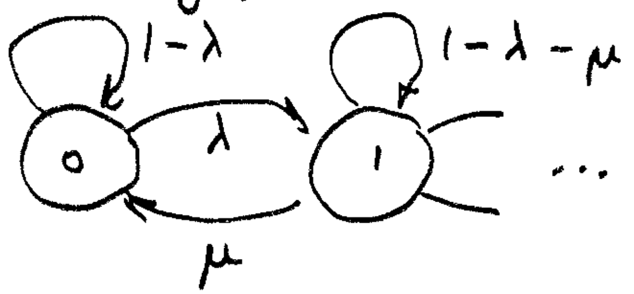
where δ a small positive time. Later shrink δ to zero. Let N_k # pkts in system at $k\delta$



... pkts in system

We need transition probabilities

We argue that, if δ is small, the prob of other trans., like 2 arriv or 2 depart is vanishingly small. Also, neglect $o(\delta)$ terms.



$\longrightarrow$ rate $=$ $\longleftarrow$ rate for steady state

Consider the prob of at least one arrival in τ_n

$P\{\tau_n \le r\} = 1 - e^{-\lambda r}$ ∴ Prob of no arrival

$P\{\tau_n > r\} = e^{-\lambda r}$

Prob at least one departure $P\{\tau_n \le s\} = 1 - e^{-\mu s}$

∴ Prob of no departures $P\{\tau_n > s\} = e^{-\mu s}$

$P\{\text{no arrivals}\} = P\{A(t+\delta) - A(t) = 0\} = 1 - \lambda\delta + o(\delta)$

$P\{\text{one arrival}\} = P\{A(t+\delta) - A(t) = 1\} = \lambda\delta + o(\delta)$

$P\{\text{two or more}\} = P\{A(t+\delta) - A(t) > 1\} = o(\delta)$

Note $e^{-\lambda\delta} = 1 - \lambda\delta + \dfrac{(\lambda\delta)^2}{2!} - \dfrac{(\lambda\delta)^3}{3!} \ldots$ Taylor series

$= 1 - \lambda\delta + o(\delta)$

Do same thing for $e^{-\mu\delta}$. Note $P\{\text{one arriv, one depart}\}$

$= e^{-\lambda\delta} e^{-\mu\delta} = 1 - \lambda\delta - \mu\delta + o(\delta)$

Summarize

P_{00} = prob system idle and no arr. $= 1 - \lambda\delta$

for all $i > 0$

P_{ii} = prob no arriv and no departs =
$= 1 - \lambda\delta - \lambda\mu + o(\delta)$

$P_{i,i+1}$ = prob arriv and no depart $= 1 - e^{-\lambda\delta} = \lambda\delta + o(\delta)$

$P_{i,i-1}$ = " no arriv and depart $= 1 - e^{-\mu\delta} = \mu\delta + o(\delta)$

$P_{ij} = o(\delta)$ otherwise

Now we calculate stationary distribution

N_k = # cust. in sqst at kth transition

we need $\longrightarrow$ $p_n = \lim_{k\to\infty} P\{N_k = n\} = \lim_{t\to\infty} P\{N(t) = n\}$

For steady state we must have equilibrium in state transitions

$$p_n \lambda\delta + o(\delta) = p_{n+1}\mu\delta + o(\delta)$$

or in limit as $\delta \to 0$, $p_n\lambda = p_{n+1}\mu$

(called global balance equations) can also be

$p_{n+1} = \frac{\lambda}{\mu} p_n = \rho\, p_n$ and $\boxed{p_n = \rho^n p_0}$
$(n = 1, 2, \ldots)$

If $\rho < 1$ (req. for steady state),

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \rho^n p_0 = \frac{p_0}{1-\rho} \Rightarrow \boxed{p_0 = 1-\rho}$$

Subst for p_0, $p_n = \rho^n(1-\rho)$

Now, calculate mean value

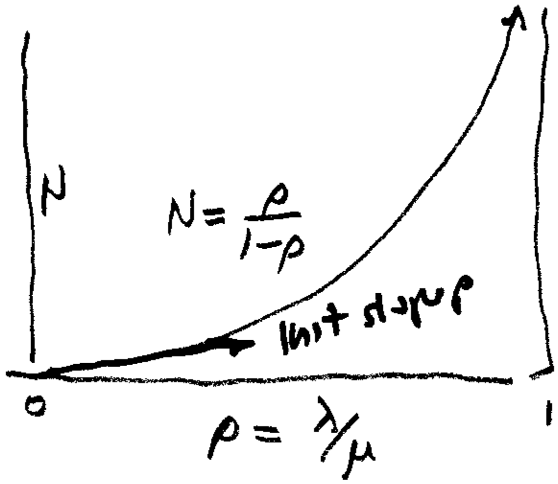
$$N = \lim_{t \to \infty} E\{N(t)\} = \sum_{n=0}^{\infty} n P_n = \sum_{n=}^{\infty} n \rho^n (1-\rho)$$

$$N = (1-\rho) \sum_{n=0}^{\infty} n \rho^{n} = \rho(1-\rho) \sum_{n=0}^{\infty} n \rho^{n-1}$$

$$N = \rho(1-\rho) \frac{\partial}{\partial \rho} \left[\sum_{n=0}^{\infty} \rho^n \right] \quad (!)$$

$$N = \rho(1-\rho) \frac{\partial}{\partial \rho} \left(\frac{1}{1-\rho} \right) = \rho(1-\rho) \frac{(-1)(-1)}{(1-\rho)^2} = \frac{\rho}{1-\rho} \; !$$

13 Apr 9



Then, by Little's Result

$$T = \frac{N}{\lambda} = \frac{\rho}{\lambda(1-\rho)}$$

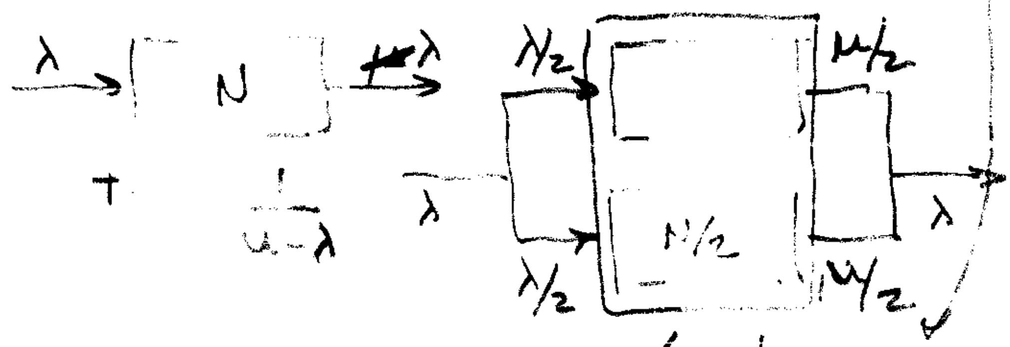
or $$T = \frac{1}{\mu - \lambda}$$

Then avg wait in queue $W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda}$

and by Little, $N_Q = \lambda W = \frac{\rho^2}{1-\rho}$

Note $\rho = 1 - p_0$ is fract of time server is busy

Example:



$$T' = \frac{1}{\mu/2 - \lambda/2} = 2T$$

for each queue

1. If two Poisson streams are merged, the resulting stream is Poisson

2. If a single Poisson stream is split with prob. p and 1 − p, resulting streams are Poisson

19 Apr 1

3.5 M/G/1 systems arrivals are Poisson
service is FIFO

Let x_i be service time of the ith arrival

and assume the r.v. $[x_1, x_2, \ldots]$ are
iid and indep. of arrival times

Let $\overline{X} = E\{X\} = \frac{1}{\mu}$ avg service time
$\overline{X^2} = E\{X^2\} =$ second moment

We will show $W = \frac{\lambda \overline{X^2}}{2(1-\rho)}$ Pollaczek Khinchin (P-K)

where W is waiting time in queue
and $\rho = \frac{\lambda}{\mu} = \lambda\overline{X}$ (also avg number in service)

and

$T = \overline{X} + W$ for system

We apply Little's result to W and T get $N_Q = \lambda W = \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}$ and $N = N_Q = \rho + N_Q$

for M/M/1, have ~~$p(\tau_n) = \mu e^{-\mu\tau_n}$~~

and $E(X) = \frac{1}{\mu}$, $E(X^2) = \frac{2}{\mu^2}$

for $p(x) = \mu e^{-\mu x}$

INSERT

$E(x) = \int_0^\infty \mu e^{-\mu x} dx$

from $P\{s_n \leq s\} = 1 - e^{-\mu s}$, $\frac{d}{ds} P\{s_n \leq s\} = (-)(-)\mu e^{-\mu s}$

19 Apr 2

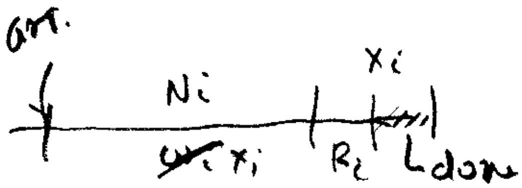
thus $W = \dfrac{\rho}{\mu(1-\rho)}$ for M/M/1 $\longrightarrow$ $\boxed{\text{ins}}$

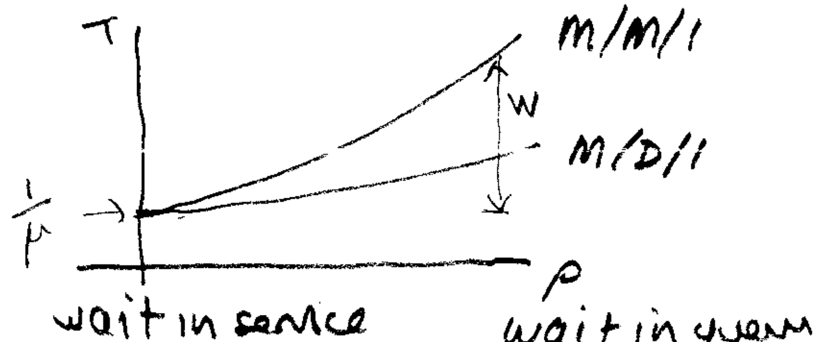
But for M/D/1 $\overline{X^2} = \dfrac{1}{\mu^2}$, $W = \dfrac{\rho}{2\mu(1-\rho)}$

because all ~~take~~ same time $\dfrac{1}{\mu}$.

then W, T, N_Q and N for M/D/1

are lower bounds w/resp. M/G/1 for any other distribution. I.e. randomness costs twice.





Define mean residual service time. let

W_i waiting time in queue for ith customer

R_i residual service time seen by the ith customer. This is the time for another customer j to complete when i arrives. $R_i = 0$ if ~~noto~~ no other customer

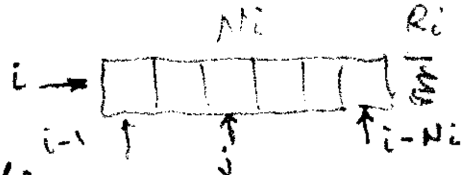
X_i service time of the ith customer

N_i number of customers found waiting in queue at time of arrival of ith customer

then $W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$ ← all other customers ahead of this one

↑ time for current customer to complete

19 Apr 3



Thaking expectations

$$E\{W_i\} = E\{R_i\} + E\Big\{\sum_{j=i-N_i}^{i-1} E\{X_j / N_i\}\Big\}$$

$$= E\{R_i\} + \overline{X}\, E\{N_i\}$$

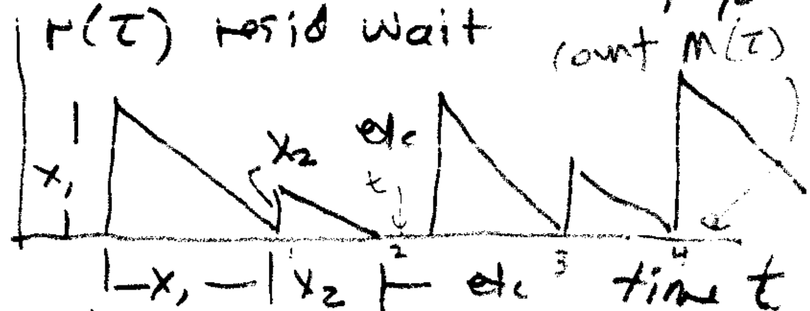
As $i \to \infty$ $\quad W = R + \frac{1}{\mu} N_Q$, where $R = \lim_{i\to\infty} E(R_i)$

where limit exists and $\lambda < \mu$

By Little's result, have $N_Q = \lambda W$ and

$W = R + \frac{\lambda W}{\mu}$ and $W = \frac{R}{1-\rho}$

Now, note



start at a time t when $r(t) = 0$, then

$$\overline{r(\tau)}_0^t = \underbrace{\frac{1}{t}\int_0^t r(\tau)\, d\tau}_{\text{mean resid service time}} = \frac{1}{t} \underbrace{\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2}_{\text{area of all triangles up until } t}$$

$M(t) \leftarrow$ # of svc completions

Write as

$$\frac{1}{t}\int_0^t r(\tau)\, d\tau = \frac{1}{2}\, \frac{M(t)}{t}\, \frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)}$$

and take limits as $t \to \infty$

19 Apr 4

have $$\underbrace{\lim_{t\to\infty}\int_0^t r(\tau)\,d\tau}_{\text{time avg of resid wait time}} = \frac{1}{2}\underbrace{\lim_{t\to\infty}\frac{M(t)}{t}}_{\substack{\lambda \\ \text{avg departure rate (= arrival rate)}}}\ \underbrace{\lim_{t\to\infty}\frac{\sum x_i^2}{M(t)}}_{\substack{\overline{X^2} \\ \text{second moment of svc time}}}$$

thus $R = \frac{1}{2}\lambda\overline{X^2}$ and $W = \frac{\lambda\overline{X^2}}{2(1-\rho)}$

KMGD

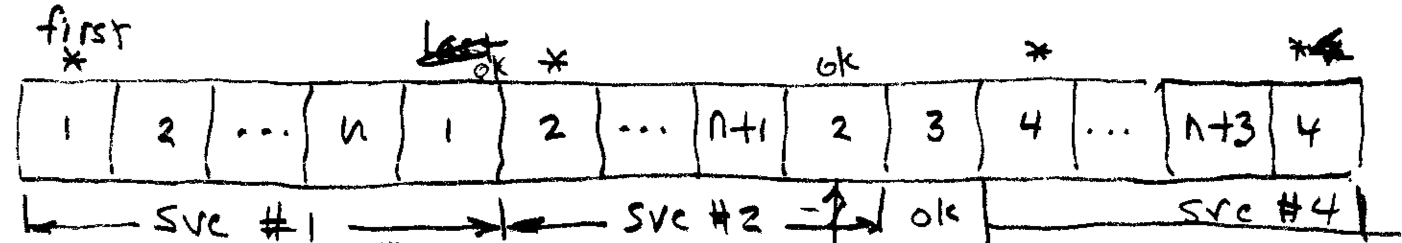
Ex 3.1.5 Delay anal of gbn   21 April

Assume that pkts are one time unit long and wait max for ack $n-1$ frames before pkt retrans.

Assume ~~pkt~~ rejected a rerr frame i, then xmtr trans frames $i+1, i+2, \ldots, i+n-1$ and then rtx ~~frame~~ frame i at time $i+n$.

Forget abt ack being lost. pkts have prob p ~~correct~~ error

Consider arrival Poisson with rate λ



$1+kn$ time units w/ prob. $(1-p)p^k$ or k ~~repts~~ retransm. thus service time behaves like M/G/1 queue.

$$P\{X = 1+kn\} = (1-p)p^k, \quad k=0,1,\ldots$$

$$\overline{X} = \sum_{k=0}^{\infty} (1+kn)(1-p)p^k = (1-p)\left(\sum_{k=0}^{\infty} p^k + n\sum_{k=0}^{\infty} kp^k\right)$$

$$\overline{X^2} = \sum_{k=0}^{\infty} (1+kn)^2(1-p)p^k = (1-p)\left(\sum_{k=0}^{\infty} p^k + 2n\sum_{k=0}^{\infty} kp^k + n^2\sum_{k=0}^{\infty} k^2p^k\right)$$

Note
$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}, \quad \sum_{k=0}^{\infty} kp^k = \frac{p}{(1-p)^2}, \quad \sum_{k=0}^{\infty} k^2p^k = \frac{p+p^2}{(1-p)^3}$$

$\frac{d}{dp}$ $\frac{d^2}{dp^2}$

NB. $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$ 21 Apr a

$$\frac{d}{dp}\sum_{k=0}^{\infty} p^k = \sum_{k=0}^{\infty} k p^{k-1} = \frac{1}{p}\sum_{k=0}^{\infty} k p^k$$

$$\frac{1}{p}\sum_{k=0}^{\infty} k p^k = \frac{(-1)(-1)}{(1-p)^2}$$

mult p both sides $\boxed{\sum_{k=0}^{\infty} k p^k = \frac{p}{(1-p)^2}}$

$$\frac{d}{dp}\sum_{k=0}^{\infty} k p^k = \sum_{k=0}^{\infty}\frac{d}{dp} k p^k = \sum_{k=0}^{\infty} k^2 p^{k-1} = \frac{1}{p}\sum_{k=0}^{\infty} k^2 p^k$$

$$\frac{d}{dp}\frac{p}{(1-p)^2} = p(1-p)^{-2} = p(1-p)^{-3}(-1) + 1(1-p)^{-2}$$

$$= \frac{2p}{(1-p)^3} + \frac{1}{(1-p)^2} = \frac{2p + (1-p)}{(1-p)^3} = \frac{p+1}{(1-p)^3}$$

$$\therefore \frac{1}{p}\sum_{k=0}^{\infty} k^2 p^k = \frac{p+1}{(1-p)^3} \quad \text{and} \quad \boxed{\sum_{k=0}^{\infty} k^2 p^k = \frac{p(p+1)}{(1-p)^3}}$$

21 April 2

Subst. $\overline{X} = (1-p)\left[\frac{1}{1-p} + n\frac{p}{(1-p)^2}\right]$

$$\overline{X} = 1 + \frac{np}{1-p}$$

$$\overline{X^2} = (1-p)\left[\frac{1}{1-p} + 2n\frac{p}{(1-p)^2} + n^2\,\frac{p+p^2}{(1-p)^3}\right]$$

$$= 1 + \frac{2np}{1-p} + \frac{n^2(p+p^2)}{(1-p)^2}$$

Then PK gives avg time in queue

$$W = \frac{\lambda\,\overline{X^2}}{2(1-\rho)}$$ rho

$$= \frac{\lambda\,\overline{X^2}}{2(1-\lambda\overline{X})} \quad \text{because } \overline{X} = \frac{1}{\mu}$$

and $T = \overline{X} + W$

Skip 3.5.1 – 3.5.2

~~E(x) E(x²)~~ 19 Mar 24

$$W = \frac{\lambda \overline{x^2}}{2(1-\rho)} \qquad \bar{x} = \frac{1}{\mu}, \quad \overline{x^2} = \frac{2}{\mu^2}, \quad \frac{2\lambda}{2\mu^2(1-\rho)} = W = \frac{\rho}{\mu(1-\rho)}$$

hr

$$\frac{N}{\lambda} = W = T - \frac{1}{\mu} = \frac{\rho}{\lambda(1-\rho)} - \frac{1}{\mu} \cancel{\mu\rho - \lambda - \rho\lambda} -$$

$$= \frac{\mu\rho - \lambda(1-\rho)}{\lambda\mu(1-\rho)} = \cancel{\mu^2\lambda}\ \frac{\cancel{\lambda - \lambda} + \lambda\rho}{\lambda\mu(1-\rho)} = \frac{\rho}{\mu(1-\rho)}$$

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26 ~~Feb~~ Apr

Priority Queueing 3.5.3

Consider k nonpreemptive priority class and n queues, have λ_k, $\frac{1}{\mu_k} = \overline{X}_k$, $\overline{X_k^2}$

N_Q^k avg # in queue for priority k

W_k " queueing time " " "

$\rho_k = \frac{\lambda_k}{\mu_k}$ utilization for " "

R mean residual service time

we assume $\rho = \sum_{k=1}^{n} \rho_k < 1$.

for highest priority (class 1), have $W_1 = R + \frac{1}{\mu_1} N_Q^1$

as per P-K. Since $N_Q^1 = \lambda_1 W_1$

have $W_1 = R + \rho_1 W_1$, $W_1 = \frac{R}{1-\rho_1}$

But ~~$W_2 = R + \rho_1 W_1$~~ +

$$W_2 = \underbrace{W_1}_{\text{wait for } W_1} + \underbrace{\frac{1}{\mu_2} N_Q^2}_{\text{wait for } W_2} + \underbrace{\frac{1}{\mu_1} \lambda_1 W_2}_{\text{wait for customers queue 1 that arrive while 2 is waiting in queue}}$$

$W_1 = \frac{R}{1-\rho_1}$

$$W_2 = R + \frac{1}{\mu_1} N_Q^1 + \frac{1}{\mu_2} N_Q^2 + \frac{1}{\mu_1} \lambda_1 W_2$$

by Little $N_Q^k = \lambda_k W_k$

$$W_2 = R + \rho_1 W_1 + \rho_2 W_2 + \rho_1 W_2 \Rightarrow W_2 = \frac{R + \rho_1 W_1}{1 - \rho_1 - \rho_2}$$

subst

$$W_2 = \frac{R(\cancel{1+\rho_1})?}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

check

26 Apr 2

We generalize

$$W_k = \frac{R}{(1-\rho_1 \cdots -\rho_{k-1})(1-\rho_1-\rho_2-\cdots-\rho_k)}$$

delay for kth cust. $T_k = \frac{1}{\mu_k} + W_k$

also

$$R = \frac{1}{2}\sum_{i=1}^{n} \lambda_i \overline{X_i^2}$$

then

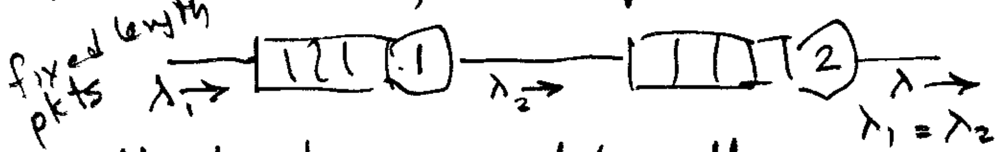
$$W_k = \frac{\frac{1}{2}\sum_{i=1}^{n} \lambda_i \overline{X_i^2}}{(1-\rho_1-\rho_2 \cdots -\rho_{k-1})(1-\rho_1-\rho_2 \cdots -\rho_k)}$$

skip ~~preemption~~ resume $T_k = \frac{1}{\mu_k} + W_k$

skip 3.5.4

3.6 Networks of Transmission Lines 29 Apr 96

Problem is that not always does interdeparture time be Poisson, so simple models don't work. e.g:



all pkts have equal length $\therefore \mu_1 = \mu_2$. and (1) is M/D/1. so PK formula works. Since $\rho < 1$, $\frac{1}{\mu_1} \le \frac{1}{\mu_2} = d$

But, arrivals at the second queue (2) have no waiting time, since each must be serviced before arrival of next pkt.

Even if pkt lengths are exponential, the arrivals and service times are not independent. In fact, there is no known analytical soln for even two tandem M/M/1 systems.

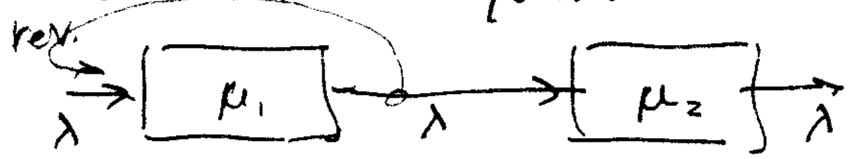
We note that, if ρ very small the interdeparture distribution approaches the interarrival distribution ($\frac{1}{\mu} = \bar{x} \to 0$). If ρ very close to 1, the interdeparture distribution approaches the service distribution (queue never empties)

30 Apr

Time Reversability - Burke's Thm.

~~We can~~
Consider the sequence of arrivals and departures. We know the mean rate of each must be the same. If we take the departure sequence, ~~and~~ reverse it and present to as arrivals, the departure sequence will be the same as the arrival sequence <u>reversed</u>. ~~Also, the number of customers in the system at each arrival and departure instant will be time reversed relative to the original system.~~

Now consider tandem queues



In this case we assume the service times of first and second queues are <u>statistically independent</u> (not like Kleinrock, where service time is dependent on packet length, which is the same for each queue

Burke's Thm
Consider an M/M/1 (and others) in equillib. with arrival rate λ Then

1. The departure process is Poisson with rate λ

2. At each time t the ~~#~~ number N of customers in the system is independent of the sequence of departure times prior to t. (memoryless)

Burke's Thm 8 may

Consider a M/M/1 queue with arrival rate λ and service rate μ. We note that

$A(t)$ is interarrival distribution $= 1 - e^{-\lambda t}$

$D(t)$ is " departure " we want find this

$B(t)$ is service distribution $1 - e^{-\mu t}$

Consider pdf of $A(t)$: $a(t) = \frac{d}{dt} A(t) = \lambda e^{-\lambda t}$

" $B(t)$: $b(t) = \frac{d}{dt} B(t) = \mu e^{-\mu t}$

And Laplace transform:

$$A(s) \equiv \mathcal{L}\{a(t)\} = \int_0^\infty a(t) e^{-st} dt = \int_0^\infty \lambda e^{-\lambda t} e^{-st} dt$$

$$= \int_0^\infty \lambda \underbrace{e^{-(s+\lambda)}}_{\frac{1}{s+\lambda}} dt$$

$$'' = \frac{\lambda}{s+\lambda}$$

also $B(s) = \frac{\mu}{s+\mu}$

There are two cases upon departure

1. no customer in service, must wait for new arrival plus service time. this means the sum of two r.v. or convolve $a(t) * b(t)$ or multiply transforms

$$D'(s) = A(s)\, B(s) = \frac{\lambda}{s+\lambda} \frac{\mu}{s+\mu}$$

2. next customer in service

$$D''(s) = B(s) = \frac{\mu}{s+\mu}$$

now, prob that ~~a~~ a departure leaves no customer in queue is equal to prob that a new arrival finds no customer in queue is $1-\rho$. $(=p_0)$

$$\therefore\ D(s) = (1-\rho)\, D'(s) + \rho\, D''(s) = (1-\rho)\frac{\lambda}{s+\lambda}\,\frac{\mu}{s+\mu} + \rho\,\frac{\mu}{s+\mu}$$

$$= \frac{\lambda\mu}{(s+\lambda)(s+\mu)} - \frac{\lambda}{\mu}\,\frac{\lambda\mu}{(s+\lambda)(s+\mu)} + \frac{\lambda}{\mu}\,\frac{\mu}{(s+\mu)}$$

$$= \frac{\lambda\mu^2 - \lambda^2\mu + (s+\lambda)\lambda\mu}{(s+\lambda)(s+\mu)\mu}$$

$$= \frac{\lambda\mu^{\cancel{2}} \cancel{-\lambda^2\mu} + s\lambda\cancel{\mu} \cancel{+\lambda^2\mu}}{(\quad)(\quad)\cancel{\mu}}$$

$$D(s) = \frac{\lambda(s+\mu)}{(s+\mu)(s+\lambda)} = \frac{\lambda}{s+\lambda}\quad !$$

Therefore ~~at~~ departure pdf = arrival pdf and for tandem queues the arrival process for the second queue is Poisson w/ param λ..

~~Kleinrock's assumption must be for other than M/M/1. ~~ ~~~~

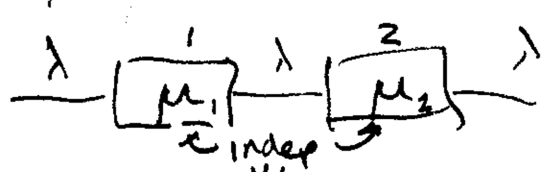
Burke's Thm

Consider an M/M/1, M/M/m, M/M/∞ system w/ arrival rate λ and steady state

a) departure process is Poisson w/ rate λ

b) at each time t, the

Two tandem queues

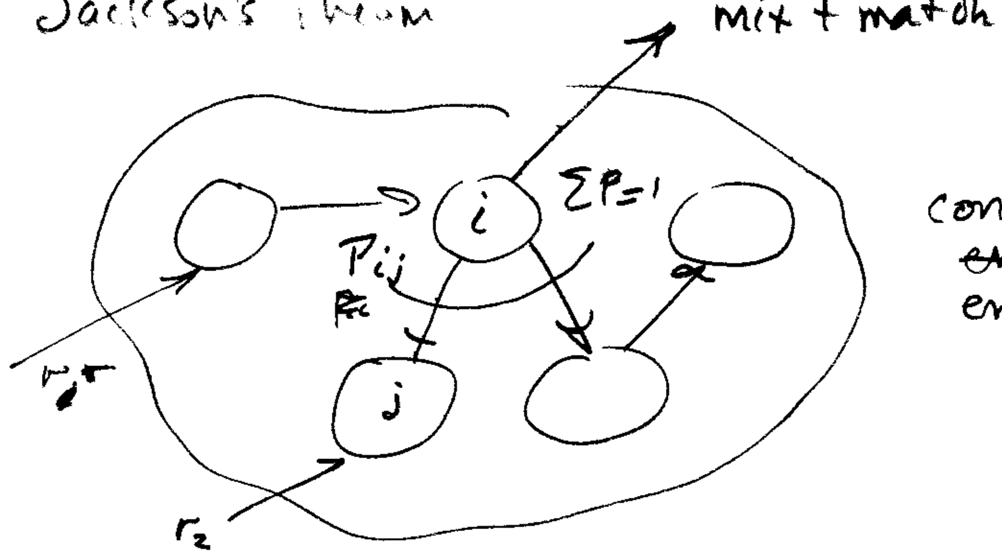


$$P\{n \text{ in } 1, m \text{ in } 2\} = \rho_1^{n}(1-\rho_1)\rho_2^{m}(1-\rho_2)$$

$$P\{n \text{ in } 1\}\, P\{m \text{ in } 2\}$$

Jackson's theorem

5 May 1



consider acyclic
~~eventually exit~~
eventually exit

Consider net of K queues (FIFO, single server)

let r_j = rate of arrivals from outside to j^{th} node

λ_i " " " from ith node

then

$$\lambda_j = r_j + \sum_{i=1}^{K} \lambda_i P_{ij}$$

P_{ij} = prob cust at node i goes to node j

P_{ij} = prob of ~~ith~~ pkt from ith node to jth.

And assume all queues are M/M/1. Let n_i be # customers in queue i and $\rho_i < 1$

$$\rho_j = \frac{\lambda_j}{\mu_j}$$

λ_j — as above

μ_j — service rate

n_j = # in jth queue

Jackson's Theorem, $n = \sum_{j=1}^{K} n_j$ state vector

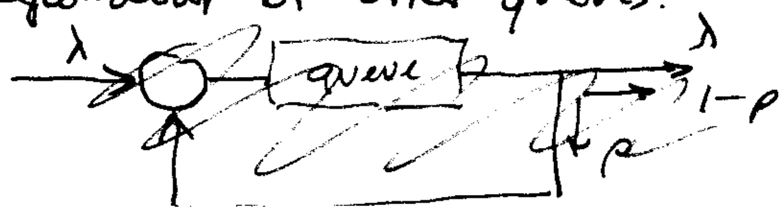
$$P(n) = P_1(n_1) P_2(n_2) \ldots P_K(n_K) \quad \text{(indep)}$$

and

$$P_j(n_j) = \rho_j^{n_j}(1-\rho_j) \quad n_j \geq 0$$

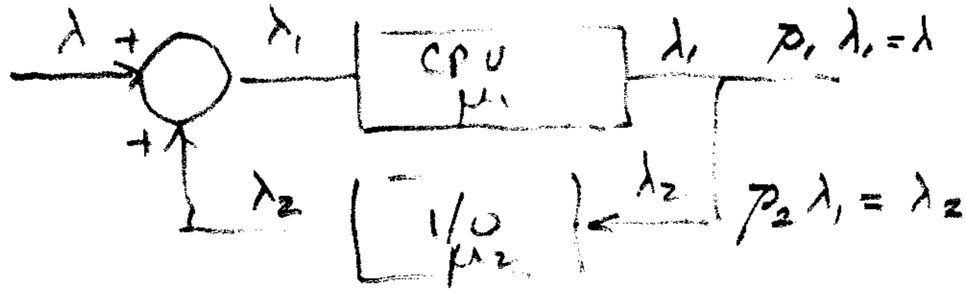
same as MM1 !

iE the numbers of customers in the queues are distributed as if each queue were M/M/1 and independent of other queues.



5 Mar 02

Example of Jackson's Theorem.



Jobs enter at rate λ. With prob p_1 they exit the system; $p_2 = 1 - p_1$ they go to I/O, then re-enter system.
(service times iid)

$\lambda_1 = \lambda + \lambda_2$, $\lambda_2 = p_2 \lambda_1$. Solve,

get $\lambda_1 = \dfrac{\lambda}{p_1}$; $\lambda_2 = \dfrac{\lambda p_2}{p_1}$

Let $\rho_1 = \dfrac{\lambda_1}{\mu_1}$, $\rho_2 = \dfrac{\lambda_2}{\mu_2}$, then

$$P_1(n_1) = \rho_1^{n_1}(1-\rho_1)$$
$$P_2(n_2) = \rho_2^{n_2}(1-\rho_2)$$

$$P(n_1, n_2) = \rho_1^{n_1}(1-\rho_1)\,\rho_2^{n_2}(1-\rho_2) \quad \text{by Jackson}$$

Then $N_1 = \dfrac{\rho_1}{1-\rho_1}$ $N_2 = \dfrac{\rho_2}{1-\rho_2}$ by M/M/1

and total # in sys $N = N_1 + N_2$

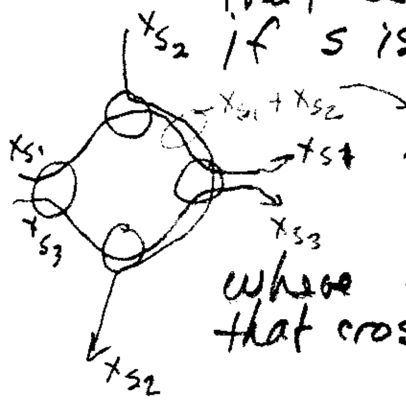
$T = \dfrac{N}{\lambda}$

NB, if $\lambda << \mu_1$ and $p_2 \to 1$, each new arrival causes many [illegible] and [illegible]
[illegible].

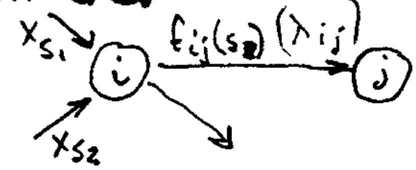
26 Apr

3.6.1 Kleinrock assumption

Let λ_{ij} be arrivals ~~to~~ at ith end of link that connects node i and node j. Then if s is arrival rate of stream



$$\lambda_{ij} = \sum_{\substack{\text{all } s \\ \text{crossing } i,j}} f_{ij}(s)\, x_s$$



where f_{ij} is the fraction of packets of streams x_s that cross i,j.

However, if many streams (independent) cross i,j, maybe departures are Poisson. This is the Kleinrock Independence Assumption.

- arrivals to the network must be independent Poisson streams
- message lengths must be exp. distrib.
- densely connected network (nb. VC and DG approx) (no loops)
- moderate to heavy traffic loads

Then by M/M/1 or use PK $N_{ij} = \dfrac{\lambda_{ij}}{\mu_{ij}-\lambda_{ij}}$ and $N_{\text{global}} = \sum_{(i,j)} \dfrac{\lambda_{ij}}{\mu_{ij}-\lambda_{ij}}$

By Littles thm. $T = \dfrac{1}{\sum_s x_s} \Big[\sum_{i,j} \underbrace{\dfrac{\lambda_{ij}}{\mu_{ij}-\lambda_{ij}}}_{N_{ij}} + \underbrace{\lambda_{ij} d_{ij}}_{\text{avg prop delay}} \Big]$

total arrival rate $\gamma \rightarrow \sum_s x_s$

Little's result for queue

Let p be a path then

$$T_p = \sum_{\substack{\text{all } i,j \\ \text{on path } p}} \Big(\underbrace{\frac{\lambda_{ij}}{\mu_{ij}(\mu_{ij}-\lambda_{ij})}}_{\text{queue}} + \underbrace{\frac{1}{\mu_{ij}}}_{\text{transm}} + \underbrace{d_{ij}}_{\text{prop}} \Big)$$

next P

$\mu_{ij}(\mu_{ij}-\lambda_{ij})$

26 Apr 2

p = path

$$T_p = \frac{1}{\lambda_p} \sum \left(\frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}} + \lambda_{ij} d_{ij} \right),$$ but for single path $\lambda_p = \sum \lambda_{ij}$, so

$$T_p = \sum \Big(\underbrace{\frac{1}{\mu_{ij} - \lambda_{ij}}}_{\text{from } M/M/1} + d_{ij} \Big)$$

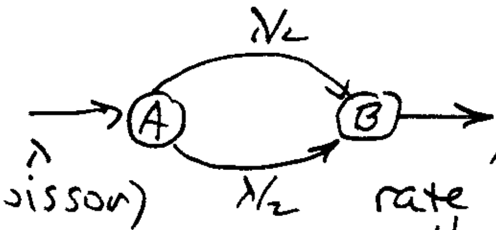
but $$W_Q = \overbrace{\frac{1}{\mu - \lambda}}^{T} - \underbrace{\frac{1}{\mu}}_{\bar{x}}$$

$$W_Q = \frac{\mu - \mu + \lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu - \lambda}$$

Therefore $$T_p = \sum \Big(\underbrace{\frac{\rho_{ij}}{\mu_{ij} - \lambda_{ij}}}_{\text{queue}} + \underbrace{\frac{1}{\mu_{ij}}}_{\text{trans}} + \underbrace{d_{ij}}_{\text{prop.}} \Big)$$

Tandem / Parallel queues 30 Apr 2

Consider an example



rate μ for both paths, exp. distr.

We can divide the traffic two ways

a) random decision. By prev. args, arrival at B on either path is Poisson. Thus each of two links acts as M/M/1 and avg delay

$$T_R = \frac{1}{\mu - \lambda/2} = \frac{2}{2\mu - \lambda}$$ consistent with Kleinrock

b) assign arrival to shortest queue. This is equiv to common queue and assign to first link that becomes idle. This is an M/M/2 system with arrival rate λ and each link acting like a server. The delay is

$$T_M = \frac{2}{(2\mu - \lambda)(1+\rho)}$$ in other words less than above by $\frac{1}{1+\rho}$ up to twice